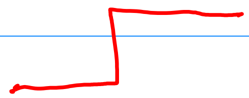
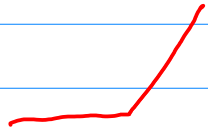
Time Responses (H.1) Standard Forms

20150529

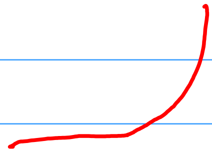
step fn $u(t) \Longleftrightarrow \frac{1}{s}$



ramp fn $t\,u(t) \Longleftrightarrow \frac{1}{s^2}$



parabola fn $t^2 u(t) \Longleftrightarrow \frac{2}{s^3}$



p

v

a

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$$

$u(t)$ $\quad \frac{1}{s}$ $\qquad C(s) = G(s) \cdot \frac{1}{s}$

$t\,u(t)$ $\quad \frac{1}{s^2}$ $\qquad C(s) = G(s) \cdot \frac{1}{s^2}$

$\frac{t^2}{2}u(t)$ $\quad \frac{1}{s^3}$ $\qquad C(s) = G(s) \cdot \frac{1}{s^3}$

Steady State : $\boxed{t \to \infty}$

$$y'' + ay' + b = x(t)$$

$$y_h = c_1 e^{-m_1 t} + c_2 e^{-m_2 t}$$ $m_1 > 0$, $m_2 > 0$

$t \to \infty$ $y_h \to 0$

$y_p =$ $x(t)$와 이웃한 모습 : undetermined coefficient

* Time Response

First Order System

$$\boxed{G(s) = \frac{a}{s+a}}$$

pole : $-a$ 한개

stable → $-a < 0$

$a > 0$

time constant $\frac{1}{a} = \tau$

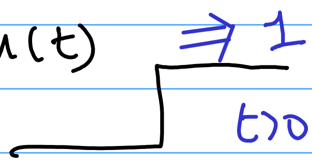
$$\frac{Y(s)}{X(s)} = \frac{a}{s+a}$$

$(s+a)Y(s) = a\,X(s)$

$$\boxed{y' + a\,y(t) = a\,x(t)}$$

Unit step response $y(t)$?

$x(t) = u(t)$



$$\boxed{y' + a\,y(t) = 1 \cdot a} \quad (t>0)$$

$y_h = C_1 e^{-at}$

$y_p = 1$

$y_p = A$

$A' + aA = a$

$A = 1$

$y = y_h + y_p = C_1 e^{-at} + 1$

$$\boxed{y(0^+) = 0}$$

$C_1 e^0 + 1 = 0$

$C_1 = -1$

$$\boxed{y = 1 - e^{-at}}$$

* Time Response

2nd order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\frac{-b \pm \sqrt{b^2 - 4a}}{2a}$$

$$\frac{-b \pm \sqrt{b^2 - ac}}{a}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

$$= -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

$\omega_n > 0$

$D = \zeta^2 - 1 > 0$ 두 실근 (Overdamping)

$$s_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1}\,\omega_n$$

$$s_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1}\,\omega_n$$

$D = \zeta^2 - 1 = 0$ 중근 (Critically Damping)

$$s_1 = s_2 = -\zeta\omega_n + \sqrt{\overset{0}{\zeta^2 - 1}}\,\omega_n = -\zeta\omega_n$$

$D = \zeta^2 - 1 < 0$ (Underdamping)

$$s_1 = -\zeta\omega_n + j\sqrt{1 - \zeta^2}\,\omega_n = \sigma + j\omega$$

$$s_2 = -\zeta\omega_n - j\sqrt{1 - \zeta^2}\,\omega_n = \sigma - j\omega$$

$D = \zeta^2 - 1 > 0$ 두 실근 (Overdamping)

$$G(s) = \frac{\omega_n^2}{(s-s_1)(s-s_2)}$$

$$\boxed{\begin{aligned} s_1 &= -\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n \\ s_2 &= -\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n \end{aligned}}$$

$D = \zeta^2 - 1 = 0$ 중근 (Critically Dampin)

$$G(s) = \frac{\omega_n^2}{(s-s_1)^2}$$

$$\boxed{s_1 = s_2 = -\zeta\omega_n}$$

$D = \zeta^2 - 1 < 0$ (Underdampin)

$$G(s) = \frac{\omega_n^2}{(s-s_1)(s-s_2)}$$

$$\boxed{\begin{aligned} s_1 &= -\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n \\ s_2 &= -\zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n \end{aligned}}$$

$$(s-s_1)(s-s_2) = s^2 - (s_1+s_2)s + s_1 s_2$$

Verification

$(a+b)(a-b) = a^2 - b^2$
$(a+jb)(a-jb) = a^2 - (jb)^2$
$= a^2 + b^2$

$D = \zeta^2 - 1 > 0$ 두 실근 (Overdampin)

$s_1 + s_2 = -2\zeta\omega_n$

$s_1 \cdot s_2 = (-\zeta\omega_n)^2 - \left(\sqrt{\zeta^2-1}\,\omega_n\right)^2$

$= \zeta^2\omega_n^2 - (\zeta^2 - 1)\omega_n^2$

$= \omega_n^2$

$$\boxed{\begin{aligned} s_1 &= -\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n \\ s_2 &= -\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n \end{aligned}}$$

$D = \zeta^2 - 1 = 0$ 중근 (Critically Dampin)

$s_1 + s_2 = 2s_1 = -2\zeta\omega_n$

$s_1 \cdot s_2 = s_1^2 = \zeta^2\omega_n^2 = \omega_n^2$

$\boxed{\zeta^2 = 1}$

$$\boxed{s_1 = s_2 = -\zeta\omega_n}$$

$D = \zeta^2 - 1 < 0$ (Underdampin)

$s_1 + s_2 = -2\zeta\omega_n$

$s_1 \cdot s_2 = (-\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2$

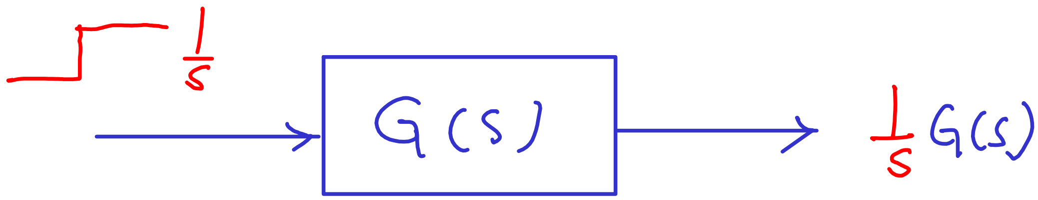
$= \zeta^2\omega_n^2 + (1-\zeta^2)\omega_n^2$

$= \omega_n^2$

$$\boxed{\begin{aligned} s_1 &= -\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n \\ s_2 &= -\zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n \end{aligned}}$$

$$\boxed{\begin{aligned} (s-s_1)(s-s_2) &= s^2 - (s_1+s_2)s + s_1 s_2 \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 \end{aligned}}$$

Step response $= \mathcal{L}^{-1}\left\{\frac{1}{s}G(s)\right\}$



$D = \zeta^2 - 1 > 0$ 두 실근 (Overdamping)

$$\frac{1}{s}G(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{k_0}{s} + \frac{k_1}{(s-s_1)} + \frac{k_2}{(s-s_2)}$$

$D = \zeta^2 - 1 = 0$ 중근 (Critically Damping)

$$\frac{1}{s}G(s) = \frac{\omega_n^2}{s(s-s_1)^2} = \frac{k_0}{s} + \frac{k_1}{(s-s_1)} + \frac{k_2}{(s-s_1)^2}$$

$D = \zeta^2 - 1 < 0$ (Underdamping)

$$\frac{1}{s}G(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{k_0}{s} + \frac{k_1}{s-s_1} + \frac{k_2}{s-s_2}$$

or $$= \frac{k_0}{s} + \frac{k_1 s + k_2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Use $\Rightarrow$ $$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) - \zeta^2\omega_n^2 + \omega_n^2}$$

$$= \frac{1}{(s + \zeta\omega_n)^2 + (1-\zeta^2)\omega_n^2}$$

Overdamping $\zeta^2 > 1$

$$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{A}{s} + \frac{B}{(s-s_1)} + \frac{C}{(s-s_2)}$$

$$s_1 = -\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n \qquad \frac{1}{s_1} = \frac{-\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n}{(\zeta\omega_n)^2 - (\sqrt{\zeta^2-1}\,\omega_n)^2} = \frac{s_2}{\omega_n^2}$$

$$s_2 = -\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n \qquad \frac{1}{s_2} = \frac{-\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n}{(\zeta\omega_n)^2 - (\sqrt{\zeta^2-1}\,\omega_n)^2} = \frac{s_1}{\omega_n^2}$$

$$s_1 - s_2 = 2\sqrt{\zeta^2-1}\,\omega_n$$

$$A = \left.\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right|_{s=0} = 1$$

$$B = \left.\frac{\omega_n^2}{s(s-s_2)}\right|_{s=s_1} = \left(\frac{\omega_n}{s_1}\right)^2 \frac{1}{s_1-s_2} = \frac{s_2}{2\sqrt{\zeta^2-1}\,\omega_n}$$

$$C = \left.\frac{\omega_n^2}{s(s-s_1)}\right|_{s=s_2} = \left(\frac{\omega_n}{s_2}\right)^2 \frac{1}{s_2-s_1} = \frac{s_1}{-2\sqrt{\zeta^2-1}\,\omega_n}$$

$$u(t) + Be^{-s_1 t} + Ce^{-s_2 t}$$

$$= u(t) + \frac{1}{2\sqrt{\zeta^2-1}\,\omega_n}\Big[(-\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n)e^{(-\zeta\omega_n+\sqrt{\zeta^2-1}\,\omega_n)t} - (-\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n)e^{(-\zeta\omega_n-\sqrt{\zeta^2-1}\,\omega_n)t}\Big]$$

$$= u(t) + \frac{1}{2\sqrt{\zeta^2-1}}\Big[(-\zeta - \sqrt{\zeta^2-1})e^{(-\zeta\omega_n+\sqrt{\zeta^2-1}\,\omega_n)t} - (-\zeta + \sqrt{\zeta^2-1})e^{(-\zeta\omega_n-\sqrt{\zeta^2-1}\,\omega_n)t}\Big]$$

Critical Damping $\zeta = 1$

$$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s(s-s_1)^2} = \frac{A}{s} + \frac{B}{(s-s_1)^2} + \frac{C}{(s-s_2)}$$

$$A = \left.\frac{\omega_n^2}{(s-s_1)^2}\right|_{s_1=0} = \frac{\omega_n^2}{s_1^2} = \frac{\omega_n^2}{\zeta^2\omega_n^2} = \frac{1}{\zeta^2} = 1 \quad \boxed{\zeta = 1}$$

$$B = \left.\frac{\omega_n^2}{s}\right|_{s=-\zeta\omega_n} = \frac{\omega_n^2}{-\zeta\omega_n} = -\frac{\omega_n}{\zeta} = -\omega_n$$

$$C = \left.\left(\frac{d}{ds}\frac{-\omega_n^2}{s}\right)\right|_{s=-\zeta\omega_n} = -\left.\left(\frac{\omega_n^2}{s^2}\right)\right|_{s=-\zeta n} = -\frac{1}{\zeta^2} = -1$$

$$y(t) = u(t) - \omega_n t e^{-\omega_n t} - e^{-\omega_n t}$$

Underdamping $\zeta^2 < 1$ ①

$$\boxed{\frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{k_0}{s} + \frac{k_1 s + k_2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}}$$

$$k_0 s^2 + 2k_0\zeta\omega_n s + k_0\omega_n^2 + k_1 s^2 + k_2 s = \omega_n^2$$

$$\underbrace{(k_0 + k_1)}_{=0} s^2 + \underbrace{(2k_0\zeta\omega_n + k_2)}_{=0} s + \underbrace{k_0}_{=1}\omega_n^2 = \omega_n^2$$

$$k_1 = -1 \qquad -2\zeta\omega_n = k_2$$

$$\boxed{\frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{1}{s} - \frac{s}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} - \frac{2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}}$$

$$\frac{1}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{1}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) - \zeta^2\omega_n^2 + \omega_n^2}$$

$$= \frac{1}{(s + \zeta\omega_n)^2 + (1-\zeta^2)\omega_n^2}$$

$$= \frac{1}{(s + \zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2}$$

$$\boxed{= \frac{1}{s} - \left\{ \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 - \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} + \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 - \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} \right\}}$$

Underdamping $\zeta^2 < 1$ ②

$$\boxed{= \frac{1}{s} - \left\{ \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} + \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} \right\}}$$

$$\boxed{\begin{aligned} &\frac{s}{s^2+k^2} \Longleftrightarrow \cos(kt) \\ &\frac{(s+a)}{(s+a)^2+k^2} \Longleftrightarrow e^{-at}\cos(kt) \end{aligned}} \qquad \boxed{\begin{aligned} &\frac{k}{s^2+k^2} \Longleftrightarrow \sin(kt) \\ &\frac{k}{(s+a)^2+k^2} \Longleftrightarrow e^{-at}\sin(kt) \end{aligned}}$$

$$\left\{ \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} + \frac{\zeta\left(\sqrt{1-\zeta^2}\,\omega_n\right)\frac{1}{\sqrt{1-\zeta^2}}}{(s+\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} \right\}$$

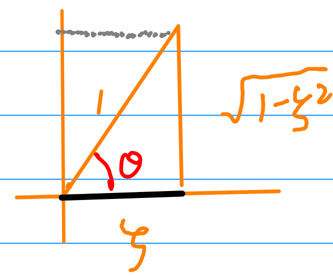
$$= e^{-\zeta\omega_n t}\left\{ \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \right\}$$

$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left\{ \sqrt{1-\zeta^2}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \zeta \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \right\}$$

$$\boxed{\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left\{\left(\sqrt{1-\zeta^2}\right)\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + (\zeta \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}}$$

① underdamping $\zeta^2 < 1$

$\cos\theta = \zeta$

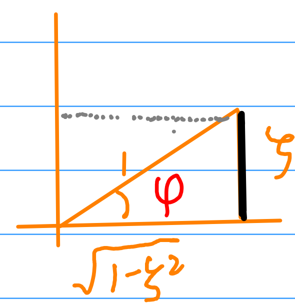


$$\boxed{\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left\{\sin\theta \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \cos\theta \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}}$$

$\sin(\alpha+\beta)$

② underdamping $\zeta^2 < 1$

$\sin\varphi = \zeta$



$$\boxed{\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left\{\cos\varphi \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \sin\varphi \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}}$$

$\cos(\alpha-\beta)$

$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \left\{ \sin\theta \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \cos\theta \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \right\}$$

① $$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \theta\right) \qquad \cos\theta = \zeta$$

② $$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \varphi\right) \qquad \sin\varphi = \zeta$$

$$\theta + \varphi = \frac{\pi}{2}$$

Underdamping $\zeta^2 < 1$ ③

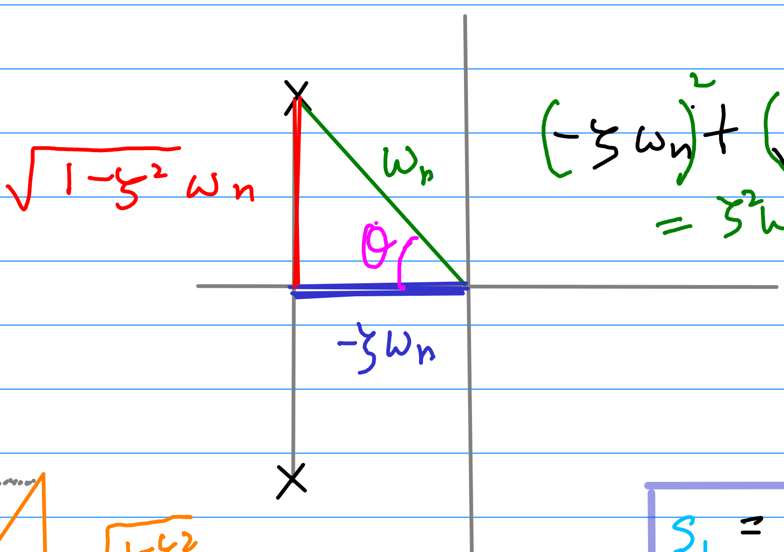
$$\frac{1}{s}G(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$= \frac{1}{s} - \left\{ \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 - \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} + \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 - \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2} \right\}$$

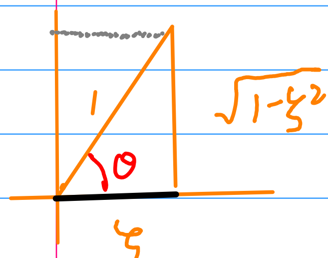
$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \left\{ \sqrt{1-\zeta^2}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \zeta \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \right\}$$

$$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \theta\right) \qquad \cos\theta = \zeta$$

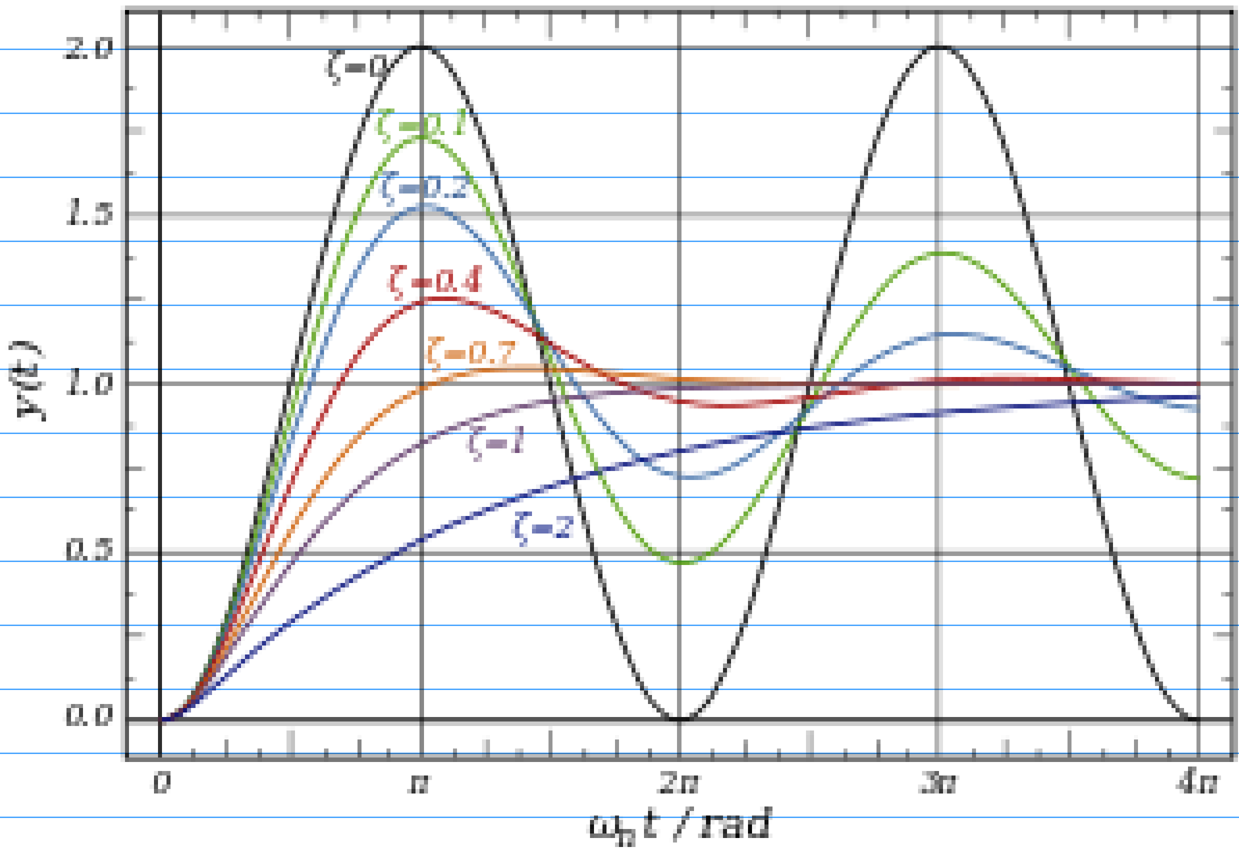
$$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \varphi\right) \qquad \sin\varphi = \zeta$$



$$(-\zeta\omega_n)^2 + \left(\sqrt{1-\zeta^2}\,\omega_n\right)^2$$
$$= \zeta^2\omega_n^2 + (1-\zeta^2)\omega_n^2 = \omega_n^2$$



$$s_1 = -\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n$$
$$s_2 = -\zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n$$

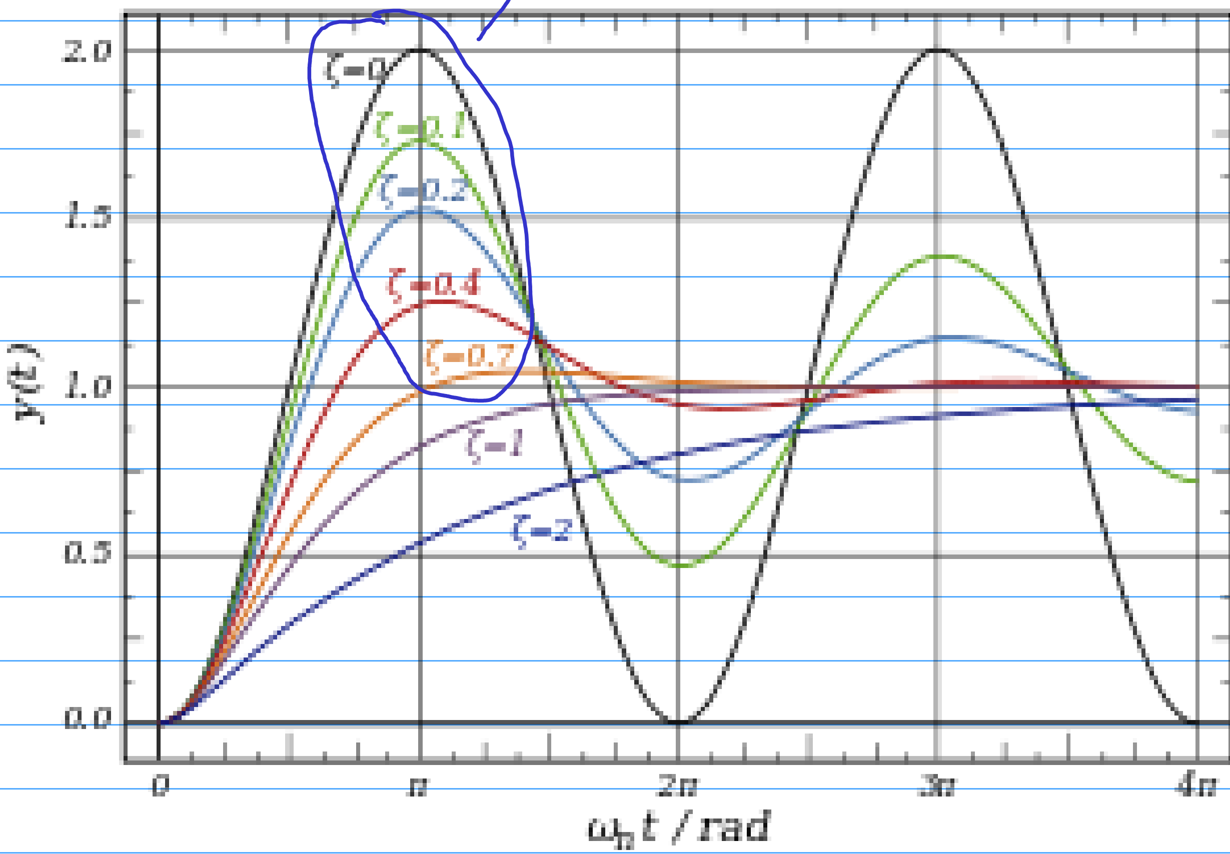


$$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \theta\right) \qquad \cos\theta = \zeta$$

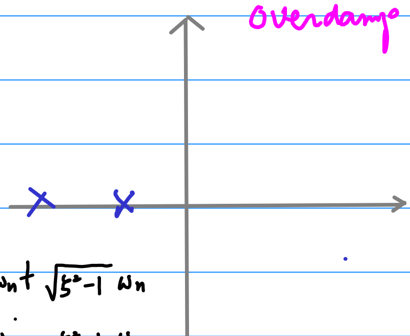
$$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \varphi\right) \qquad \sin\varphi = \zeta$$

step response $\frac{1}{s}$

underdamping ($\zeta < 1$)



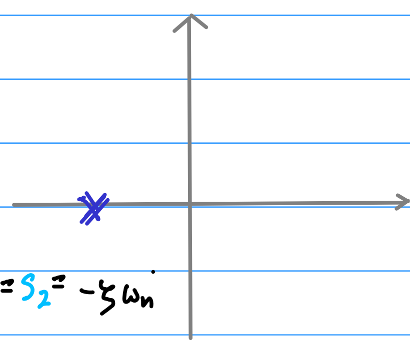
$D > 0$ $\zeta > 1$ overdamp



$s_1 = -\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n$

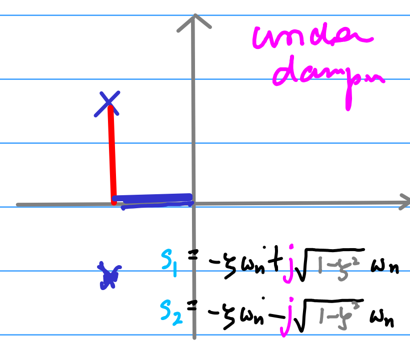
$s_2 = -\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n$

$D = 0$ $\zeta = 1$



$s_1 = s_2 = -\zeta\omega_n$

$D < 0$ $\zeta < 1$ under damping



$s_1 = -\zeta\omega_n + j\sqrt{1-\zeta^2}\,\omega_n$

$s_2 = -\zeta\omega_n - j\sqrt{1-\zeta^2}\,\omega_n$

impulse Response

전달 함수

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

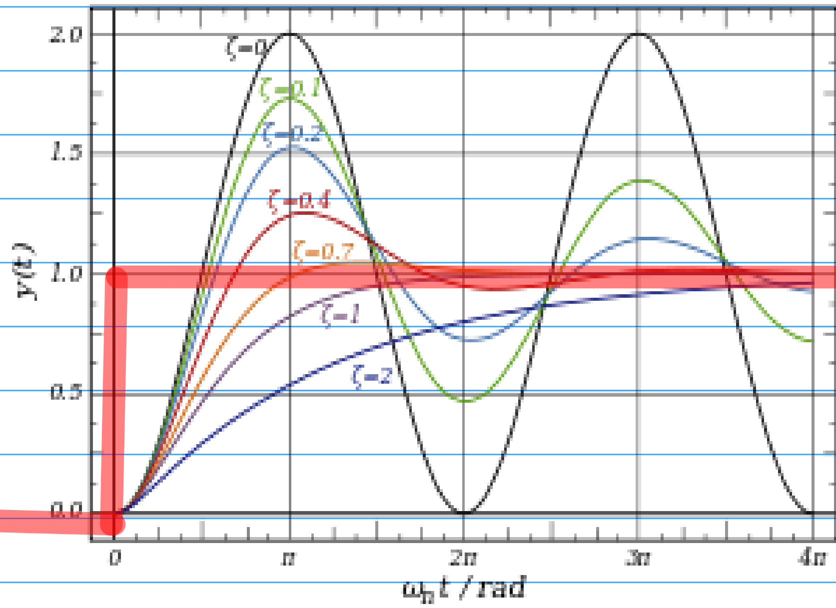
Step Response

$$Y(s) = \frac{\omega_n^2}{s \cdot (s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

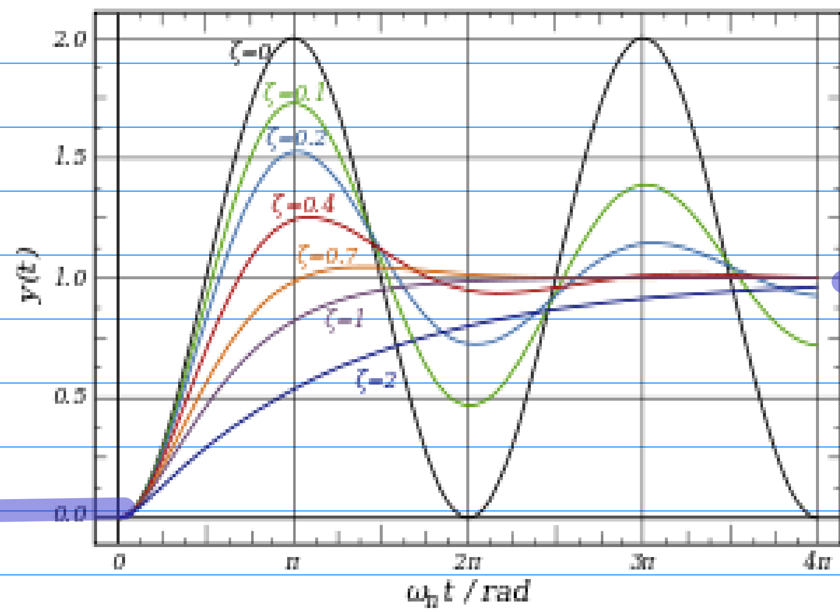
(pole : 전달함수의 분모를 0 으로 만드는 s)

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Step Response .... Transient Response ⟷ Steady State Rsp



$u(t)$



Steady state

$t \to \infty$

Steady state response

Transient Response

$$\sin(\alpha+\beta) = \sin\alpha \boxed{\cos\beta} + \cos\alpha \boxed{\sin\beta}$$

$$\cos(\alpha+\beta) = \cos\alpha \boxed{\cos\beta} - \sin\alpha \boxed{\sin\beta}$$

$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\underbrace{\left(\sqrt{1-\zeta^2}\right)}_{=\sin\theta}\boxed{\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right)} + \underbrace{(\zeta)}_{=\cos\theta}\boxed{\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)}\right]$$

$$= \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\theta\right) \qquad \cos\theta = \zeta$$

$$r(t) \longrightarrow \boxed{G(s)} \longrightarrow c(t)$$

1

$$C(s) = G(s)\,R(s) = G(s)\,\frac{1}{s}$$

$$= \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{1}{s} - \frac{s}{(s^2+2\zeta\omega_n s+\omega_n^2)} - \frac{2\zeta\omega_n}{(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\theta\right)$$