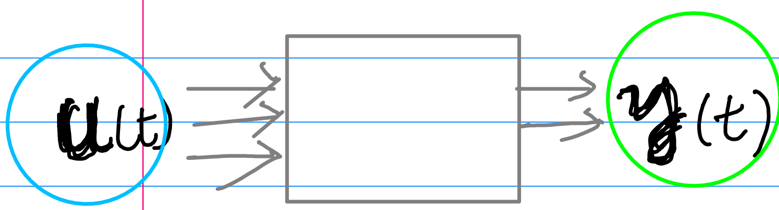
State Space (H1)
Transfer Function

P 167. 5.5 State Space → Transfer fn



ZSR 의미

초기 조건 = 0

inputs

state eq. $\dot{x}(t) = A\,x(t) + B\,u(t)$

inputs

output eq $y(t) = C\,x(t) + D\,u(t)$

$x(0)$. initial conditions ⇒ 0

transfer fn 구할 때는

State eq $\dot{x}(t) = A\,x(t) + B\,u(t)$

$$sX(s) - x(0) = AX(s) + B\,U(s)$$

$$(sI - A)\,X(s) = B\,U(s) + x(0)$$

Output eq $y(t) = C\,x(t) + D\,u(t)$

$$Y(s) = C\,X(s) + D\,U(s)$$

$$(sI - A)\,X(s) = \left(B\,U(s) + X(0)\right)$$

$$X(s) = (sI - A)^{-1}\left(B\,U(s) + X(0)\right)$$

$$X(s) = (sI - A)^{-1} B\,\underbrace{U(s)}_{\text{input}} + (sI - A)^{-1}\underbrace{X(0)}_{\text{초기치}}$$

Transfr fn ⇒ 0

$$Y(s) = C\,X(s) + D\,U(s)$$

$$Y(s) = C(sI - A)^{-1} B\,U(s) + C(sI - A)^{-1} X(0) + D\,U(s)$$

$$\Phi(s)$$

$$Y(s) = \left[C\,\underbrace{(sI - A)^{-1}}_{\Phi(s)} B + D\right] U(s) + C\,\underbrace{(sI - A)^{-1}}_{\Phi(s)} X(0)$$

$$Y(s) = \left[C\,\Phi(s)\,B + D\right] U(s) + C\,\Phi(s)\,X(0)$$

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{X}(s) = \mathbf{B}\,\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\underbrace{\mathbf{U}(s)}_{\text{input}}$$

$$\mathbf{Y}(s) = \mathbf{C}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{U}(s) + \mathbf{D}\,\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \left[\mathbf{C}\,\underbrace{(s\mathbf{I} - \mathbf{A})^{-1}}_{\phi(s)}\,\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$

전달함수 $$H(s) = C\,\overbrace{(sI - A)^{-1}}^{\Phi(s)}\,B \qquad \frac{1}{s+a}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

임펄스응답 $$h(t) = C\,\underbrace{e^{At}}_{\Phi(t)}\,B \qquad e^{-at}$$

$\Phi(t)$ ~~~ 상태천이행렬

State transition Matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$At = t\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 t & a_2 t & a_3 t \\ b_1 t & b_2 t & b_3 t \\ c_1 t & c_2 t & c_3 t \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{a_1 t} & e^{a_2 t} & e^{a_3 t} \\ e^{b_1 t} & e^{b_2 t} & e^{b_3 t} \\ e^{c_1 t} & e^{c_2 t} & e^{c_3 t} \end{bmatrix}$$

$$\Phi(s) = (sI - A)^{-1} = \frac{adj(sI - A)}{|sI - A|}$$

$$Y(s) = [C\Phi(s)B + D]U(s) + C\Phi(s)X(0)$$

전달함수 구하기 위해서 $X(0) \Leftarrow 0$

$$G(s) = \frac{Y(s)}{U(s)} = [C\Phi(s)B + D]$$

$$\Phi(s) = (sI - A)^{-1} = \frac{adj(sI - A)}{|sI - A|}$$

pole

$$\boxed{AP = P\Lambda \quad \therefore \begin{cases} A = P\Lambda P^{-1} \\ \Lambda = P^{-1}AP \end{cases}}$$

$\underset{n \times n}{A}$

$\lambda_1, \lambda_2, \cdots, \lambda_n$: n개의 distinct eigen value

$\downarrow$

$p_1, \; p_2, \; \cdots, \; p_n$ eigenvector.

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} = \underset{n \times n}{P} \qquad \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \Lambda$$

$\Lambda \leftarrow A$	$A \leftarrow \Lambda$
$\Lambda = P^{-1}AP$	$A = P\Lambda P^{-1}$
$\Lambda^k = P^{-1}A^kP$	$A^k = P\Lambda^k P^{-1}$
$e^{\Lambda} = \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!}$	$e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!}$
$e^{\Lambda} \doteq P^{-1}e^{A}P$	$e^{A} \doteq Pe^{\Lambda}P^{-1}$

$$\Lambda \leftarrow A$$

$$(sI-\Lambda) = P^{-1}(sI-A)P$$

$$(sI-\Lambda)^{-1} = P^{-1}\underbrace{(sI-A)}^{-1}P$$

$$A \leftarrow \Lambda$$

$$(sI-A) = P(sI-\Lambda)P^{-1}$$

$$(sI-A)^{-1} = P\underbrace{(sI-\Lambda)}^{-1}P^{-1}$$

$$e^{\Lambda t} = P^{-1}e^{At}P$$

$$e^{\Lambda} = \sum_{k=0}^{\infty}\frac{\Lambda^k}{k!}$$

$$e^{\Lambda} = P^{-1}e^{A}P$$

$$e^{At} = Pe^{\Lambda t}P^{-1}$$

$$= \varphi(t)$$

$$e^{A} = \sum_{k=0}^{\infty}\frac{A^k}{k!}$$

$$e^{A} = Pe^{\Lambda}P^{-1}$$

Trans. Func

$H(s)$ | $\overbrace{C\,(sI-A)^{-1}\,B}^{\Phi(s)}$ | $\frac{1}{s+a} = (s+a)^{-1}$

$\Updownarrow$ | $\Updownarrow$

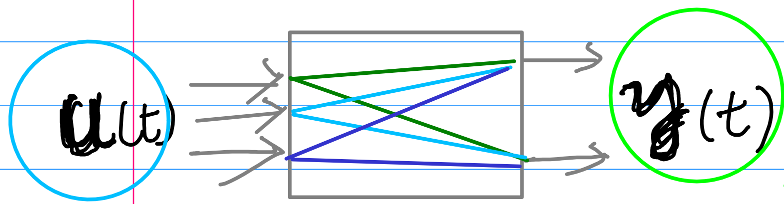
$h(t)$ | $\underbrace{C\,e^{At}\,B}_{\phi(t)}$ | e^{-at}

Impulse Res

상태전이 행열

State transition Matrix

State Space Representation



ZSR 의미

$\boxed{\text{init cond} = 0}$

State eq. $\dot{x}(t) = A\,x(t) + B\,u(t)$ (inputs)

output eq. $y(t) = C\,x(t) + D\,u(t)$ (inputs)

$x(0)$ · initial conditions $\Rightarrow 0$

transfer fn 구할때는

State eq. $\dot{x}(t) = A\,x(t) + B\,u(t)$

$$sX(s) = AX(s) + BU(s)$$

$$\underline{(sI - A)\,X(s) = BU(s)}$$

Output eq. $y(t) = C\,x(t) + D\,u(t)$

$$\underline{Y(s) = CX(s) + DU(s)}$$

State eq $\dot{X}(t) = A\,X(t) + B\,u(t)$

Output eq $y(t) = C\,X(t) + D\,u(t)$

$$sX(s) = AX(s) + B\,U(s)$$

$$(sI - A)\,X(s) = B\,U(s)$$

$$X(s) = (sI - A)^{-1} B\,U(s)$$

$$Y(s) = C\,X(s) + D\,U(s)$$

$$Y(s) = C(sI - A)^{-1} B\,U(s) + D\,U(s)$$

$$X(s) = \boxed{\overbrace{(sI - A)^{-1}}^{\Phi(s) \, =}} B \, U(s)$$

$$Y(s) = C(sI - A)^{-1} B \, U(s) + D \, U(s)$$

$$\boxed{\begin{aligned} X(s) &= \Phi(s) B \, U(s) \\ Y(s) &= C \, X(s) + D \, U(s) \end{aligned}}$$

initial cond vector

$$\boxed{\begin{aligned} X(s) &= \Phi(s) B \, U(s) + \underline{\Phi(s) x(0)} \\ Y(s) &= C \, X(s) + D \, U(s) \end{aligned}}$$

$$\Phi(s) = (sI - A)^{-1} \qquad A = P\Lambda P^{-1}$$
$$\Lambda = P^{-1}AP$$

$$\Phi(s) \Updownarrow \varphi(t) \rightsquigarrow \text{State transition matrix}$$

$$\varphi(t) = \mathcal{L}^{-1}\{\Phi(s)\} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

$$(sI - A) = sPP^{-1} - P\Lambda P^{-1} = P(sI - \Lambda)P^{-1}$$

$$(sI - A) = P(sI - \Lambda)P^{-1}$$

$$(sI - A)^{-1} = \left(P(sI - \Lambda)P^{-1}\right)^{-1}$$

$$= (P^{-1})^{-1}(sI - \Lambda)^{-1}(P)^{-1}$$

$$\boxed{\begin{aligned} \Phi(s) &= P(sI - \Lambda)^{-1}P^{-1} \\ &\Updownarrow \\ \varphi(t) &= \mathcal{L}^{-1}\{P(sI - \Lambda)^{-1}P^{-1}\} \end{aligned}}$$

$$A = P \Lambda P^{-1}$$

$$\Lambda = P^{-1} A P$$

$$e^{A} = P e^{\Lambda} P^{-1}$$

$$(sI - A) = P(sI - \Lambda)P^{-1}$$

$$(sI - A)^{-1} = P(sI - \Lambda)^{-1} P^{-1}$$

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \Lambda^{2} = \begin{bmatrix} 2^{2} & 0 \\ 0 & 3^{2} \end{bmatrix}$$

$$\Lambda^{-1} = \begin{bmatrix} 2^{-1} & 0 \\ 0 & 3^{-1} \end{bmatrix} \qquad \Lambda \Lambda^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$sI - \Lambda = \begin{bmatrix} s-\lambda_1 & & & 0 \\ & s-\lambda_2 & & \\ & & \ddots & \\ 0 & & & s-\lambda_n \end{bmatrix}$$

$$(sI - \Lambda)^{-1} = \begin{bmatrix} \frac{1}{s-\lambda_1} & & & 0 \\ & \frac{1}{s-\lambda_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{s-\lambda_n} \end{bmatrix}$$

$$\varphi(t) = \mathcal{L}^{-1}\left\{ P(sI - \Lambda)^{-1} P^{-1} \right\} = P\,\mathcal{L}^{-1}\left\{ (sI - \Lambda)^{-1} \right\} P^{-1}$$

$$= P\,\mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s-\lambda_1} & & & 0 \\ & \frac{1}{s-\lambda_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{s-\lambda_n} \end{bmatrix} \right\} P^{-1}$$

$$\varphi(t) = \mathcal{L}^{-1}\left\{ P(sI - \Lambda)^{-1} P^{-1} \right\} = P\,\mathcal{L}^{-1}\left\{ (sI - \Lambda)^{-1} \right\} P^{-1}$$

$$= P\,\mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s-\lambda_1} & & & 0 \\ & \frac{1}{s-\lambda_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{s-\lambda_n} \end{bmatrix} \right\} P^{-1}$$

$$= P \begin{bmatrix} \mathcal{L}^{-1}\left\{\frac{1}{s-\lambda_1}\right\} & & & 0 \\ & \mathcal{L}^{-1}\left\{\frac{1}{s-\lambda_2}\right\} & & \\ & & \ddots & \\ 0 & & & \mathcal{L}^{-1}\left\{\frac{1}{s-\lambda_n}\right\} \end{bmatrix} P^{-1}$$

$$= P \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

$$\boxed{\varphi(t) = P\, e^{\Lambda t}\, P^{-1}}$$

$$\boxed{\begin{aligned} \Phi(s) &=: P(sI-\Lambda)^{-1}P^{-1} \\ &\Updownarrow \\ \varphi(t) &= \mathcal{L}^{-1}\{P(sI-\Lambda)^{-1}P^{-1}\} \end{aligned}}$$

$$\boxed{\begin{aligned} \Phi(s) &=: (sI-A)^{-1} \Rightarrow P(sI-\Lambda)^{-1}P^{-1} \\ &\Updownarrow \qquad \| \\ \varphi(t) &= e^{At} \Rightarrow P\,e^{\Lambda t}\,P^{-1} \end{aligned}}$$

$$\frac{1}{s-a} = (s-a)^{-1}$$

$$e^{at}$$

Convolution $\Longleftrightarrow$ Multiplication

$$f(t) \Longleftrightarrow F(s)$$

$$g(t) \Longleftrightarrow G(s)$$

$$\boxed{\begin{aligned} & f(t) * g(t) \\ &= \int_0^t f(\tau) g(t-\tau)\, d\tau \end{aligned}} \Longleftrightarrow \boxed{F(s)\, G(s)}$$

Multiplication

$$\boxed{\begin{aligned} X(s) &= \boxed{\Phi(s)}\, \boxed{B\, U(s)} + \underline{\Phi(s)\, x(0)} \\ Y(s) &= C\, X(s) + D\, U(s) \end{aligned}}$$

Convolution

$$x(t) = \varphi(t) * B u(t) + \varphi(t) x(0)$$

$$= \int_0^t \varphi(t-\tau) B u(\tau)\, d\tau + \varphi(t) x(0)$$

$$y(t) = C \int_0^t \varphi(t-\tau) B u(\tau)\, d\tau + C \varphi(t) x(0) + D u(t)$$

$$\Phi(s) = P(sI - \Lambda)^{-1}P^{-1} = (sI - A)^{-1}$$

$$\Updownarrow$$

$$\varphi(t) = P e^{\Lambda t} P^{-1} = e^{At}$$

given

$$\boxed{x(t) = \int_0^t \varphi(t-\tau) B u(\tau)\, d\tau + \varphi(t)\, \underline{x(0)}}$$

state transition matrix

properties of state transition matrix

$$\varphi(0) = P e^{\Lambda 0} P^{-1} = e^{A 0} = I$$

$$\boxed{\begin{aligned}\varphi^{-1}(t) &= \left(P e^{\Lambda t} P^{-1}\right)^{-1} = \left(P^{-1}\right)^{-1}\left(e^{\Lambda t}\right)^{-1}\left(P\right)^{-1} \\ &= P\left(e^{\Lambda t}\right)^{-1} P^{-1} = P\left(e^{\Lambda}\right)^{(-1)(t)} P^{-1} \\ \varphi(-t) &= P e^{\Lambda(-t)} P^{-1} \Leftarrow \left(e^{At}\right)^{-1}\end{aligned}}$$

$$\varphi(t_2-t_1) = P e^{\Lambda\cdot(t_2-t_1)} P^{-1}$$

$$\varphi(t_1-t_0) = P e^{\Lambda\cdot(t_1-t_0)} P^{-1}$$

$$\begin{aligned}\varphi(t_2-t_1)\,\varphi(t_1-t_0) &= P e^{\Lambda\cdot(t_2-t_1)} P^{-1} P e^{\Lambda\cdot(t_1-t_0)} P^{-1} \\ &= P e^{\Lambda\cdot(t_2-t_1)} e^{\Lambda\cdot(t_1-t_0)} P^{-1} \\ &= P e^{\Lambda\cdot(t_2-t_0)} P^{-1}\end{aligned}$$

$$\Phi(s) = P(sI - \Lambda)^{-1}P^{-1} \Leftarrow (sI - A)^{-1}$$

$$\Updownarrow$$

$$\varphi(t) = P e^{\Lambda t} P^{-1} \Leftarrow e^{At}$$

$$\varphi^{-1}(t) = \left(P e^{\Lambda t} P^{-1}\right)^{-1} = \left(P^{-1}\right)^{-1}\left(e^{\Lambda t}\right)^{-1}\left(P\right)^{-1}$$

$$= P\left(e^{\Lambda t}\right)^{-1}P^{-1} = P\left(e^{\Lambda}\right)^{(-1)(t)}P^{-1}$$

$$\varphi(-t) = P e^{\Lambda(-t)} P^{-1} \Leftarrow \left(e^{At}\right)^{-1}$$

Inverse Matrix

Replace with $-t$

$$\varphi^{-1}(t) = \varphi(-t)$$

$$\left(e^{At}\right)^{-1} = \left(e^{A(-t)}\right)$$

$$\left(P e^{\Lambda t} P^{-1}\right)^{-1} = P e^{\Lambda(-t)} P^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} t^k & & & 0 \\ & \sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!} t^k & & \\ & & \ddots & \\ 0 & & & \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} t^k \end{bmatrix}.$$

$$\left(e^{\Lambda t}\right)^{-1} = \begin{bmatrix} \left(\sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} t^k\right)^{-1} & & & 0 \\ & \left(\sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!} t^k\right)^{-1} & & \\ & & \ddots & \\ 0 & & & \left(\sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} t^k\right)^{-1} \end{bmatrix}.$$

$$\boxed{\begin{aligned} \varphi^{-1}(t) &= \left(P\, e^{\Lambda t}\, P^{-1}\right)^{-1} = \left(P^{-1}\right)^{-1} \left(e^{\Lambda t}\right)^{-1} \left(P\right)^{-1} \\ &= P \left(e^{\Lambda t}\right)^{-1} P^{-1} = P \left(e^{\Lambda}\right)^{(-1)(t)} P^{-1} \\ \varphi(-t) &= P\, e^{\Lambda(-t)}\, P^{-1} \Longleftarrow \left(e^{\Lambda t}\right)^{-1} \end{aligned}}$$

$$\left(e^{\Lambda t}\right)^{-1} = \begin{bmatrix} \left(\sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} t^k\right)^{-1} & & & \\ & \left(\sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!} t^k\right)^{-1} & & \\ & & \ddots & \\ & & & \left(\sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} t^k\right)^{-1} \end{bmatrix}$$

$$\left(\sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} t^k\right)^{-1} = \frac{1}{\left(\sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} t^k\right)} = \frac{1}{e^{\lambda_1 t}} = e^{\lambda_1(-t)}$$

$$\left(e^{\Lambda t}\right)^{-1} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!} (-t)^k & & & \\ & \sum_{k=0}^{\infty} \frac{\lambda_2^k}{k!} (-t)^k & & \\ & & \ddots & \\ & & & \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} (-t)^k \end{bmatrix}$$

$$\varphi(t) = P e^{\Lambda t} P^{-1}$$

$$\varphi^{k}(t) = \overbrace{P e^{\Lambda t} P^{-1} P e^{\Lambda t} P^{-1} \cdots P e^{\Lambda t} P^{-1}}^{k}$$

$$= P \left(e^{\Lambda t}\right)^{k} P^{-1}$$

$$\varphi(kt) = P e^{\Lambda kt} P^{-1}$$

$$\varphi(0) = e^{\mathbf{A}0} = \mathbf{I}$$

$$\varphi(t) = e^{-\mathbf{A}t.(-1)} = \left(e^{-\mathbf{A}t}\right)^{-1} = \left(\varphi(-t)\right)^{-1}$$

$$\varphi^{-1}(t) = \varphi(-t)$$

$$\varphi(t_1 + t_2) = e^{\mathbf{A}(t_1 + t_2)} = e^{\mathbf{A}t_1} e^{\mathbf{A}t_2}$$

$$= \varphi(t_1)\,\varphi(t_2)$$

$$\left(\varphi(t)\right)^n = \varphi(nt)$$

$$\varphi(t_2 - t_1)\,\varphi(t_1 - t_0) = \varphi(t_2 - t_0)$$

$$= \varphi(t_1 - t_0)\,\varphi(t_2 - t_1)$$

$$\boxed{\begin{aligned} X(s) &= \Phi(s)\,B\,U(s) + \underline{\Phi(s)\,x(0)} \\ Y(s) &= C\,X(s) + D\,U(s) \end{aligned}}$$

$$\boxed{\begin{aligned} x(t) &= \int_0^t \phi(t-\tau)\,B\,u(\tau)\,d\tau + \underline{\phi(t)\,x(0)} \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}}$$

Zero input $u(t) = 0$ 이면

$$x(t) = \underline{\phi(t)\,x(0)}$$

초기치

transition

$$A = P \Lambda P^{-1}$$
$$\Lambda = P^{-1} A P$$

$$\Lambda = \bar{A} = P^{-1} A P$$

$$\dot{x}(t) = A x(t) + B u(t)$$

$$P \dot{\bar{x}}(t) = A P \bar{x}(t) + B u(t)$$

$$\dot{\bar{x}}(t) = P^{-1} A P \bar{x}(t) + P^{-1} B u(t)$$

$$\boxed{\dot{\bar{x}}(t) = \bar{A} \bar{x}(t) + \bar{B} u(t)}$$

$$y(t) = C x(t) + D u(t)$$

$$y(t) = C P \bar{x}(t) + D u(t)$$

$$\boxed{y(t) = \bar{C} \bar{x}(t) + \bar{D} u(t)}$$

$$|sI - \bar{A}| = |sI - P^{-1}AP|$$

$$= |sP^{-1}P - P^{-1}AP|$$

$$= |P^{-1}(sI - A)P|$$

$$= |P^{-1}|\,|(sI - A)|\,|P|$$

$$= |P^{-1}|\,|P|\,|(sI - A)|$$

$$= |P^{-1} \cdot P|\,|(sI - A)|$$

$$= |(sI - A)|$$