Computational Aspects (1A)

of Fourier Analysis Types

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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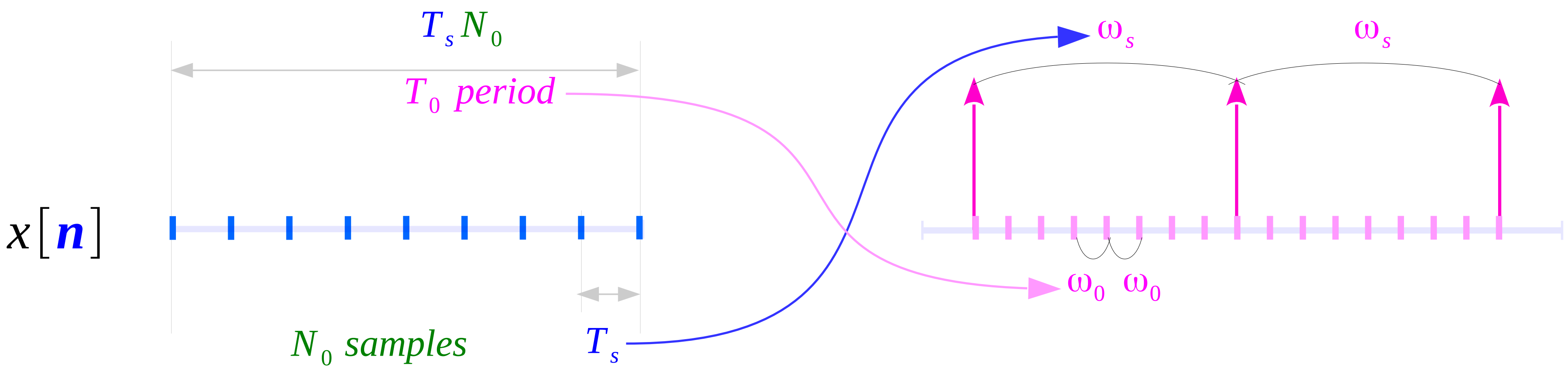
ω_s and ω_0

	$T_s = 1 \cdot T_s$	$T_0 = N_0 \cdot T_s$
	replication frequency	*frequency resolution*
$\omega = \frac{2\pi}{T}$ **Continuous Time**	$\omega_s = \frac{2\pi}{T_s}$	$\omega_0 = \frac{2\pi}{T_0}$
$\omega = \frac{\hat{\omega}}{T_s}$ **Discrete Time**	$\hat{\omega}_s = \frac{2\pi}{1}$	$\hat{\omega}_0 = \frac{2\pi}{N_0}$
	normalized	*normalized*

ω_s and ω_0

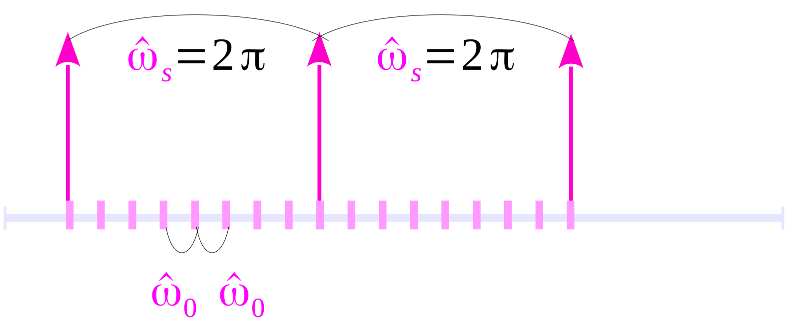


$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\hat{\omega}_0 = \frac{2\pi}{N_0}$$

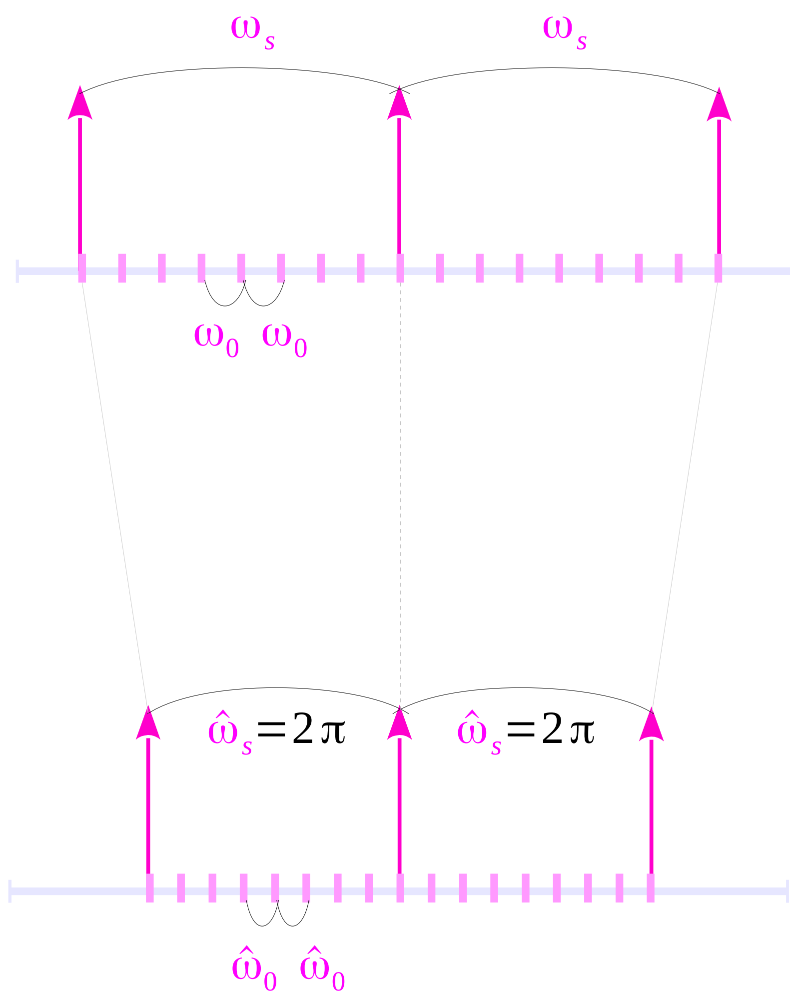
$$\hat{\omega}_s = \frac{2\pi}{1}$$



Normalized ω_s and ω_0

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_s = \frac{2\pi}{T_s}$$



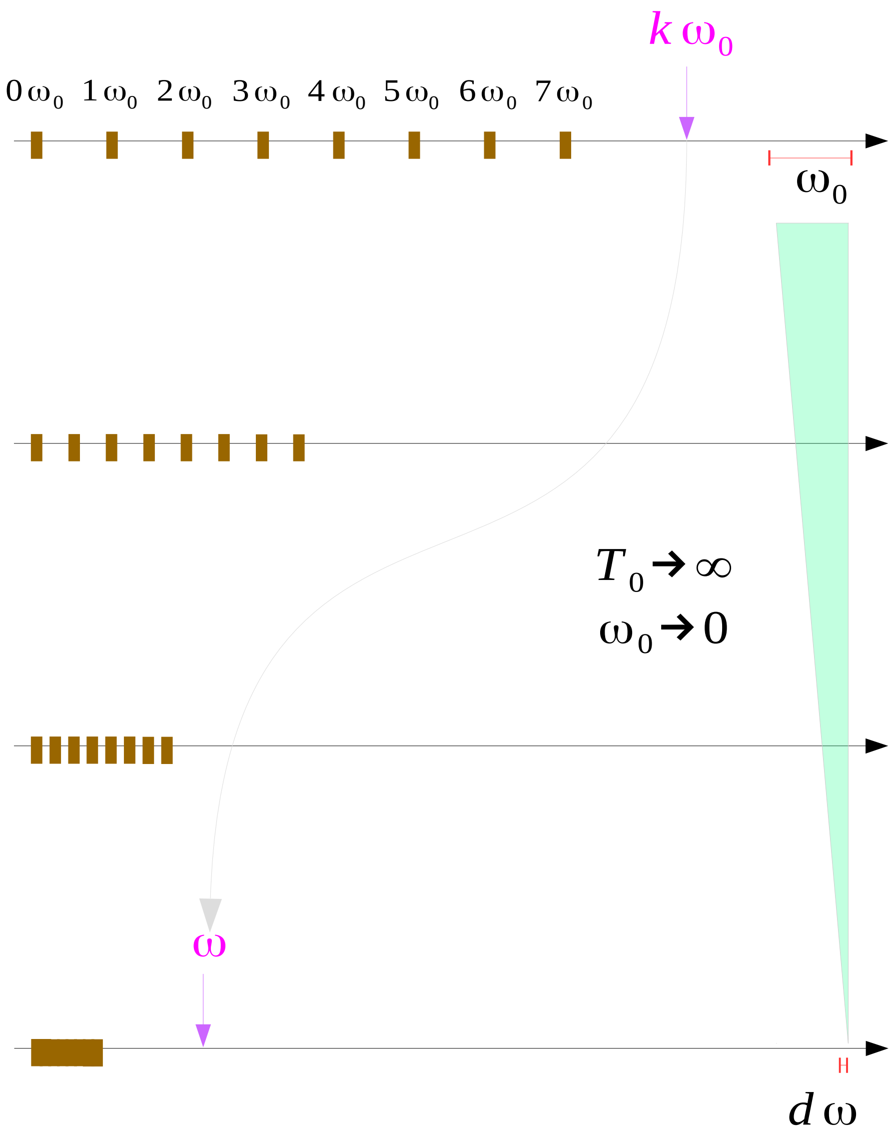
$$\hat{\omega}_0 = \omega_0 T_s = \frac{2\pi}{N_0 T_s} T_s$$

$$\hat{\omega}_s = \omega_s T_s = \frac{2\pi}{T_s} T_s$$

$$\hat{\omega}_0 = \frac{2\pi}{N_0}$$

$$\hat{\omega}_s = \frac{2\pi}{1}$$

CTFS → CTFT



$$x_{T_0}(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+j\omega_0 k t} \cdot 1$$

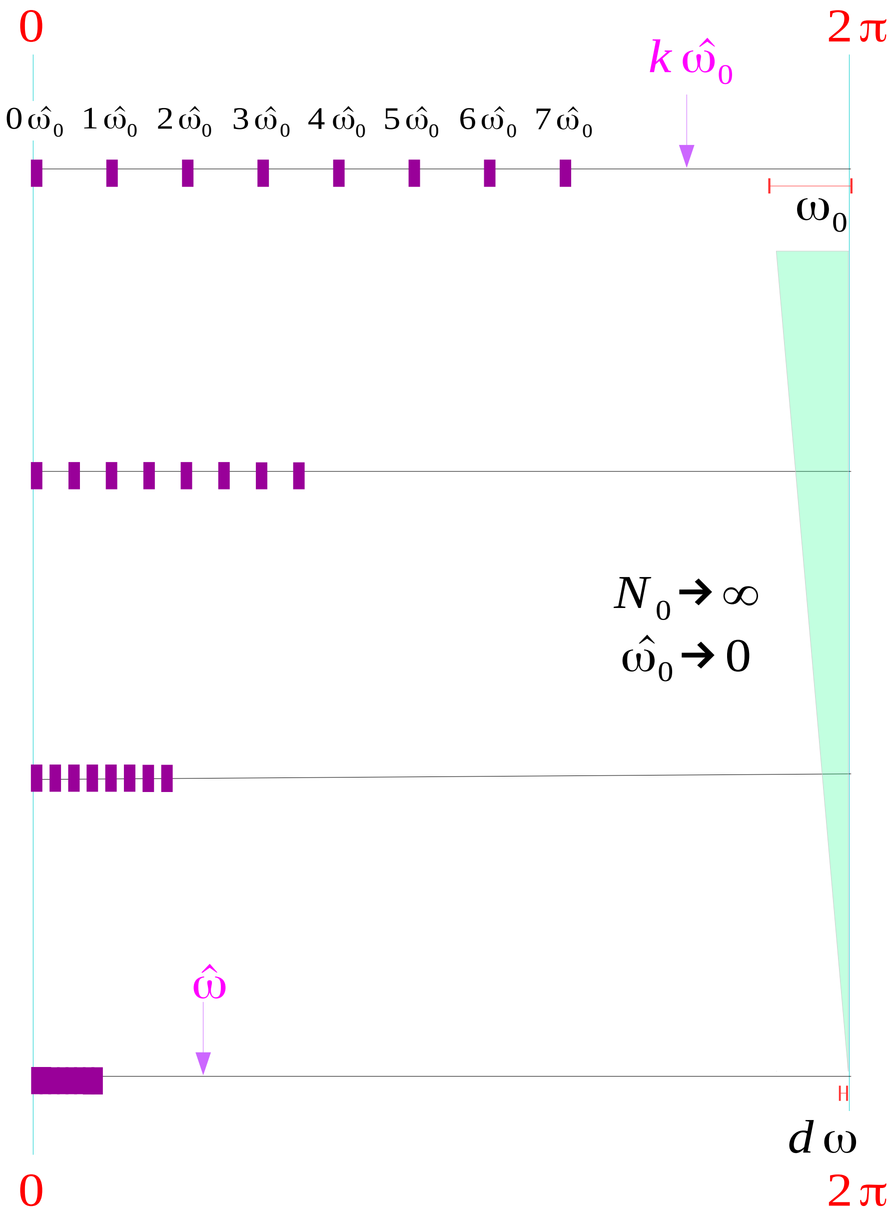
$$= \sum_{k=-\infty}^{+\infty} C_k \, e^{+j\omega_0 k t} \cdot \left(\frac{T_0}{2\pi}\right) \cdot \left(\frac{2\pi}{T_0}\right)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} C_k T_0 \, e^{+j\omega_0 k t} \cdot \left(\frac{2\pi}{T_0}\right)$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} C_k T_0 \, e^{+j\omega_0 k t} \cdot \omega_0$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \, e^{+j\omega t} \, d\omega$$

DTFS → DTFT



$$x_{N_0}[n] = \sum_{k=0}^{N_0} \gamma_k \, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot 1$$

$$= \sum_{k=0}^{N_0} \gamma_k \, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \left(\frac{N_0}{2\pi}\right) \cdot \left(\frac{2\pi}{N_0}\right)$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N_0} \gamma_k N_0 \, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \left(\frac{2\pi}{N_0}\right)$$

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=0}^{N_0} \gamma_k N_0 \, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \left(\frac{2\pi}{N_0}\right)$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) \, e^{+j\hat{\omega}n} \, d\hat{\omega}$$

CT $x(t)$ PT $\frac{1}{T}\int_0^T \quad dt$

DT $x[n]$ PT $\frac{1}{N}\sum_{n=0}^{N-1}$

PT $\frac{1}{T}\int_0^T 1\, dt = \frac{T}{T}$

PT $\frac{1}{N}\sum_{n=0}^{N-1} 1 = \frac{N}{N}$

$X(j\omega) \approx T \cdot C_k$ CF $\left(\frac{1}{2\pi}\right) \cdot T \cdot \left(\frac{2\pi}{T}\right)$

$X(j\hat{\omega}) \approx N \cdot \gamma_k$ CF $\left(\frac{1}{2\pi}\right) \cdot N \cdot \left(\frac{2\pi}{N}\right)$

CF $X(j\omega)$ AF $\frac{1}{2\pi}\int_{-\infty}^{+\infty} \quad d\omega$

CF $X(j\hat{\omega})$ PF $\frac{1}{2\pi}\int_{-\pi}^{+\pi} \quad d\hat{\omega}$

DTFS and DFT coefficients relationship

Discrete Time Fourier Series DTFS

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = N\cdot\frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \frac{1}{N} X[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = N\,\gamma[k] \qquad\qquad \gamma[k] = \frac{1}{N} X[k]$$

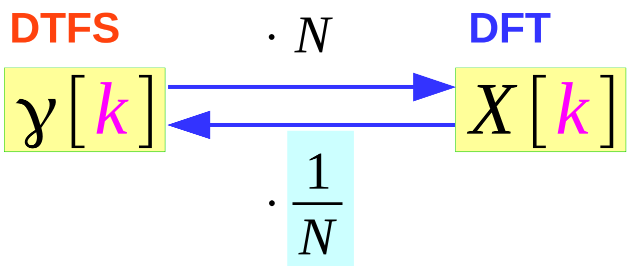
Discrete Fourier Transform DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}kn}$$

Converting DTFS and DFT Coefficients

$$DFT(x[n]) = N\,DTFS(x[n])$$

$$X[k] = N\,\gamma[k]$$



$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$DTFS(x[n]) = \frac{1}{N}\,DFT(x[n])$$

$$\gamma[k] = \frac{1}{N}\,X[k]$$

Fourier Transform Types

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

Discrete Time Fourier **Series**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier **Transform**

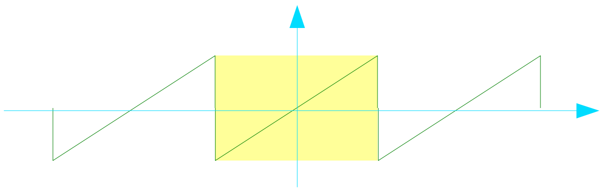
$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

Continuous Time – **CTFS** Computation

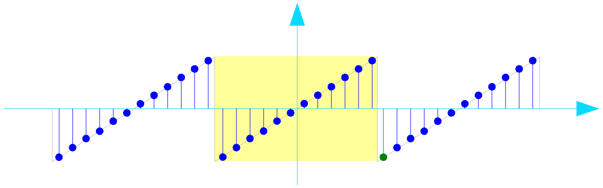
Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad \int x(t)dt \approx \sum_n x[n]\cdot T_s$$

$$\frac{1}{T}\cdot T_s = \frac{1}{N T_s}\cdot T_s = \frac{1}{N} \qquad \omega_0 t = \left(\frac{2\pi}{N T_s}\right)(n T_s)$$



$$C_k \approx \frac{1}{N}\, \boldsymbol{DFT}\{x(n T_s)\}$$



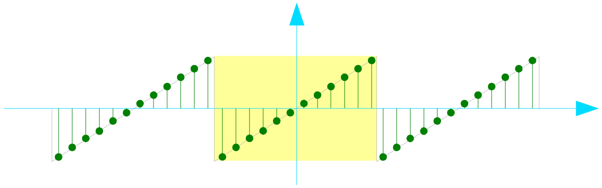
$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \sum_n x[n]$$

Discrete Time – **DTFS** computation

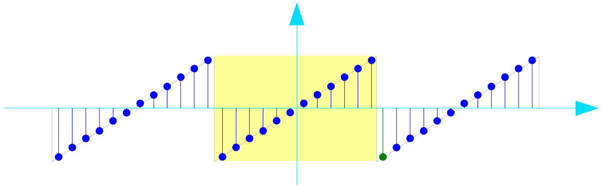
Discrete Time Fourier **Series**

$$\gamma[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$

$$\sum_{n} x[n]$$



$$\gamma[k] = \frac{1}{N} \boldsymbol{DFT}\{x[n]\}$$



$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$$\sum_{n} x[n]$$

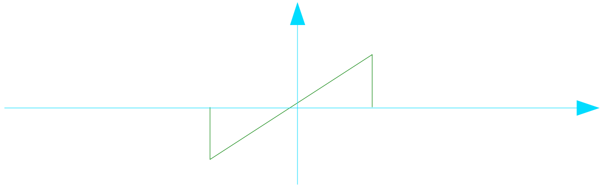
Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

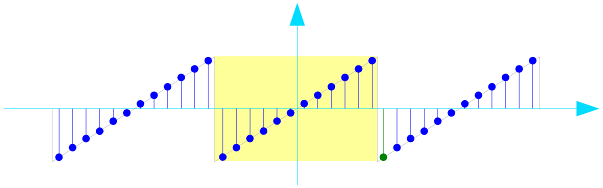
$$\int_{-\infty}^{+\infty} x(t)\,dt \approx \sum_{n=0}^{N-1} x[n]\cdot T_s$$

T_s

$$k\omega_0 t = k\left(\frac{2\pi}{N T_s}\right)(n T_s)$$



$$X(jk\omega_0) \approx T_s\, \boldsymbol{DFT}\{x(nT_s)\}$$



$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

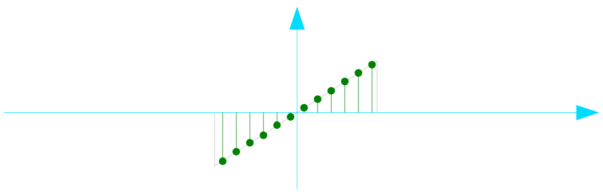
$$\sum_{n=0}^{N-1} x[n]$$

Discrete Time – **DTFT** computation

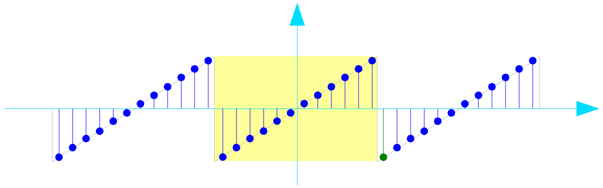
Discrete Time Fourier Transform

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \qquad \sum_{n=-\infty}^{+\infty} x[n]$$

$$k\,\hat{\omega}_0\, n = k\left(\frac{2\pi}{N}\right) n$$



$$X(jk\,\hat{\omega}_0) \approx \boldsymbol{DFT}\{x[n]\}$$

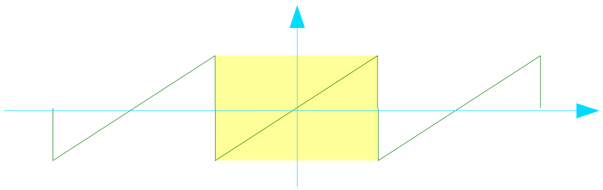


$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \sum_{n=0}^{N-1} x[n]$$

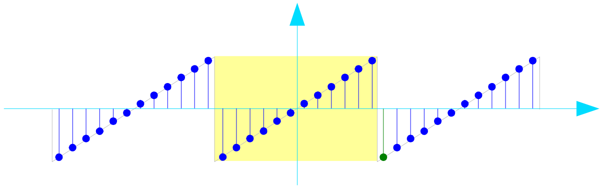
Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad \int x(t)dt \approx \sum_n x[n]\cdot T_s$$

$$\frac{1}{T}\cdot T_s = \frac{1}{N T_s}\cdot T_s = \frac{1}{N} \qquad \omega_0 t = \left(\frac{2\pi}{N T_s}\right)(n T_s)$$



$$C_k \approx \frac{1}{N}\, \boldsymbol{DFT}\{x(nT_s)\}$$

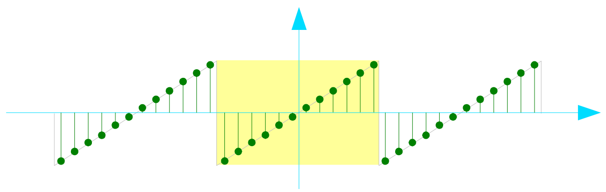


$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \sum_n x[n]$$

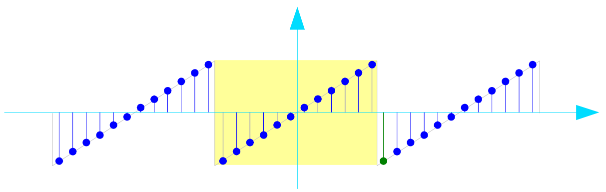
Discrete Time – **DTFS** computation

Discrete Time Fourier **Series**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \qquad \sum_n x[n]$$



$$\gamma[k] = \frac{1}{N}\, \boldsymbol{DFT}\{x[n]\}$$



$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \sum_n x[n]$$

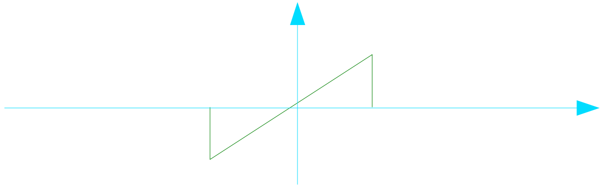
Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

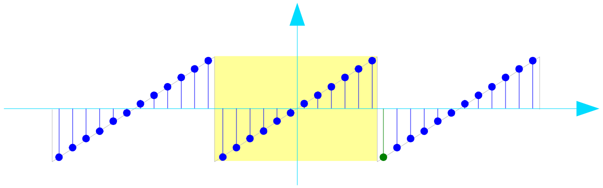
$$\int_{-\infty}^{+\infty} x(t)\,dt \approx \sum_{n=0}^{N-1} x[n]\cdot T_s$$

T_s

$$k\omega_0 t = k\left(\frac{2\pi}{N T_s}\right)(n T_s)$$



$$X(jk\omega_0) \approx T_s\, \boldsymbol{DFT}\{x(nT_s)\}$$



$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

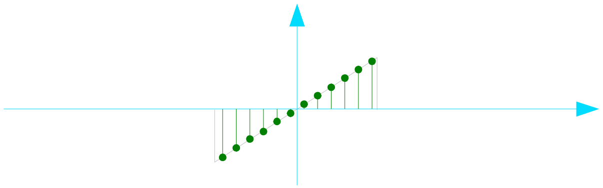
$$\sum_{n=0}^{N-1} x[n]$$

Discrete Time – **DTFT** computation

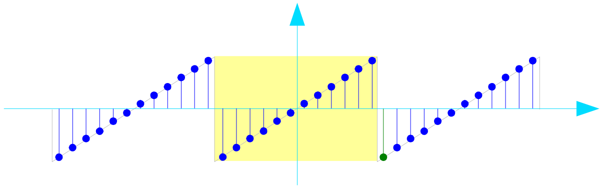
Discrete Time Fourier Transform

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \qquad \sum_{n=-\infty}^{+\infty} x[n]$$

$$k\,\hat{\omega}_0 n = k\left(\frac{2\pi}{N}\right)n$$



$$X(j k \hat{\omega}_0) \approx \boldsymbol{DFT}\{x[n]\}$$

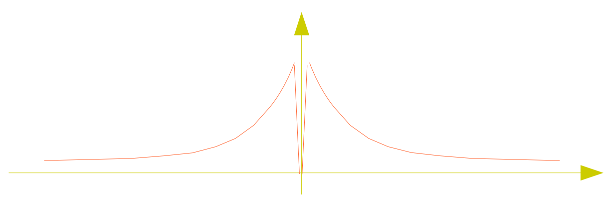


$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad \sum_{n=0}^{N-1} x[n]$$

Continuous Time – ICTFT computation

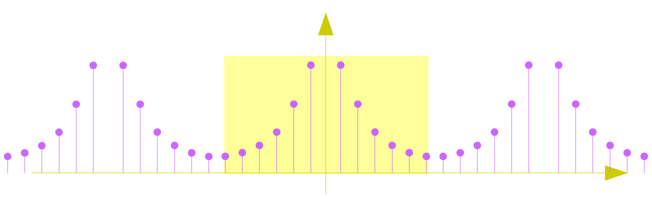
Continuous Time Fourier Transform

$$\int X(j\omega)\, d\omega \approx \sum_{k} X[k] \cdot \omega_0 \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$



$$\frac{1}{2\pi} \cdot \omega_0 = \frac{1}{T} = \frac{1}{T_s} \frac{1}{N} \qquad k\,\omega_0\, t = k \left(\frac{2\pi}{N\, T_s}\right)(n\, T_s)$$

$$x(n\, T_s) \approx \frac{1}{T_s} \boldsymbol{IDFT} \{ X(j\, k\, \omega_0) \}$$

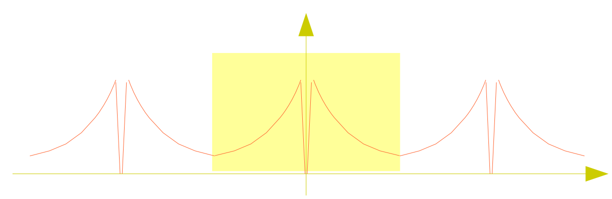


$$\sum_{k} X[k] \qquad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N} k n}$$

Discrete Time – IDTFT computation

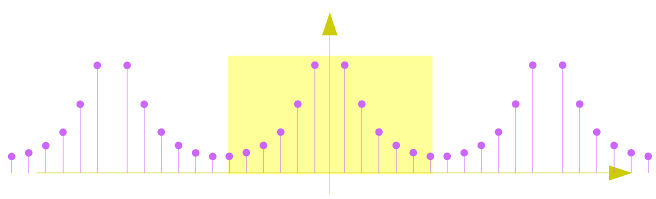
Discrete Time Fourier **Transform**

$$\int X(j\hat{\omega})\,d\hat{\omega} \approx \sum_k X[k]\cdot\hat{\omega}_0 \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\,e^{+j\hat{\omega}n}\,d\hat{\omega}$$



$$\frac{1}{2\pi}\cdot\hat{\omega}_0 = \frac{1}{N} \qquad k\,\hat{\omega}_0\,n = k\left(\frac{2\pi}{N}\right)n$$

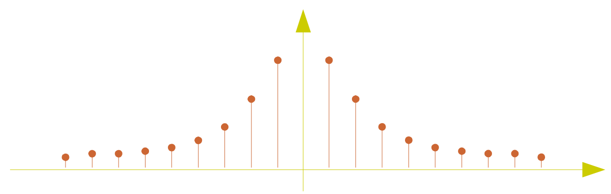
$$x[n] \approx \boldsymbol{IDFT}\{X(jk\hat{\omega}_0)\}$$



$$\sum_k X[k] \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\,e^{+j\frac{2\pi}{N}kn}$$

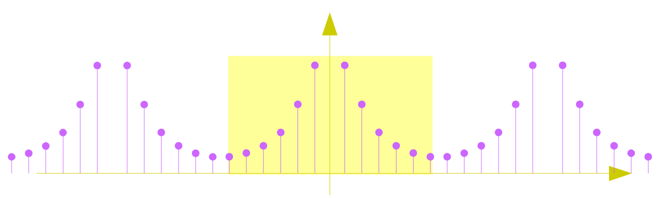
Continuous Time Fourier Series

$$\sum_{k=-\infty}^{+\infty} C_k \qquad x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$



$$k\omega_0 t = k\left(\frac{2\pi}{N T_s}\right)(n T_s)$$

$$x(nT_s) \approx N \, \boldsymbol{IDFT}\{C_k\}$$

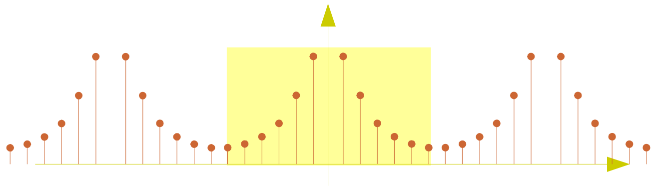


$$\sum_{k=0}^{N-1} X[k] \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \, e^{+j\frac{2\pi}{N}kn}$$

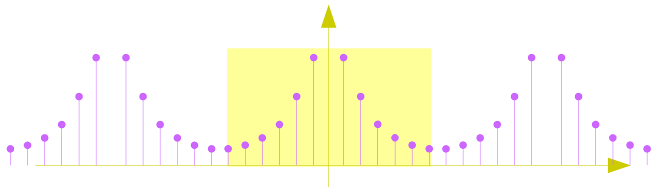
Discrete Time Fourier Series

$$\sum_{k=0}^{N-1} \gamma[k]$$

$$x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$



$$x[n] = N\, \boldsymbol{IDFT}\{\gamma_k\}$$



$$\sum_{k=0}^{N-1} X[k]$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}kn}$$

Computations using DFT

CTFS — **Periodic x(t)**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k \approx \frac{1}{N} DFT\{x(nT_s)\}$$

$k\omega_0$

$$@ \quad k\omega_0 = k\left(\frac{2\pi}{T}\right) \quad rad/sec$$

CTFT — **Aperiodic x(t)**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(jk\omega_0) \approx T_s\, DFT\{x(nT_s)\}$$

$\omega \leftarrow k\omega_0$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right) \quad rad/sec$$

DTFS — **Periodic x[n]**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$

$$\gamma[k] = \frac{1}{N} DFT\{x[n]\}$$

$k\hat{\omega}_0$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right) \quad rad/sec$$

DTFT — **Aperiodic x[n]**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

$$X(jk\hat{\omega}_0) \approx DFT\{x[n]\}$$

$\hat{\omega} \leftarrow k\hat{\omega}_0$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right) \quad rad/sec$$

Forward Computations using DFT

CTFS — Periodic x(t)

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$\frac{1}{T}\cdot T_s = \frac{1}{N}$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$k\,\omega_0$

$$C_k \approx \frac{1}{N}\,\boldsymbol{DFT}\{x(nT_s)\}$$

CTFT — Aperiodic x(t)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$1\cdot T_s = T_s$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$\omega \leftarrow k\,\omega_0$

$$X(jk\omega_0) \approx T_s\,\boldsymbol{DFT}\{x(nT_s)\}$$

DTFS — Periodic x[n]

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$

$\frac{1}{N}$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$k\,\hat{\omega}_0$

$$\gamma[k] = \frac{1}{N}\,\boldsymbol{DFT}\{x[n]\}$$

DTFT — Aperiodic x[n]

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

1

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

$\hat{\omega} \leftarrow k\,\hat{\omega}_0$

$$X(jk\hat{\omega}_0) \approx \boldsymbol{DFT}\{x[n]\}$$

Inverse Computations using DFT

ICTFS — Periodic x(t)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$1 \cdot N = N$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{+j\frac{2\pi}{N}kn}$$

$$t \leftarrow nT_s$$

$$x(nT_s) \approx N \, \mathbf{IDFT}\{C_k\}$$

ICTFT — Aperiodic x(t)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \, e^{+j\omega t} d\omega$$

$$\frac{1}{2\pi} \cdot \omega_0 = \frac{1}{N T_s}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{+j\frac{2\pi}{N}kn}$$

$$t \leftarrow nT_s$$

$$x(nT_s) \approx \frac{1}{T_s} \mathbf{IDFT}\{X(jk\omega_0)\}$$

IDTFS — Periodic x[n]

$$x[n] = \sum_{k=0}^{N-1} \gamma[k] \, e^{+jk\hat{\omega}_0 n}$$

$$1 \cdot N = N$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{+j\frac{2\pi}{N}kn}$$

$$nT_s$$

$$x[n] = N \, \mathbf{IDFT}\{\gamma_k\}$$

IDTFT — Aperiodic x[n]

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(j\hat{\omega}) \, e^{+j\hat{\omega} n} d\hat{\omega}$$

$$\frac{1}{2\pi} \cdot \hat{\omega}_0 = \frac{1}{N}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{+j\frac{2\pi}{N}kn}$$

$$nT_s$$

$$x[n] \approx \mathbf{IDFT}\{X(jk\hat{\omega}_0)\}$$

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