Dispersion (3A)

- 1-D Dispersion

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1/18/14

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Dispersionless Wave

Dispersionless Wave — wave speed is independent of ω and k

Wave Equation

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$

A form of possible solutions

$$\psi(x,t) = A e^{i(kx - \omega t)}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 A e^{i(kx-\omega t)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 A e^{i(kx-\omega t)}$$

A trivial dispersion relation:

$$\omega^2 = c^2 k^2$$

wave velocity

$$v = \frac{\omega}{k} = \pm c$$

Phase Velocity of a Dispersionless Wave

A trivial dispersion relation:

$$\omega^2 = c^2 k^2$$

wave velocity

$$v = \frac{\omega}{k} = \pm c$$

The speed of $\sin(k x - \omega t)$

How fast a point with constant phase $(k x - \omega t)$ moves

$$(k x - \omega t) = const \quad \Rightarrow \quad \frac{d}{dt}(k x - \omega t) = 0 \quad \Rightarrow \quad k \frac{dx}{dt} - \omega = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k}$$

phase velocity

Dispersionful Wave

A Dispersionless System

A linear relationship between ω and k

$$v_p = \frac{\omega}{k} = c$$

$$v_g = \frac{d\omega}{dk} = 0$$

All of the wave components move with the same speed v_p

Any function of the form $f(x-ct)$: dispersionless

A Dispersionful System

A non-linear relationship between ω and k

$$v_p = \frac{\omega}{k} \neq c$$

$$v_g = \frac{d\omega}{dk} \neq 0$$

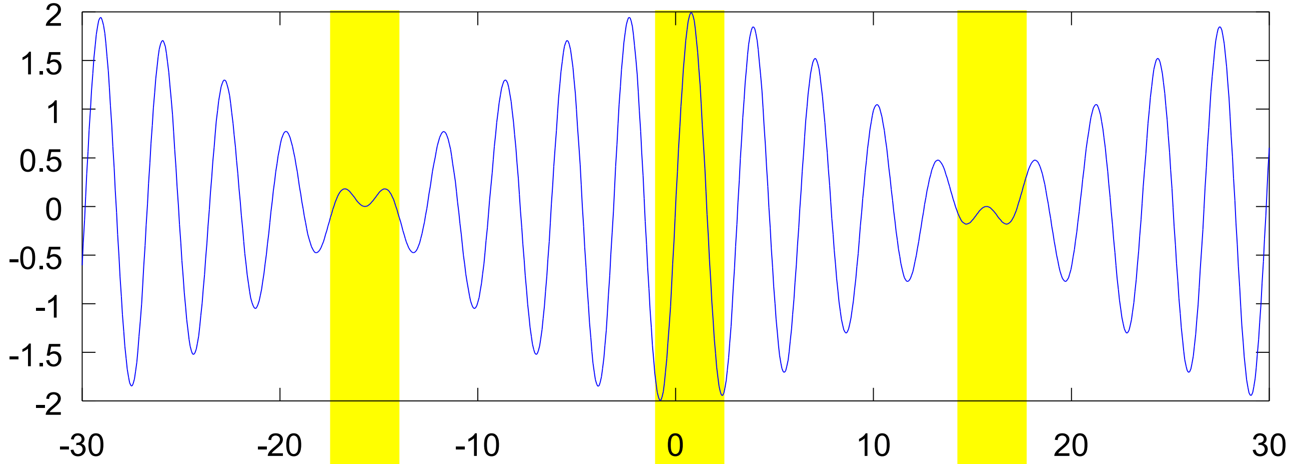
The different sinusoidal waves that make up the bump travel at different speeds

Which value of k is chosen to get the group velocity?
The value of k where the bump dominates – at the peak of the Fourier Transform of the bump

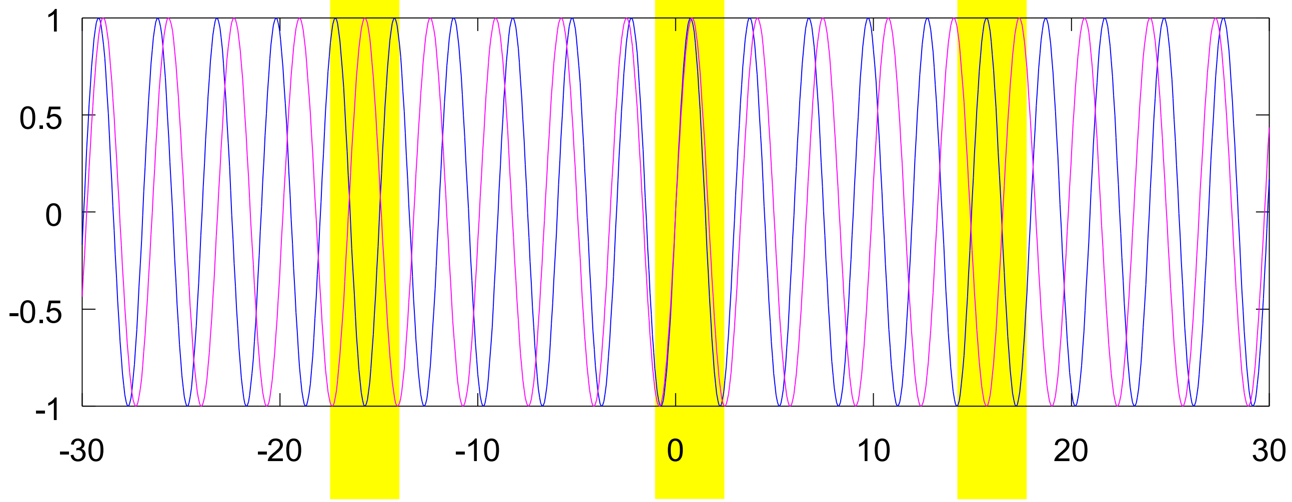
$$v_g = \frac{d\omega}{dk}$$

group velocity

Dispersionful Wave Example



$\sin(2.1\,t) + \sin(1.9\,t)$



$\sin(2.1\,t)$
$\sin(1.9\,t)$

Group Velocity Derivation: Method I (1)

$$\psi_1(x,t) = A\cos(\omega_1 t - k_1 x)$$
$$\psi_2(x,t) = A\cos(\omega_2 t - k_2 x)$$

$$\omega_\Sigma = \frac{\omega_1 + \omega_2}{2} \qquad \omega_\Delta = \frac{\omega_1 - \omega_2}{2}$$

$$\omega_1 = \omega_\Sigma + \omega_\Delta \qquad \omega_2 = \omega_\Sigma - \omega_\Delta$$

$$k_\Sigma = \frac{k_1 + k_2}{2} \qquad k_\Delta = \frac{k_1 - k_2}{2}$$

$$k_1 = k_\Sigma + k_\Delta \qquad k_2 = k_\Sigma - k_\Delta$$

$$\psi_1(x,t) = A\cos((\omega_\Sigma + \omega_\Delta)t - (k_\Sigma + k_\Delta)x)$$
$$= A\cos((\omega_\Sigma t - k_\Sigma x) + (\omega_\Delta t - k_\Delta x))$$

$$\psi_2(x,t) = A\cos((\omega_\Sigma - \omega_\Delta)t - (k_\Sigma - k_\Delta)x)$$
$$= A\cos((\omega_\Sigma t - k_\Sigma x) - (\omega_\Delta t - k_\Delta x))$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\psi_1(x,t) + \psi_2(x,t)$$
$$= 2A\cos(\omega_\Sigma t - k_\Sigma x)\cos(\omega_\Delta t - k_\Delta x)$$
$$= 2A\cos(\omega_\Delta t - k_\Delta x)\cos(\omega_\Sigma t - k_\Sigma x)$$

slow moving envelope — fast moving actual sum

$$\omega_\Sigma \gg \omega_\Delta \qquad k_\Sigma \gg k_\Delta$$

Group Velocity Derivation: Method I (2)

$$\psi_1(x,t) = A\cos(\omega_1 t - k_1 x)$$
$$\psi_2(x,t) = A\cos(\omega_2 t - k_2 x)$$

$$\omega_\Sigma = \frac{\omega_1 + \omega_2}{2} \qquad \omega_\Delta = \frac{\omega_1 - \omega_2}{2}$$

$$k_\Sigma = \frac{k_1 + k_2}{2} \qquad k_\Delta = \frac{k_1 - k_2}{2}$$

$$\omega_\Sigma \gg \omega_\Delta \qquad k_\Sigma \gg k_\Delta$$
$$\omega_\Sigma \approx \omega_1 \approx \omega_2 \qquad k_\Sigma \approx k_1 \approx k_2$$

$$v_p = \frac{\omega}{k} \qquad v_q = \frac{d\omega}{dk}$$

$$\psi_1(x,t) + \psi_2(x,t)$$
$$= 2A\cos(\omega_\Delta t - k_\Delta x)\cos(\omega_\Sigma t - k_\Sigma x)$$

slow moving envelope — fast moving actual sum

The phase velocity of fast moving wave

$$\frac{\omega_\Sigma}{k_\Sigma} \approx \frac{\omega_1}{k_1} \approx \frac{\omega_2}{k_2}$$

The phase velocity of the envelope wave
wave formed by two waves

$$\frac{\omega_\Delta}{k_\Delta} = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

➡ The group velocity

c *when* $\omega = ck$

$\frac{d\omega}{dk}$ *when* $\omega(k)$ $\quad (k_1 - k_2) \to 0$

Group Velocity Derivation: Method I (3)

$$\psi_1(x,t) + \psi_2(x,t)$$
$$= 2A\cos(\omega_\Delta t - k_\Delta x)\cos(\omega_\Sigma t - k_\Sigma x)$$

slow moving envelope

fast moving actual sum

$$v_p = \frac{\omega}{k}$$

$$v_q = \frac{d\omega}{dk}$$

The fast wiggles move wrt the envelope

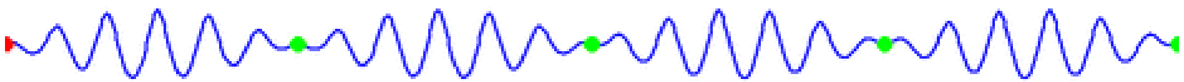
$$v_p > v_g$$

The little wiggles op into existence at the **left** end of an envelope bump

$$v_p < v_g$$

The little wiggles op into existence at the **right** end of an envelope bump

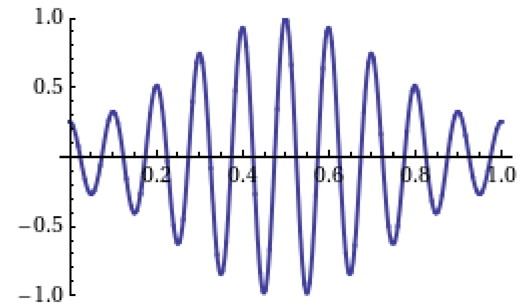
Group Velocity Derivation: Method I (4)



$v_p > v_g$

The little wiggles pop into existence at the left end of an envelope bump

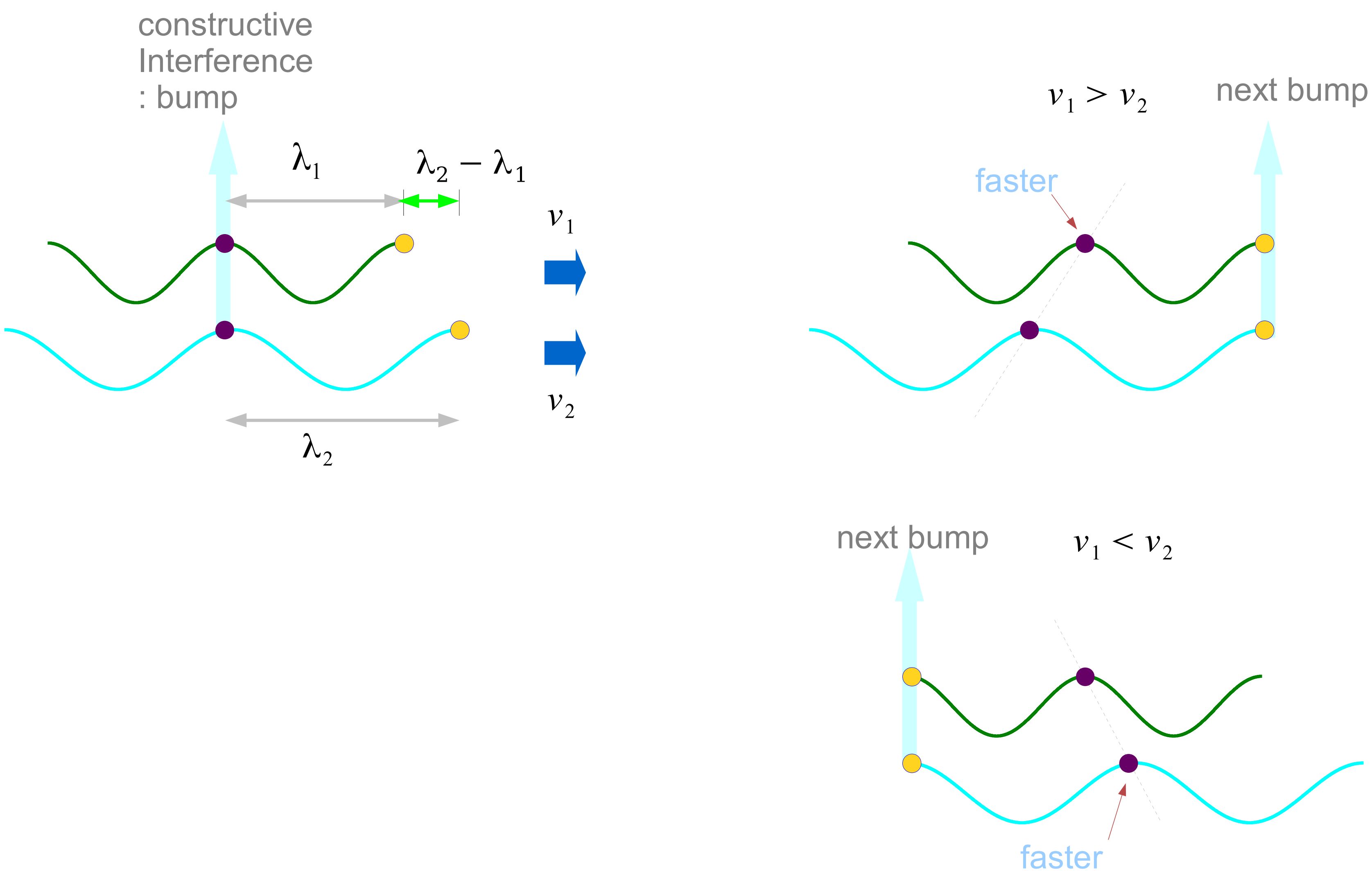
They grow and then shrink as they move through the bump, until finally they disappear when they reach the right end of the bump



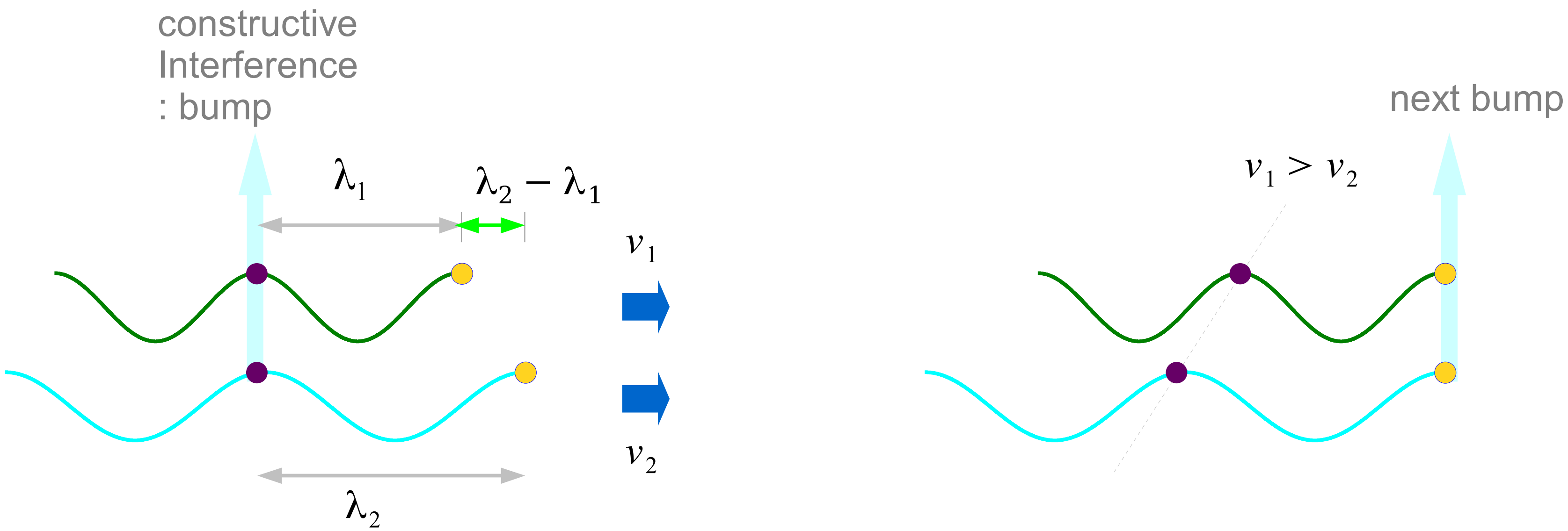
$v_g > 0$

$v_p < 0$

Group Velocity Derivation: Method II (1)



Group Velocity Derivation: Method II (2)



initial distance	$\lambda_2 - \lambda_1$	next alignment	$x = \lambda_1 + v_1 t$ $x = \lambda_2 + v_2 t$
relative velocity	$v_1 - v_2 \qquad (v_1 > v_2)$		
time lapse	$t = \dfrac{(\lambda_2 - \lambda_1)}{(v_1 - v_2)}$	bump speed	$\dfrac{x}{t}$

$$\frac{x}{t} = \frac{\lambda_1 + v_1 t}{t} = \frac{\lambda_1}{t} + v_1 = \lambda_1\left(\frac{v_1 - v_2}{\lambda_2 - \lambda_1}\right) + v_1 = \frac{\lambda_1 v_1 - \lambda_1 v_2}{\lambda_2 - \lambda_1} + \frac{\lambda_2 v_1 - \lambda_1 v_1}{\lambda_2 - \lambda_1}$$

Group Velocity Derivation: Method II (3)

$$\frac{x}{t} = \frac{\lambda_1 + v_1 t}{t} = \frac{\lambda_1}{t} + v_1 = \lambda_1\left(\frac{v_1 - v_2}{\lambda_2 - \lambda_1}\right) + v_1 = \frac{\lambda_1 v_1 - \lambda_1 v_2}{\lambda_2 - \lambda_1} + \frac{\lambda_2 v_1 - \lambda_1 v_1}{\lambda_2 - \lambda_1}$$

$$= \frac{\lambda_2 v_1 - \lambda_1 v_2}{\lambda_2 - \lambda_1} \qquad v = \frac{\omega}{k} \qquad k = \frac{2\pi}{\lambda}$$

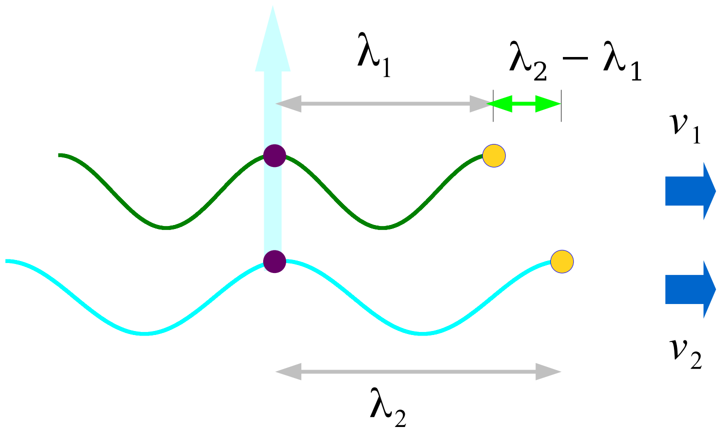
$$= \frac{\frac{2\pi}{k_2}\frac{\omega_1}{k_1} - \frac{2\pi}{k_1}\frac{\omega_2}{k_2}}{\frac{2\pi}{k_2} - \frac{2\pi}{k_1}} = \frac{\frac{2\pi}{k_1 k_2}(\omega_1 - \omega_2)}{\frac{2\pi}{k_1 k_2}(k_1 - k_2)}$$

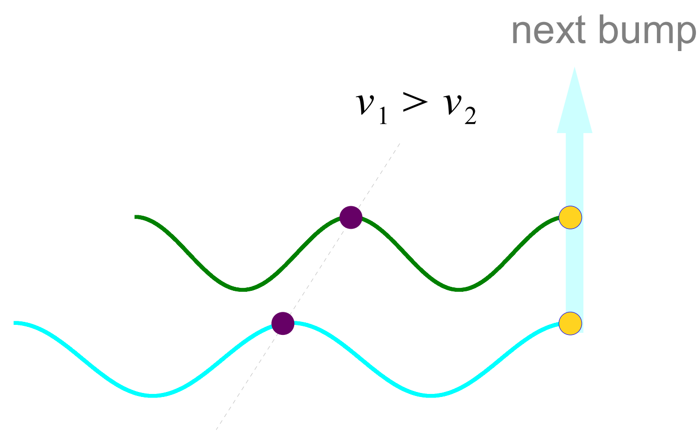
$$= \frac{(\omega_1 - \omega_2)}{(k_1 - k_2)} = v_g$$

$$\frac{x}{t} = \frac{\lambda_1 + v_1 t}{t} = \frac{\lambda_1}{t} + v_1 = \lambda_1\left(\frac{v_1 - v_2}{\lambda_2 - \lambda_1}\right) + v_1 = \frac{\lambda_1 v_1 - \lambda_1 v_2}{\lambda_2 - \lambda_1} + \frac{\lambda_2 v_1 - \lambda_1 v_1}{\lambda_2 - \lambda_1}$$

Group Velocity Derivation: Method II (4)

A pair of waves





$k_1 \approx k_2$
$\omega_1 \approx \omega_2$

⇒

$\lambda_1 \approx \lambda_2$
$v_1 \approx v_2$

⇒

$$v_g = \frac{(\omega_1 - \omega_2)}{(k_1 - k_2)} \uparrow$$

The nearly equal wavelengths

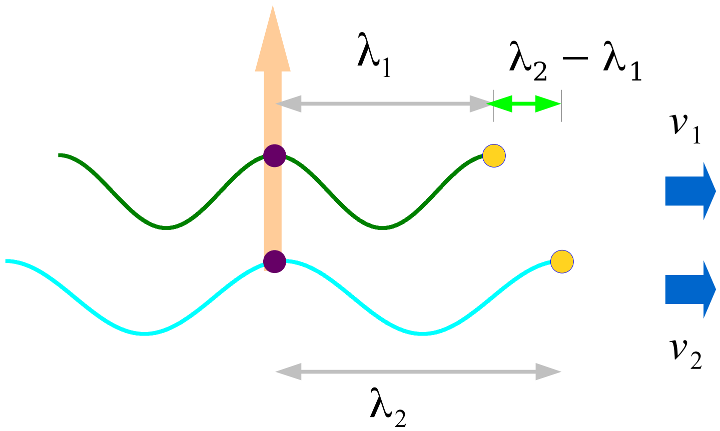
$\lambda_2 - \lambda_1$ very small

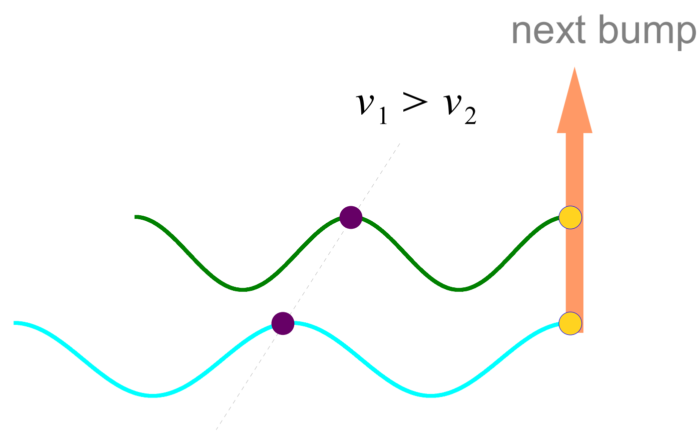
- the location of the alignment jumps ahead
- by a distance of one wavelength
- in essentially no time

this means that the effective speed is large
(at least as large as the λ_1 / t)

Group Velocity Derivation: Method II (5)

A pair of waves

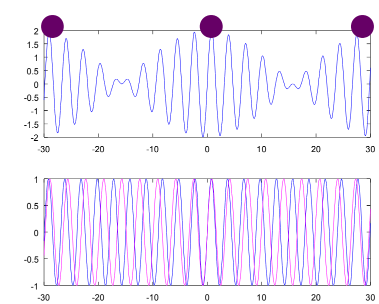




In between alignment of peaks
the bump disappears,
then appears in the negative direction
then disappears again
before reappearing at the next bump

But on average, the bump effectively moves with velocity

$$v_g = \frac{(\omega_1 - \omega_2)}{(k_1 - k_2)}$$

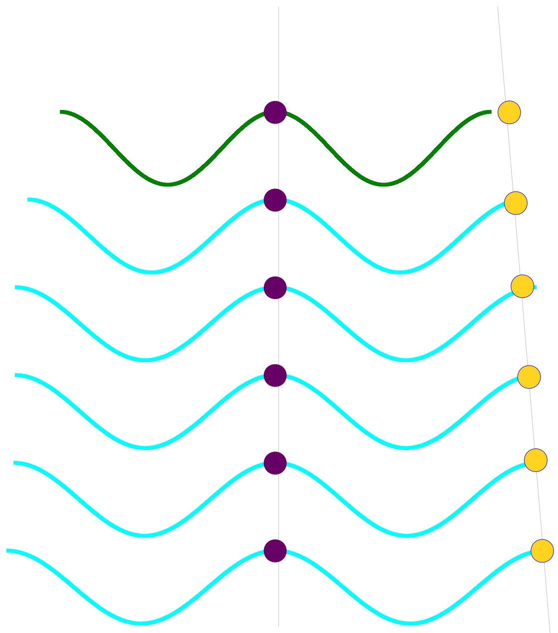


Consistent with the fact that
the wiggly wave doesn't always touch the midpoint
– the highest point of the envelop bump.
(in fact rarely does)

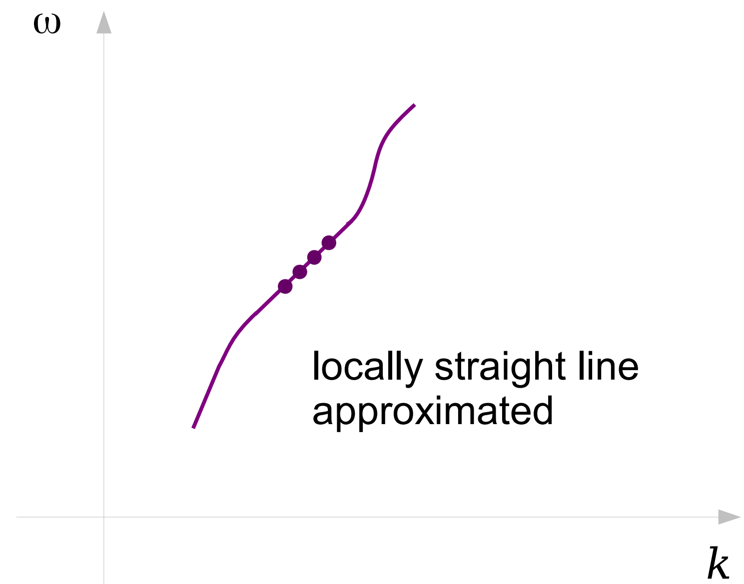
Group Velocity Derivation: Method II (6)

A large number of waves

roughly the same values of
k and ω



$$v_{p1} = \omega_1 / k_1$$
$$v_{p2} = \omega_2 / k_2$$
$$v_{p3} = \omega_3 / k_3$$
$$v_{p4} = \omega_4 / k_4$$
$$v_{p5} = \omega_5 / k_5$$
$$v_{p6} = \omega_6 / k_6$$



different phase velocities

the same group velocity

$$x = v_g t \qquad v_g = x / t$$

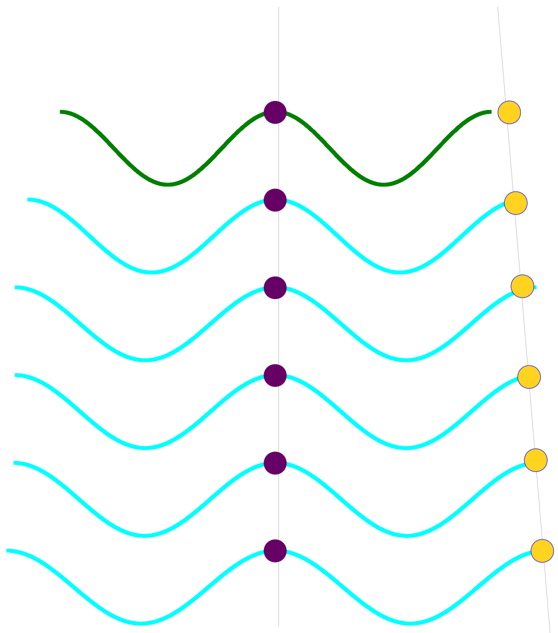
$$\frac{d\omega}{dk} \approx \frac{(\omega_i - \omega_j)}{(k_i - k_j)}$$

The various waves all travel with different phase velocity $v_p = \omega / k$
The group velocity depends only on the differences in ω and k
Not on the actual values of ω and k

Group Velocity Derivation: Method II (7)

A large number of waves

roughly the same values of k and ω



$$v_{p1} = \omega_1 / k_1$$

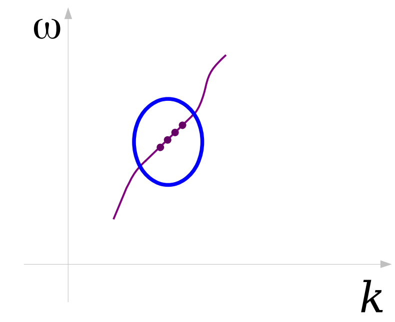
$$v_{p2} = \omega_2 / k_2$$

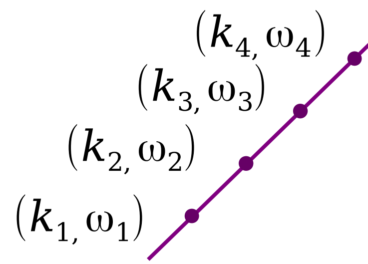
$$v_{p3} = \omega_3 / k_3$$

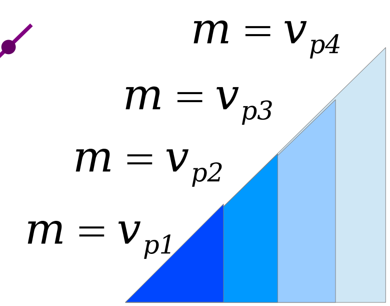
$$v_{p4} = \omega_4 / k_4$$

$$v_{p5} = \omega_5 / k_5$$

$$v_{p6} = \omega_6 / k_6$$







the same slope

$$x = v_g t \qquad v_g = x / t$$

$$\frac{d\omega}{dk} \approx \frac{(\omega_i - \omega_j)}{(k_i - k_j)}$$

Group Velocity Derivation: Method III (1)

Fourier Analysis

A wave consists of components with many different frequencies

Bump at a certain place

Bump at the origin $x = 0,\ t = 0$

The phases of the various components must be equal at the bump

$$\omega_i \cdot 0 - k_i \cdot 0 + \phi_i = \omega_j \cdot 0 - k_j \cdot 0 + \phi_j$$

$$\phi_i = \phi_j$$

ϕ Independent of k

for constructive interference

$$\omega_i t - k_i x + \phi_i$$

the same phase

ϕ Independent of k

$$\frac{d\phi}{dk} = 0 \Rightarrow \frac{d\omega}{dk} = v_g$$

ϕ Independent of t

$$\frac{d\phi}{dt} = 0 \Rightarrow \frac{dx}{dt} = v_p$$

Group Velocity Derivation: Method III (2)

ϕ Independent of k

$$\frac{d}{dk}(\omega t - k x + \phi) = 0 \qquad \frac{d\phi}{dk} = 0$$

$$\frac{d\omega}{dk} t - x = 0 \qquad v_g = \frac{d\omega}{dk} = \frac{x}{t}$$

ϕ Independent of t

$$\frac{d}{dt}(\omega t - k x + \phi) = 0 \qquad \frac{d\phi}{dt} = 0$$

$$\omega - k \frac{dx}{dt} = 0 \qquad v_p = \frac{dx}{dt} = \frac{\omega}{k}$$

Group Velocity Derivation: Method III (3)

$\frac{d\omega}{dk}$ exists ⇏ there is a bump with a group velocity v_g

if there is a bump ⇒ It is traveling with the velocity $v_g = \frac{d\omega}{dk}$

evaluated at the *k* value that dominates the bump

found by Fourier Transform of the bump

$$\omega - k\frac{dx}{dt} = 0$$

$$v_p = \frac{dx}{dt} = \frac{\omega}{k}$$

References

[1] http://en.wikipedia.org/
[2] http://www.people.fas.harvard.edu/~djmorin/book.html D Morin, “Waves”