Hilbert Inner Product Space (2B)

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Trigonometric Identities

$$\cos\theta\,\cos\phi = \frac{1}{2}\left(\cos(\theta-\phi) + \cos(\theta+\phi)\right) \qquad \frac{1}{2}\left(1 + \cos(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\sin\theta\,\sin\phi = \frac{1}{2}\left(\cos(\theta-\phi) - \cos(\theta+\phi)\right) \qquad \frac{1}{2}\left(1 - \cos(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\sin\theta\,\cos\phi = \frac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta-\phi)\right) \qquad \frac{1}{2}\left(\sin(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\cos\theta\,\sin\phi = \frac{1}{2}\left(\sin(\theta+\phi) - \sin(\theta-\phi)\right) \qquad \frac{1}{2}\left(\sin(\theta+\phi)\right) \quad \textit{when} \quad \theta=\phi$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = \pi \qquad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = \pi \qquad (n = m)$$

$\boldsymbol{n}, \boldsymbol{m}$: integer

Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x\, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x\, dx = \pi \quad (n = m)$$

$\boldsymbol{n}, \boldsymbol{m}$: integer

$$a_{\boldsymbol{k}} \leftarrow f(x) \cdot \cos \boldsymbol{k} x = a_0 \cdot \cos \boldsymbol{k} x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m} x \cdot \cos \boldsymbol{k} x + b_m \sin \boldsymbol{m} x \cdot \cos \boldsymbol{k} x)$$

$$b_{\boldsymbol{k}} \leftarrow f(x) \cdot \sin \boldsymbol{k} x = a_0 \cdot \sin \boldsymbol{k} x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m} x \cdot \sin \boldsymbol{n} x + b_m \sin \boldsymbol{m} x \cdot \sin \boldsymbol{k} x)$$

Inner Product Space

Hilbert Space real / complex inner product space

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\,dt$$

complex conjugate

$$\langle y, x \rangle = \overline{\langle x, y \rangle}$$

linear

$$\langle a x_1 + b x_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle$$

positive semidefinite

$$\langle x, x \rangle \geq 0$$

Norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Cauchy-Schwartz Inequality

$$|\langle x, y \rangle| \leq \|x\|\,\|y\|$$

Orthogonal Functions (1) $e^{j n \omega_0 t}$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) \, e^{-jk\omega_0 t} \, dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

fundamental frequency $f_0 = \frac{1}{T}$ $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

n-th harmonic frequency $f_n = n \, f_0$ $\omega_n = 2\pi f_n = \frac{2\pi n}{T}$

$$\langle e^{jm\omega_0 t}, \; e^{jn\omega_0 t} \rangle = \int_0^T e^{+j(m-n)\omega_0 t} \, dt = \begin{cases} 0 & (m \neq n) \\ T & (m = n) \end{cases}$$

m, n : integer

Orthogonal Functions (2) $e^{j n \omega_0 t}$

$$\langle e^{j m \omega_0 t}, \; e^{j n \omega_0 t} \rangle = \int_0^T e^{+j(m-n)\omega_0 t} \, dt = \begin{cases} 0 & (m \neq n) \\ T & (m = n) \end{cases}$$

m, n : integer

$$e^{+j m \omega_0 t} \cdot e^{-j n \omega_0 t} = (\cos m\omega_0 t + j \sin m\omega_0 t) \cdot (\cos n\omega_0 t - j \sin n\omega_0 t)$$

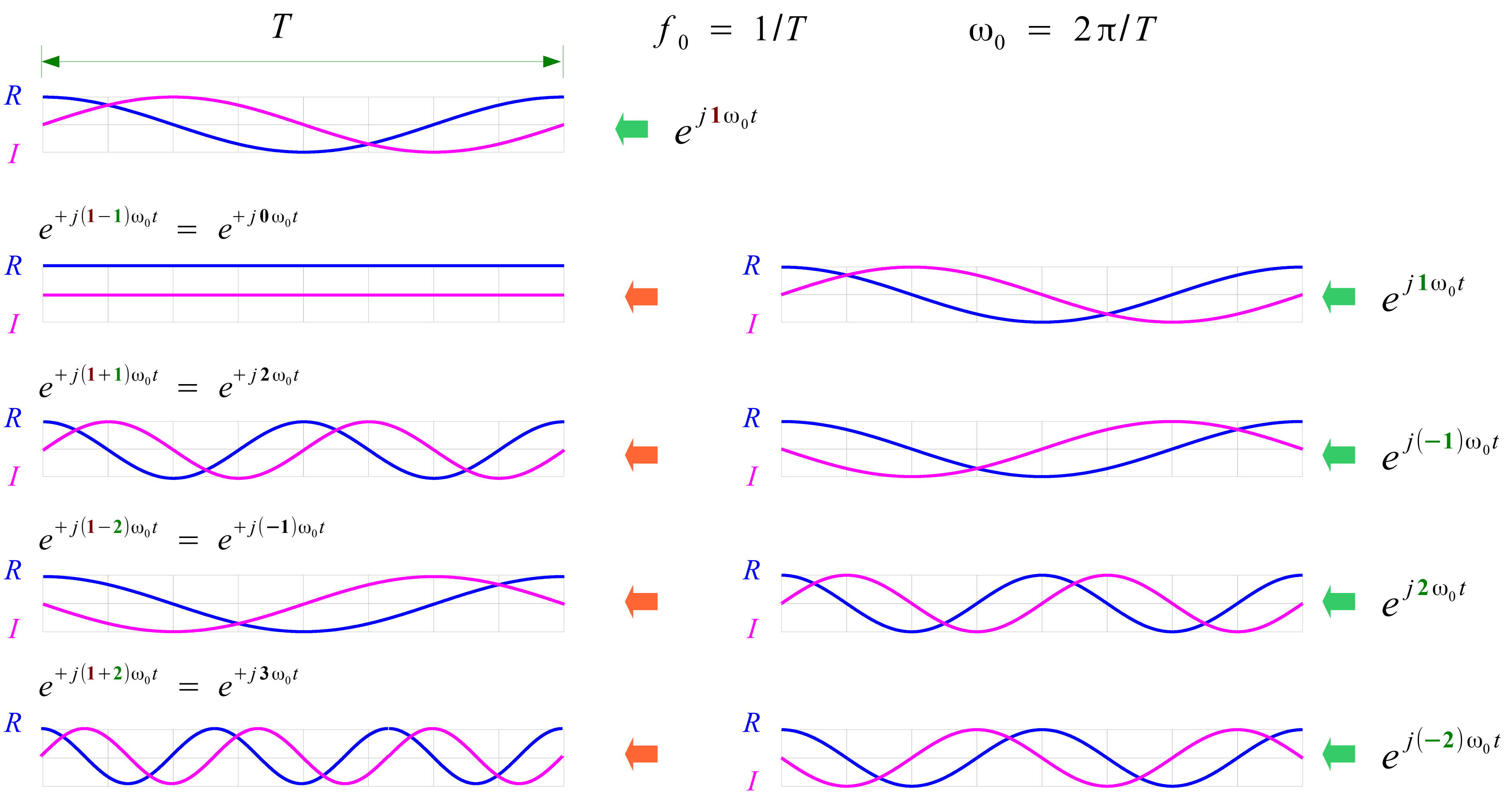
$$= \{\cos m\omega_0 t \cdot \cos n\omega_0 t + \sin m\omega_0 t \cdot \sin n\omega_0 t\}$$

$$+ j\{\sin m\omega_0 t \cdot \cos n\omega_0 t - \cos m\omega_0 t \sin n\omega_0 t\}$$

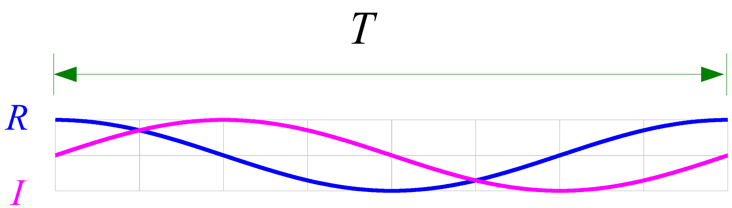
$$= \cos\{m\omega_0 t - n\omega_0 t\} + j \sin\{m\omega_0 t - \sin n\omega_0\}$$

$$= \underbrace{\cos\{(m-n)\omega_0 t\}}_{1} + j \underbrace{\sin\{(m-n)\omega_0 t\}}_{0} \qquad (m = n)$$

Inner Product Examples (1)



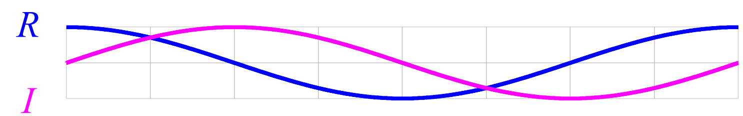
Inner Product Examples (2)



$e^{j\mathbf{1}\omega_0 t}$

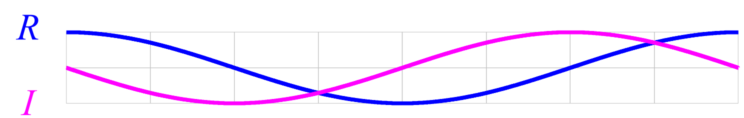
$f_0 = 1/T \qquad \omega_0 = 2\pi/T$

$$\langle e^{j\mathbf{1}\omega_0 t},\ e^{j\mathbf{1}\omega_0 t}\rangle = \int_0^T e^{+j(\mathbf{1}-\mathbf{1})\omega_0 t}\,dt = T$$



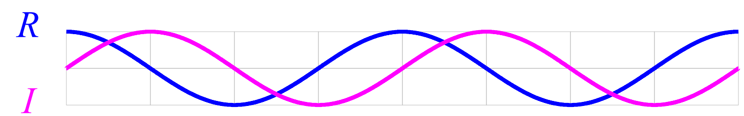
$e^{j\mathbf{1}\omega_0 t}$

$$\langle e^{j\mathbf{1}\omega_0 t},\ e^{j-\mathbf{1}\omega_0 t}\rangle = \int_0^T e^{+j(\mathbf{1}+\mathbf{1})\omega_0 t}\,dt = 0$$



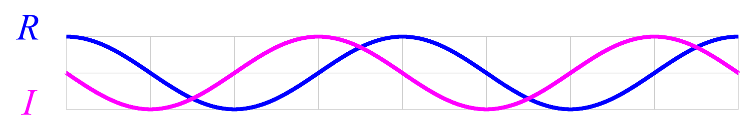
$e^{j-\mathbf{1}\omega_0 t}$

$$\langle e^{j\mathbf{1}\omega_0 t},\ e^{j\mathbf{2}\omega_0 t}\rangle = \int_0^T e^{+j(\mathbf{1}-\mathbf{2})\omega_0 t}\,dt = 0$$



$e^{j\mathbf{2}\omega_0 t}$

$$\langle e^{j\mathbf{1}\omega_0 t},\ e^{j-\mathbf{2}\omega_0 t}\rangle = \int_0^T e^{+j(\mathbf{1}+\mathbf{2})\omega_0 t}\,dt = 0$$



$e^{j-\mathbf{2}\omega_0 t}$

Orthogonal Functions

$\cos n \omega_0 t$

$$\langle \cos m\omega_0 t,\ \cos n\omega_0 t \rangle = \int_0^T \cos m\omega_0 t \cdot \cos n\omega_0 t \, dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

m, n : integer

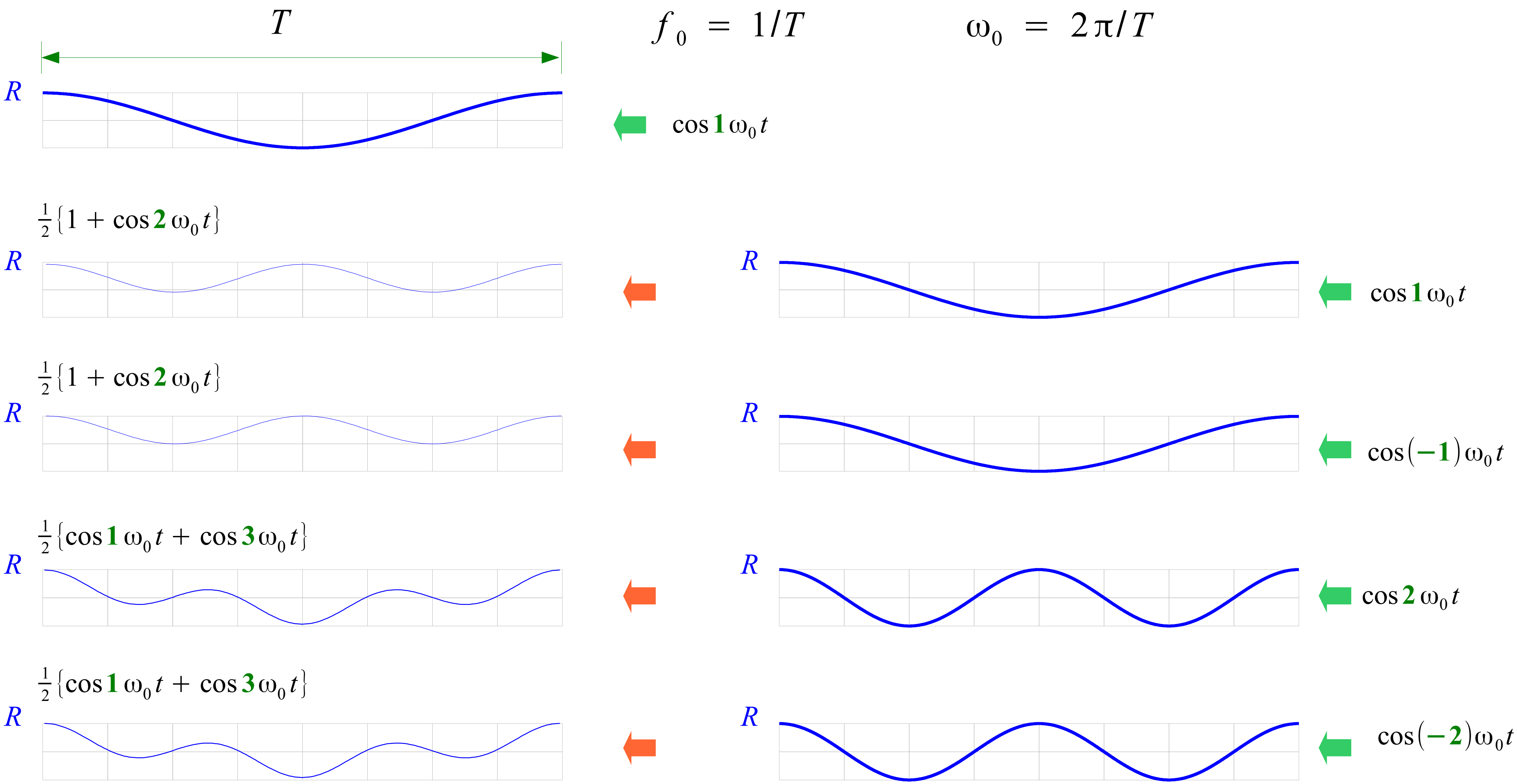
$$\cos m\omega_0 t \cdot \cos n\omega_0 t = \frac{1}{2}\{ \underline{\cos(m-n)\omega_0 t} + \cos(m+n)\omega_0 t \}$$

1

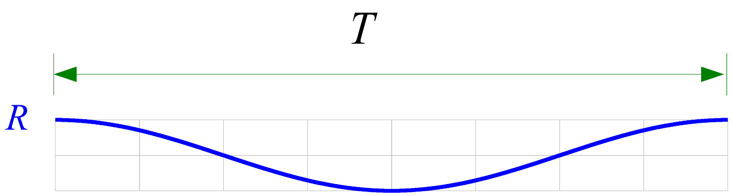
$(m = \pm n)$

Inner Product Examples (1) $\cos n \omega_0 t$

$f_0 = 1/T \qquad \omega_0 = 2\pi/T$



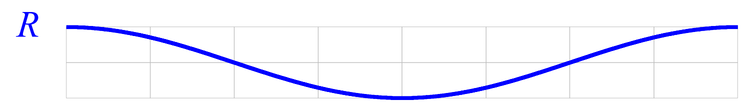
Inner Product Examples (2) $\cos n\,\omega_0 t$



$f_0 = 1/T \qquad \omega_0 = 2\pi/T$

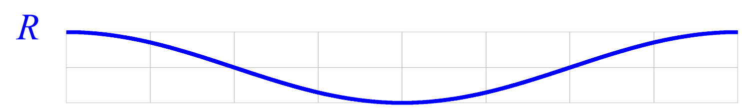
$\cos 1\,\omega_0 t$

$\langle \cos 1\,\omega_0 t,\ \cos 1\,\omega_0 t \rangle$
$= \int_0^T \frac{1}{2}\{\cos(1-1)\omega_0 t + \cos(1+1)\omega_0 t\}\,dt$
$= \int_0^T \frac{1}{2}\{1 + \cos 2\,\omega_0 t\}\,dt = \frac{T}{2}$



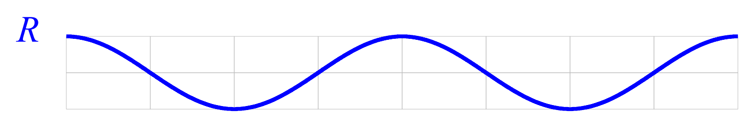
$\cos 1\,\omega_0 t$

$\langle \cos 1\,\omega_0 t,\ \cos(-1)\omega_0 t \rangle$
$= \int_0^T \frac{1}{2}\{\cos(1+1)\omega_0 t + \cos(1-1)\omega_0 t\}\,dt$
$= \int_0^T \frac{1}{2}\{1 + \cos 2\,\omega_0 t\}\,dt = \frac{T}{2}$



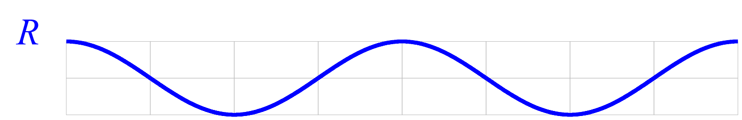
$\cos(-1)\omega_0 t$

$\langle \cos 1\,\omega_0 t,\ \cos 2\,\omega_0 t \rangle$
$= \int_0^T \frac{1}{2}\{\cos(1-2)\omega_0 t + \cos(1+2)\omega_0 t\}\,dt$
$= \int_0^T \frac{1}{2}\{\cos 1\,\omega_0 t + \cos 3\,\omega_0 t\}\,dt = 0$



$\cos 2\,\omega_0 t$

$\langle \cos 1\,\omega_0 t,\ \cos(-2)\omega_0 t \rangle$
$= \int_0^T \frac{1}{2}\{\cos(1+2)\omega_0 t + \cos(1-2)\omega_0 t\}\,dt$
$= \int_0^T \frac{1}{2}\{\cos 1\,\omega_0 t + \cos 3\,\omega_0 t\}\,dt = 0$



$\cos(-2)\omega_0 t$

Orthogonal Functions

$\sin n \omega_0 t$

$$\langle \sin m\omega_0 t,\ \sin n\omega_0 t \rangle = \int_0^T \sin m\omega_0 t \cdot \sin n\omega_0 t \, dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

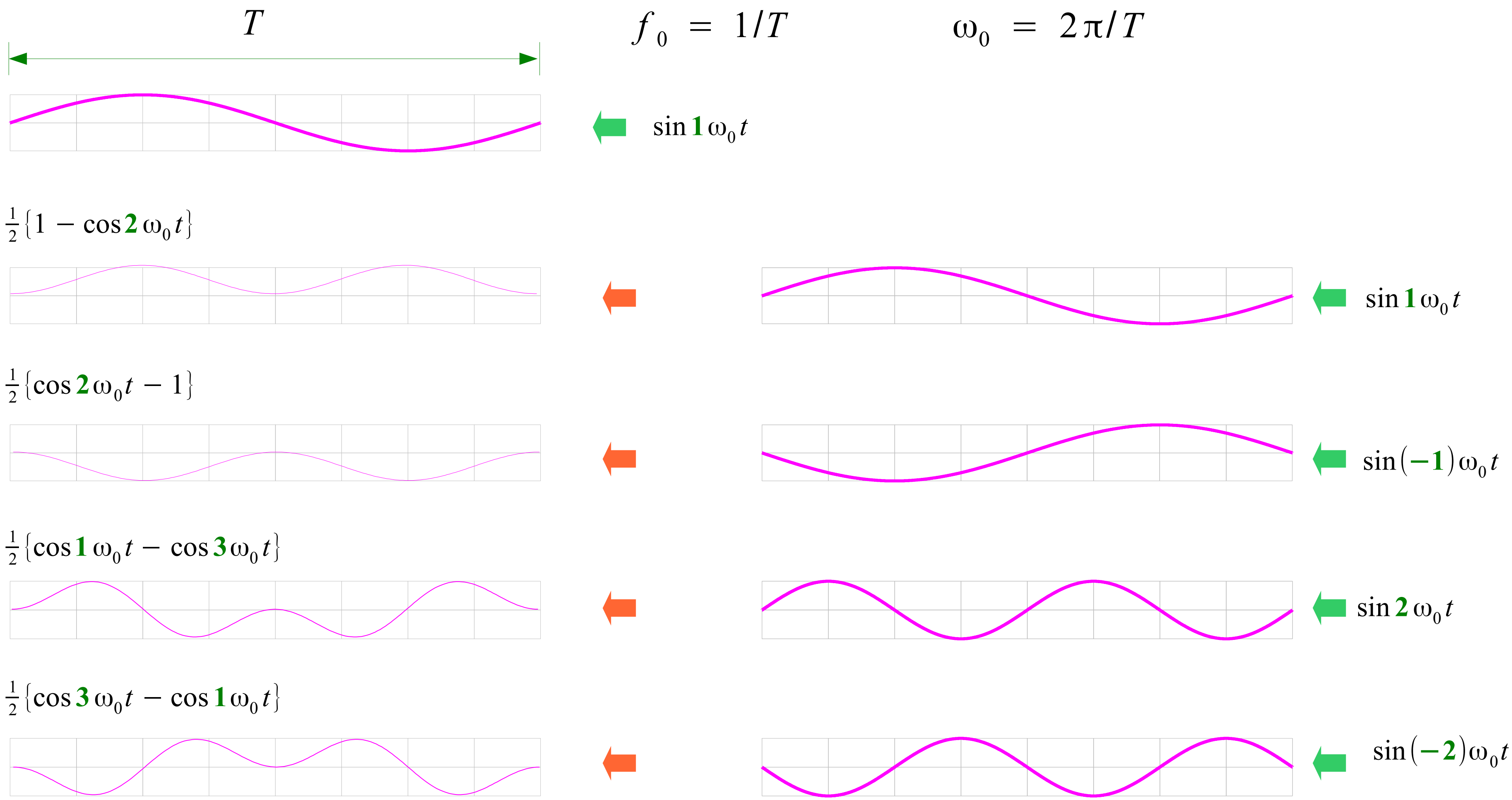
m, n : integer

$$\sin m\omega_0 t \cdot \sin n\omega_0 t = \tfrac{1}{2}\{ \underbrace{\cos(m-n)\omega_0 t}_{1} - \cos(m+n)\omega_0 t \}$$

$(m = \pm n)$

Inner Product Examples (1) $\sin n\omega_0 t$

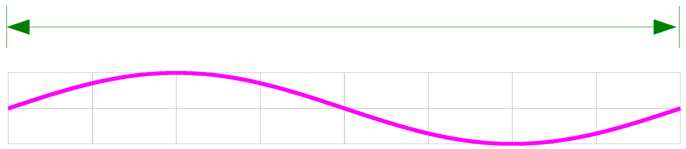
$f_0 = 1/T \qquad \omega_0 = 2\pi/T$



Inner Product Examples (2) $\sin n\omega_0 t$

T

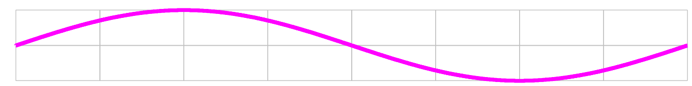
R



$f_0 = 1/T \qquad \omega_0 = 2\pi/T$

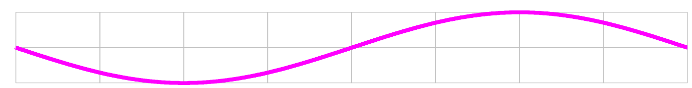
$\sin 1\omega_0 t$

$$\langle \sin 1\omega_0 t,\ \sin 1\omega_0 t\rangle$$
$$= \int_0^T \tfrac{1}{2}\{\cos(1-1)\omega_0 t - \cos(1+1)\omega_0 t\}\,dt$$
$$= \int_0^T \tfrac{1}{2}\{1 - \cos 2\omega_0 t\}\,dt = \tfrac{T}{2}$$



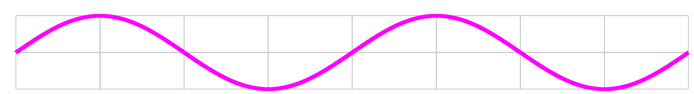
$\sin 1\omega_0 t$

$$\langle \sin 1\omega_0 t,\ \sin(-1)\omega_0 t\rangle$$
$$= \int_0^T \tfrac{1}{2}\{\cos(1+1)\omega_0 t - \cos(1-1)\omega_0 t\}\,dt$$
$$= \int_0^T \tfrac{1}{2}\{\cos 2\omega_0 t - 1\}\,dt = -\tfrac{T}{2}$$



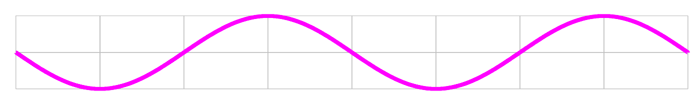
$\sin(-1)\omega_0 t$

$$\langle \sin 1\omega_0 t,\ \sin 2\omega_0 t\rangle$$
$$= \int_0^T \tfrac{1}{2}\{\cos(1-2)\omega_0 t - \cos(1+2)\omega_0 t\}\,dt$$
$$= \int_0^T \tfrac{1}{2}\{\cos 1\omega_0 t - \cos 3\omega_0 t\}\,dt = 0$$



$\sin 2\omega_0 t$

$$\langle \sin 1\omega_0 t,\ \sin(-2)\omega_0 t\rangle$$
$$= \int_0^T \tfrac{1}{2}\{\cos(1+2)\omega_0 t - \cos(1-2)\omega_0 t\}\,dt$$
$$= \int_0^T \tfrac{1}{2}\{\cos 3\omega_0 t - \cos 1\omega_0 t\}\,dt = 0$$



$\sin(-2)\omega_0 t$

Infinite Set of Orthogonal Functions

$$\{1,\ \cos\omega_0 t,\ \cos 2\omega_0 t,\ \cdots,\ \cos n\omega_0 t,\ \cdots\} \Rightarrow g(t) = \sum_{n=0}^{+\infty} a_n \cos n\omega t$$

Linear Combination: ***even** function only*

$$\{\sin\omega_0 t,\ \sin 2\omega_0 t,\ \cdots,\ \sin n\omega_0 t,\ \cdots\} \Rightarrow h(t) = \sum_{n=1}^{+\infty} b_n \sin n\omega t$$

Linear Combination: ***odd** function only*

$$\langle \cos m\omega_0 t,\ \cos n\omega_0 t\rangle = \int_0^T \cos m\omega_0 t \cdot \cos n\omega_0 t\ dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

$m > 0,\ n > 0$: integer

$$\langle \sin m\omega_0 t,\ \sin n\omega_0 t\rangle = \int_0^T \sin m\omega_0 t \cdot \sin n\omega_0 t\ dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

$m > 0,\ n > 0$: integer

Completeness (1)

$$\{1,\ \cos\omega_0 t,\ \cos 2\omega_0 t,\ \cos 3\omega_0 t,\ \cdots\ ,\ \sin\omega_0 t,\ \sin 2\omega_0 t,\ \sin 3\omega_0 t,\ \cdots\ \}$$

Linear Combination:

$$f(t) = \sum_{n=0}^{+\infty} a_n \cos n\omega t + \sum_{n=1}^{+\infty} b_n \sin n\omega t$$

*Can be **even** / **odd** function*

*When the collection of orthogonal functions are **complete**,*
every function can be expanded via a linear combination of these infinite set of functions

$$\langle \cos m\omega_0 t,\ \cos n\omega_0 t\rangle = \int_0^T \cos m\omega_0 t \cdot \cos n\omega_0 t\ dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

$m > 0$, $n > 0$: integer

$$\langle \sin m\omega_0 t,\ \sin n\omega_0 t\rangle = \int_0^T \sin m\omega_0 t \cdot \sin n\omega_0 t\ dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

$m > 0$, $n > 0$: integer

Completeness (2)

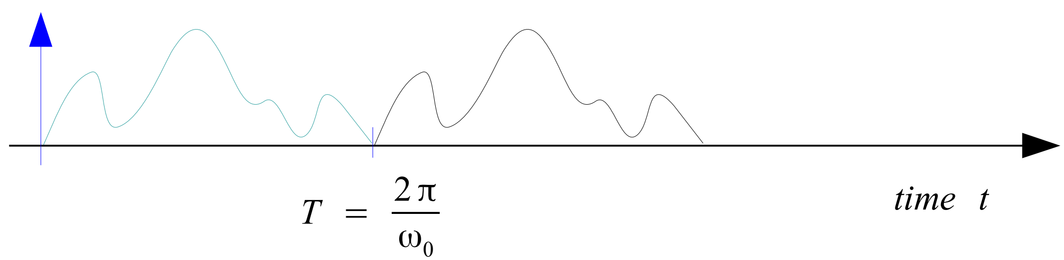
$$\{1,\ \cos\omega_0 t,\ \cos 2\omega_0 t,\ \cos 3\omega_0 t,\ \cdots,\ \sin\omega_0 t,\ \sin 2\omega_0 t,\ \sin 3\omega_0 t,\ \cdots\}$$

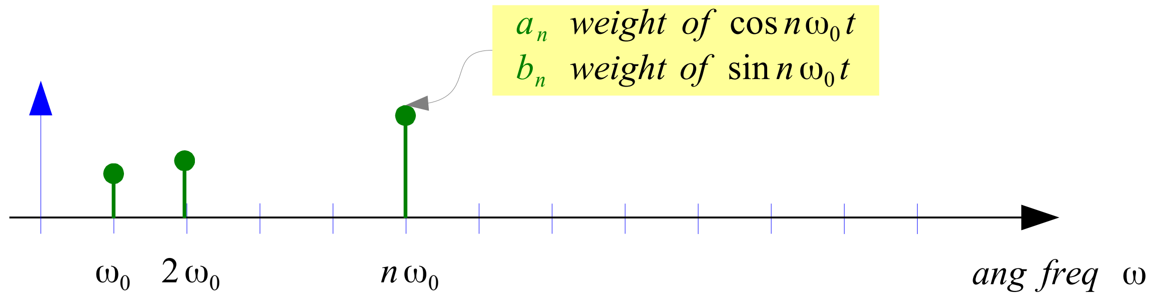
Linear Combination:

$$f(t) = \sum_{n=0}^{+\infty} a_n \cos n\omega t + \sum_{n=1}^{+\infty} b_n \sin n\omega t$$

***even* / *odd* function**
periodic function

periodic continuous time function f(t)





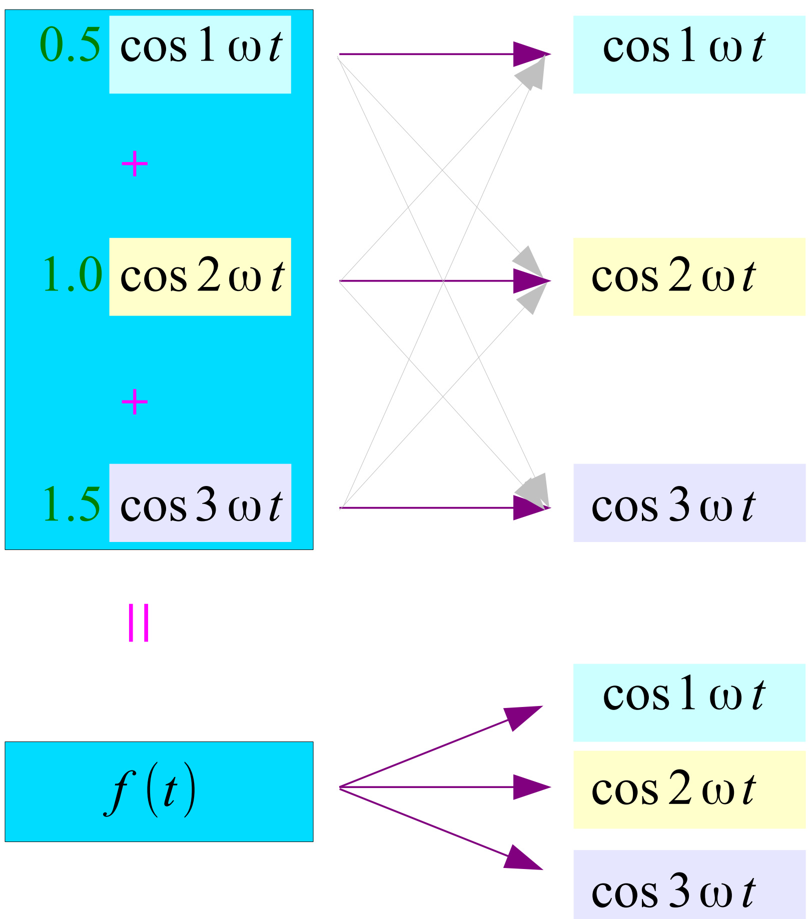
infinite but discrete set of angular frequency

Completeness (3)

$$f(t) = \sum_{n=0}^{+\infty} a_n \cos n\omega t + \sum_{n=1}^{+\infty} b_n \sin n\omega t$$

$$a_1 = 0.5, \quad a_2 = 1.0, \quad a_3 = 1.5$$

$$a_4 = a_5 = \cdots = 0 \qquad b_1 = b_2 = \cdots = 0$$



$$\langle 0.5 \cos 1\omega t, \cos 1\omega t \rangle = \tfrac{T}{2} \cdot 0.5$$
$$\langle 1.0 \cos 2\omega t, \cos 1\omega t \rangle = 0$$
$$\langle 1.5 \cos 3\omega t, \cos 1\omega t \rangle = 0$$

$$\langle 0.5 \cos 1\omega t, \cos 2\omega t \rangle = 0$$
$$\langle 1.0 \cos 2\omega t, \cos 2\omega t \rangle = \tfrac{T}{2} \cdot 1.0$$
$$\langle 1.5 \cos 3\omega t, \cos 2\omega t \rangle = 0$$

$$\langle 0.5 \cos 1\omega t, \cos 3\omega t \rangle = 0$$
$$\langle 1.0 \cos 2\omega t, \cos 3\omega t \rangle = 0$$
$$\langle 1.5 \cos 3\omega t, \cos 3\omega t \rangle = \tfrac{T}{2} \cdot 1.5$$

$$\langle 0.5 \cos 1\omega t, \cos 1\omega t \rangle = \tfrac{T}{2} \cdot 0.5$$
$$\langle 1.0 \cos 2\omega t, \cos 2\omega t \rangle = \tfrac{T}{2} \cdot 1.0$$
$$\langle 1.5 \cos 3\omega t, \cos 3\omega t \rangle = \tfrac{T}{2} \cdot 1.5$$

Hilbert Space

Complete Orthogonal System

$$\{ \quad \cdots \quad , \; e^{-jn\omega_0 t}, \quad \cdots \quad , \; e^{-j\omega_0 t}, \; 1, \; e^{+j\omega_0 t}, \quad \cdots \quad , \; e^{+jn\omega_0 t}, \quad \cdots \quad \}$$

$$\langle e^{j\boldsymbol{m}\omega_0 t}, \; e^{j\boldsymbol{n}\omega_0 t} \rangle = \int_0^T e^{+j(\boldsymbol{m}-\boldsymbol{n})\omega_0 t} \, dt = \begin{cases} 0 & (m \neq n) \\ T & (m = n) \end{cases}$$

$\boldsymbol{m}, \boldsymbol{n}$: integer

Complete BiOrthogonal System

$$\{1, \; \cos\omega_0 t, \; \cos 2\omega_0 t, \; \cos 3\omega_0 t, \quad \cdots \quad , \; \sin\omega_0 t, \; \sin 2\omega_0 t, \; \sin 3\omega_0 t, \quad \cdots \quad \}$$

$$\langle \cos \boldsymbol{m}\omega_0 t, \; \cos \boldsymbol{n}\omega_0 t \rangle = \int_0^T \cos \boldsymbol{m}\omega_0 t \cdot \cos \boldsymbol{n}\omega_0 t \, dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

$\boldsymbol{m} > \boldsymbol{0}, \; \boldsymbol{n} > \boldsymbol{0}$: integer

$$\langle \sin \boldsymbol{m}\omega_0 t, \; \sin \boldsymbol{n}\omega_0 t \rangle = \int_0^T \sin \boldsymbol{m}\omega_0 t \cdot \sin \boldsymbol{n}\omega_0 t \, dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

$\boldsymbol{m} > \boldsymbol{0}, \; \boldsymbol{n} > \boldsymbol{0}$: integer

Cauchy-Schwartz Inequality

For all vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of an inner product space

$$|\langle x, y\rangle|^2 \le \langle x, x\rangle \cdot \langle y, y\rangle$$

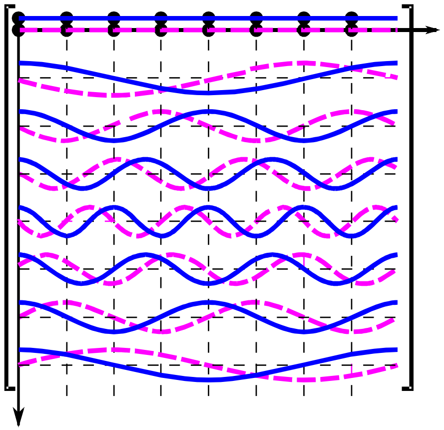
$$|\langle x, y\rangle| \le \|x\| \cdot \|y\|$$

The equality holds if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent ⇨ maximum

$$\left|\int_a^b x(t)\overline{y(t)}\,dt\right| \le \sqrt{\int_a^b x(t)\overline{x(t)}\,dt}\;\sqrt{\int_a^b y(t)\overline{y(t)}\,dt}$$

Inner product is maximum
when $y = k\,x$

Orthogonality



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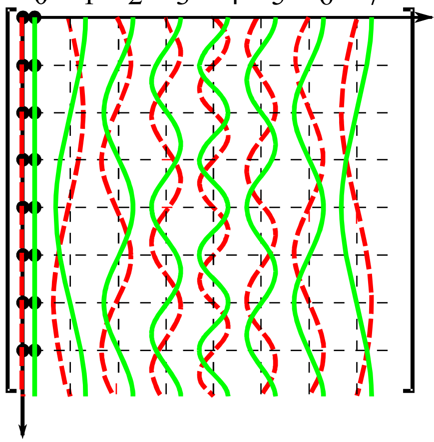


$$= \begin{bmatrix}
r_0 \cdot r_0^* & r_0 \cdot r_1^* & r_0 \cdot r_2^* & r_0 \cdot r_3^* & r_0 \cdot r_4^* & r_0 \cdot r_5^* & r_0 \cdot r_6^* & r_0 \cdot r_7^* \\
r_1 \cdot r_0^* & r_1 \cdot r_1^* & r_1 \cdot r_2^* & r_1 \cdot r_3^* & r_1 \cdot r_4^* & r_1 \cdot r_5^* & r_1 \cdot r_6^* & r_1 \cdot r_7^* \\
r_2 \cdot r_0^* & r_2 \cdot r_1^* & r_2 \cdot r_2^* & r_2 \cdot r_3^* & r_2 \cdot r_4^* & r_2 \cdot r_5^* & r_2 \cdot r_6^* & r_2 \cdot r_7^* \\
r_3 \cdot r_0^* & r_3 \cdot r_1^* & r_3 \cdot r_2^* & r_3 \cdot r_3^* & r_3 \cdot r_4^* & r_3 \cdot r_5^* & r_3 \cdot r_6^* & r_3 \cdot r_7^* \\
r_4 \cdot r_0^* & r_4 \cdot r_1^* & r_4 \cdot r_2^* & r_4 \cdot r_3^* & r_4 \cdot r_4^* & r_4 \cdot r_5^* & r_4 \cdot r_6^* & r_4 \cdot r_7^* \\
r_5 \cdot r_0^* & r_5 \cdot r_1^* & r_5 \cdot r_2^* & r_5 \cdot r_3^* & r_5 \cdot r_4^* & r_5 \cdot r_5^* & r_5 \cdot r_6^* & r_5 \cdot r_7^* \\
r_6 \cdot r_0^* & r_6 \cdot r_1^* & r_6 \cdot r_2^* & r_6 \cdot r_3^* & r_6 \cdot r_4^* & r_6 \cdot r_5^* & r_6 \cdot r_6^* & r_6 \cdot r_7^* \\
r_7 \cdot r_0^* & r_7 \cdot r_1^* & r_7 \cdot r_2^* & r_7 \cdot r_3^* & r_7 \cdot r_4^* & r_7 \cdot r_5^* & r_7 \cdot r_6^* & r_7 \cdot r_7^*
\end{bmatrix}$$

$$\langle \boldsymbol{r}_i^H , \boldsymbol{r}_i^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_i^* = N$$

$$\langle \boldsymbol{r}_i^H , \boldsymbol{r}_j^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_j^* = 0 \quad (i \neq j)$$

Complex Vector Inner Product

Hermitian inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} = \sum x_i^* y_i$$

$\boldsymbol{x}^H$: conjugate transpose

Norm of Hermitian inner products

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\boldsymbol{x}^H \cdot \boldsymbol{x}} = \sqrt{\sum x_i^* x_i}$$

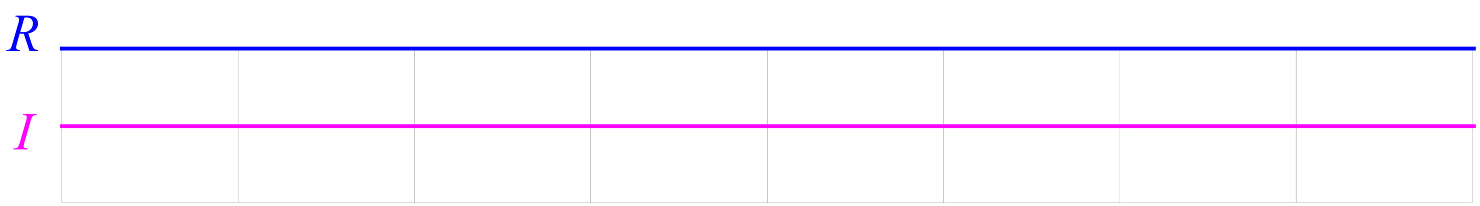
the length of a vector

$$\boldsymbol{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - j b_1 & a_2 - j b_2 & \cdots & a_n - j b_n \end{pmatrix} \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} = \sum_{i=1}^{n} a_i^2 + b_i^2$$

The 1st Row of the DFT Matrix



0 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 0, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *samples of* $\sin(-\omega t) = -\sin(\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi \cdot \left(\tfrac{0}{8}\right) \cdot f_s \cdot t$$

$X[0]$ *measures how much of the* $+0\cdot\omega$ *component is present in* x.

The 3rd Row of the DFT Matrix

R

I

2 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 2, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-2\omega t) = \cos(2\omega t)$

I ➡ *samples of* $\sin(-2\omega t) = -\sin(2\omega t)$

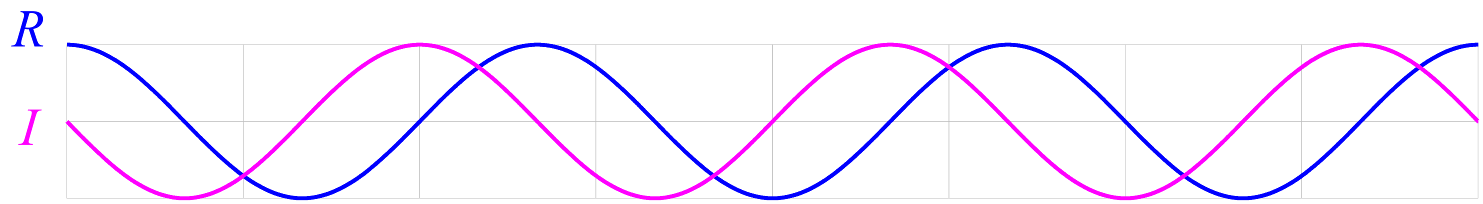
measure →

$\omega t = 2\pi f t$

$2\pi \cdot (\frac{2}{8}) \cdot f_s \cdot t$

$X[2]$ *measures how much of the* $+2\cdot\omega$ *component is present in* $\mathbf{x}$.

The 4th Row of the DFT Matrix



3 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 3, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-3\,\omega t) = \cos(3\,\omega t)$

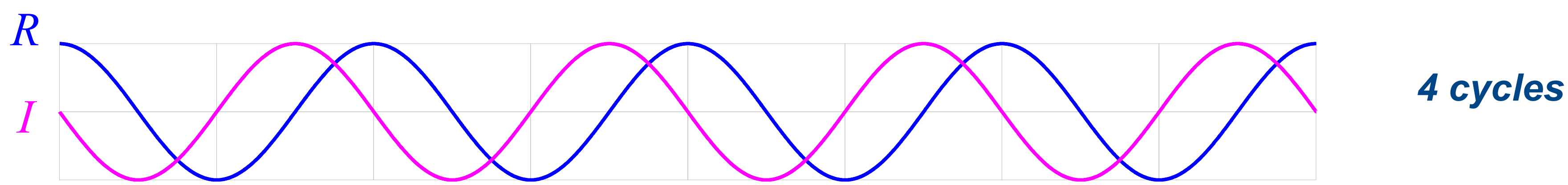
I ➡ *samples of* $\sin(-3\,\omega t) = -\sin(3\,\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{3}{8}\right)\cdot f_s\cdot t$$

$X[3]$ *measures how much of the* $+3\cdot\omega$ *component is present in* x.

The 5th Row of the DFT Matrix



$$W_8^{k\,n} = e^{-j\left(\frac{2\pi}{8}\right)k\,n} \qquad k = 4, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-4\,\omega\,t) = \cos(4\,\omega\,t)$

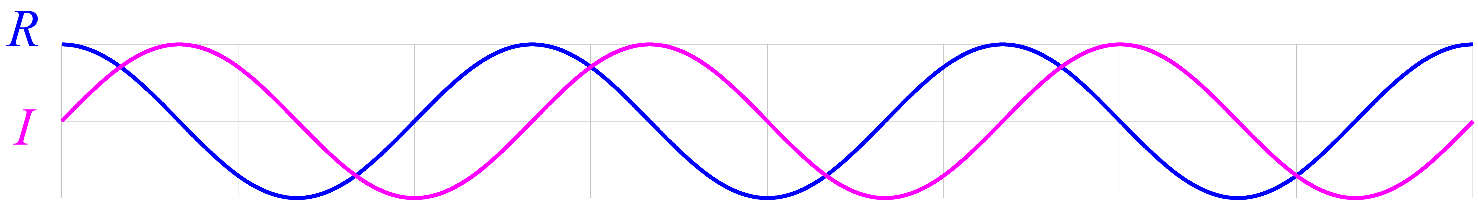
I ➡ *samples of* $\sin(-4\,\omega\,t) = -\sin(4\,\omega\,t)$

measure →

$$\omega t = 2\pi f\,t$$
$$2\pi \cdot \left(\tfrac{4}{8}\right) \cdot f_s \cdot t$$

X[4] *measures how much of the* ***+4·ω*** *component is present in* ***x***.

The 6th Row of the DFT Matrix



3 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 5, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-(-3\omega)t) = \cos(3\omega t)$

I ➡ *samples of* $\sin(-(-3\omega)t) = \sin(3\omega t)$

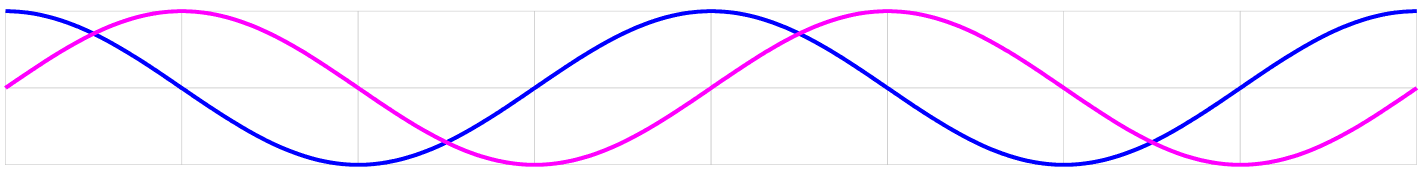
measure →

$$-\omega t = -2\pi f t$$

$$2\pi \cdot \left(\frac{-3}{8}\right) \cdot f_s \cdot t$$

$X[5]$ *measures how much of the* $-3\cdot\omega$ *component is present in* x.

The 7th Row of the DFT Matrix



2 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 2, \quad n = 0, 1, \ldots, 7$$

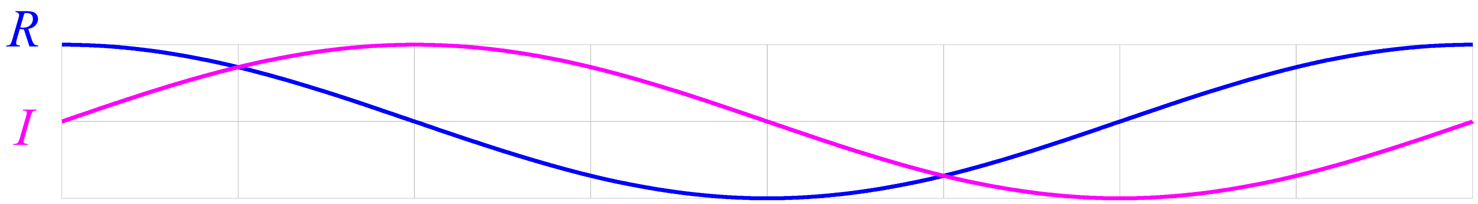
R → *samples of* $\cos(-(-2\omega)t) = \cos(2\omega t)$

I → *samples of* $\sin(-(-2\omega)t) = \sin(2\omega t)$

measure → $-\omega t = -2\pi f t$, $2\pi \cdot (\frac{-2}{8}) \cdot f_s \cdot t$

$X[6]$ *measures how much of the* $-2\cdot\omega$ *component is present in* $\mathbf{x}$.

The 8th Row of the DFT Matrix



1 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 7, \quad n = 0, 1, \ldots, 7$$

R → *samples of* $\cos(-(-\omega)t) = \cos(\omega t)$

I → *samples of* $\sin(-(-\omega)t) = \sin(\omega t)$

measure → $-\omega t = -2\pi f t$, $2\pi \cdot \left(\frac{-1}{8}\right) \cdot f_s \cdot t$

X[7] *measures how much of the* $-1 \cdot \omega$ *component is present in* ***x***.

Any Period $p = 2L$

$$g(v) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kv + b_k \sin kv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v)\, dv$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos kv\, dv$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin kv\, dv$$

$$k = 1,\ 2,\ \ldots$$

$$v :\ [-\pi,\ +\pi]$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L}\, dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L}\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x :\ [-L,\ +L]$$

$$v = \frac{\pi}{L} x$$

$$dv = \frac{\pi}{L} dx$$

Time and Frequency

$$f(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi}{L}x + b_k \sin\frac{k\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)\, dx$$

$$a_k = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{k\pi x}{L}\, dx$$

$$b_k = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\frac{k\pi x}{L}\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x :\ [-L,\ +L]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2\pi k}{T}t + b_k \sin\frac{2\pi k}{T}t\right)$$

$$a_0 = \frac{1}{T}\int_{0}^{T} x(t)\, dt$$

$$a_k = \frac{2}{T}\int_{0}^{T} x(t)\cos\frac{2\pi k t}{T}\, dt$$

$$b_k = \frac{2}{T}\int_{0}^{T} x(t)\sin\frac{2\pi k t}{T}\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$t :\ [0,\ T]$$

$$2L = T$$

Continuous Time Periodic Signal $x(t)$

Harmonic Frquency

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2\pi k}{T}t + b_k \sin\frac{2\pi k}{T}t\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos\frac{2\pi k t}{T}\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin\frac{2\pi k t}{T}\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos(2\pi \boldsymbol{k} f_0 t) + b_k \sin(2\pi \boldsymbol{k} f_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(2\pi \boldsymbol{k} f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(2\pi \boldsymbol{k} f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

resolution frequency $\quad f_0 = \frac{1}{T}$

n-th harmonic frequency $\quad f_n = \boldsymbol{n} f_0 = \boldsymbol{n}\frac{1}{T}$

Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_k \cos(k 2\pi f_0 t) + b_k \sin(k 2\pi f_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k 2\pi f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k 2\pi f_0 t)\,dt \qquad k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_k \cos(\boldsymbol{k}\,\omega_0 t) + b_k \sin(\boldsymbol{k}\,\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(\boldsymbol{k}\,\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(\boldsymbol{k}\,\omega_0 t)\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

linear frequency f

angular (radial) frequency $\omega = 2\pi f$

Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}\right)$$

$$A_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$t:\ [0,\ T]$$

Real coefficients

$$a_0,\ a_k,\ b_k,\quad k = 1,\ 2,\ \ldots$$

→ **one-sided spectrum**
only positive frequencies

Complex coefficients

$$A_0,\ A_k,\ B_k,\quad k = 1,\ 2,\ \ldots$$

→ **two-sided spectrum**
Both pos and neg frequencies

Euler Equation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$a_k \cos(k\omega_0 t) \qquad + \; b_k \sin(k\omega_0 t)$$

$$= a_k \frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + b_k \frac{1}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$

$$= \frac{(a_k - jb_k)}{2} e^{jk\omega_0 t} + \frac{(a_k + jb_k)}{2} e^{-jk\omega_0 t}$$

$$= A_k\, e^{jk\omega_0 t} \qquad + \; B_k\, e^{-jk\omega_0 t}$$

$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty}\left(A_k\, e^{jk\omega_0 t} + B_k\, e^{-jk\omega_0 t}\right)$$

zero freq → $A_0 = a_0$

pos freq → $A_k = \frac{1}{2}(a_k - jb_k)$

neg freq → $B_k = \frac{1}{2}(a_k + jb_k)$

only positive frequencies

Euler Equation (2)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

zero freq → $A_0 = a_0$

pos freq → $A_k = \frac{1}{2}(a_k - j b_k)$

neg freq → $B_k = \frac{1}{2}(a_k + j b_k)$

only positive frequencies

$$A_k = \frac{1}{T} \int_0^T x(t) \left(\cos(k\omega_0 t) - j \sin(k\omega_0 t) \right) dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) \left(\cos(k\omega_0 t) + j \sin(k\omega_0 t) \right) dt$$

$$A_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$B_k = \frac{1}{T} \int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

Complex Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = 0,\ 1,\ 2,\ \ldots$$

$$B_k = \frac{1}{T} \int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2}(a_k - j b_k)$$

$$B_k = \frac{1}{2}(a_k + j b_k)$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_k & (k < 0) \end{cases}$$

Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$\boldsymbol{k} = +1, +2, \ldots$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$\boldsymbol{k} = +1, +2, \ldots$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = g_k \cos(\phi_k)\cos(k\omega_0 t) - g_k \sin(\phi_k)\sin(k\omega_0 t)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Phasor Representation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k\, \Re\left\{ e^{+j(k\omega_0 t + \phi_k)} \right\}$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \Re\left\{ g_k \cdot e^{+j\phi_k} \cdot e^{+jk\omega_0 t} \right\}$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re\left\{ X_k e^{+jk\omega_0 t} \right\}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

$$k = 1,\ 2,\ \ldots$$

Phasor Representation (2)

$$x(t) = g_0 + \sum_{k=1}^{\infty} \frac{g_k}{2} \cdot \left(e^{+j(k\omega_0 t + \phi_k)} + e^{-j(k\omega_0 t + \phi_k)} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k}{2} e^{+j\phi_k} e^{+jk\omega_0 t} + \frac{g_k}{2} e^{-j\phi_k} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k e^{+j\phi_k}}{2} e^{+jk\omega_0 t} + \frac{g_k e^{-j\phi_k}}{2} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{g_k e^{+j\phi_k}}{2} \qquad (k > 0)$$

$$C_{-k} = \frac{g_k e^{-j\phi_k}}{2} \qquad (k < 0)$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re\{X_k e^{+jk\omega_0 t}\}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

$$k = 1,\ 2,\ \ldots$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$k = \ldots, -2, -1, 0, +1, +2, \ldots$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$|C_k|^2 + |C_{-k}|^2 = \frac{1}{2}|g_k|^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$k = \ldots, -2, -1, 0, +1, +2, \ldots$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_k \, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_k \, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

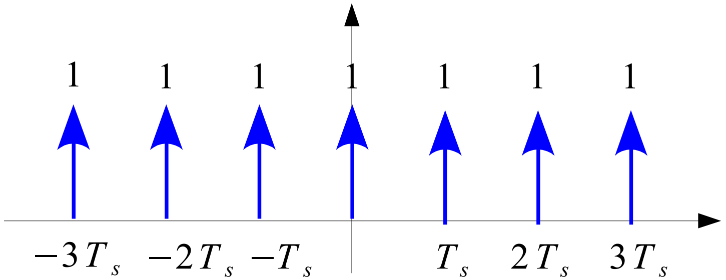
$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = |g_k| = \sqrt{a_k^2 + b_k^2}$$

CTFS of Impulse Train (1)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



Fourier Series Expansion of Impulse Train

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

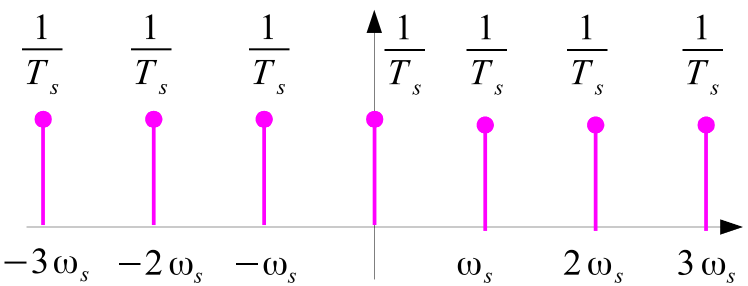
$$\omega_s = \frac{2\pi}{T_s}$$

Fourier Series Coefficients

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) \, e^{-jk\omega_s t} \, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) \, e^{-jk\omega_s 0} \, dt$$

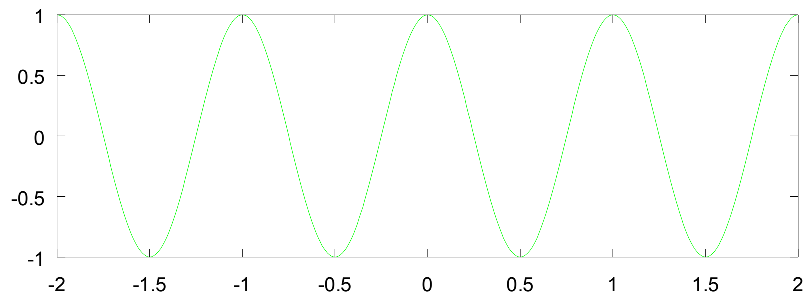
$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) \, dt = \frac{1}{T_s}$$

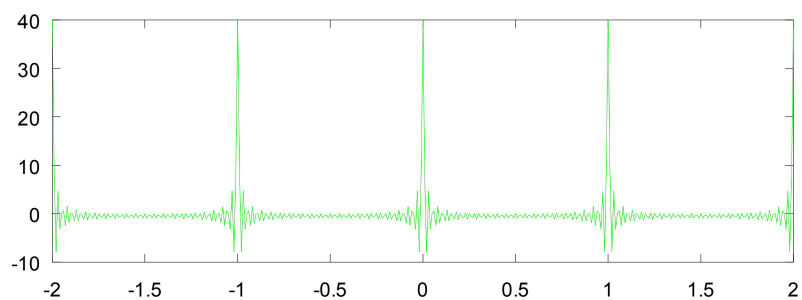


CTFS of Impulse Train (2)

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k\, e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

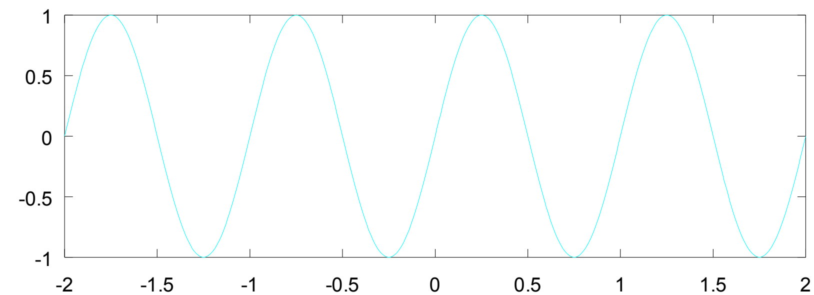
$\cos 2\pi \cdot 1 \cdot t$

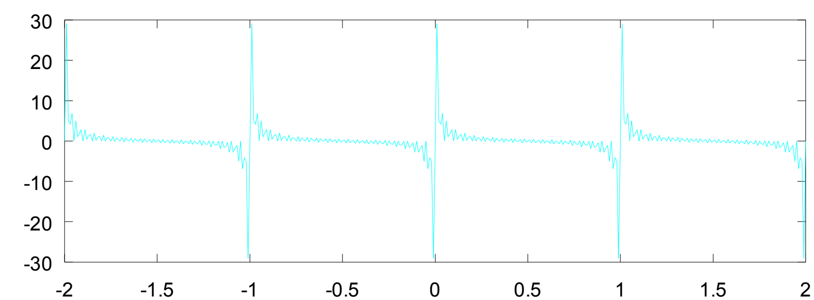




$\sum_{k=1}^{40} \cos 2\pi \cdot k \cdot t$

$\sin 2\pi \cdot 1 \cdot t$



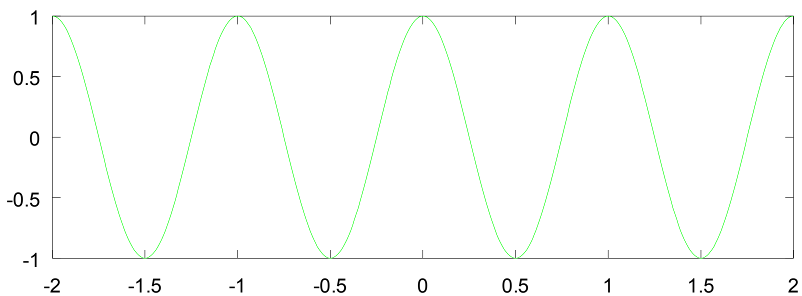


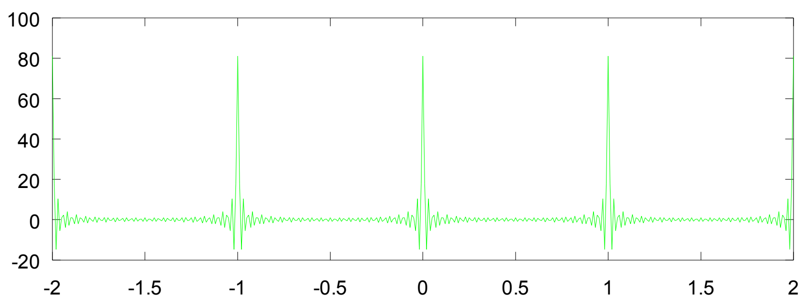
$\sum_{k=1}^{40} \sin 2\pi \cdot k \cdot t$

CTFS of Impulse Train (3)

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

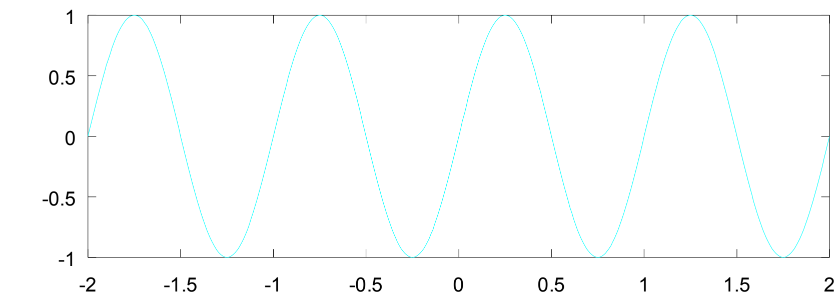
$\cos 2\pi \cdot 1 \cdot t$

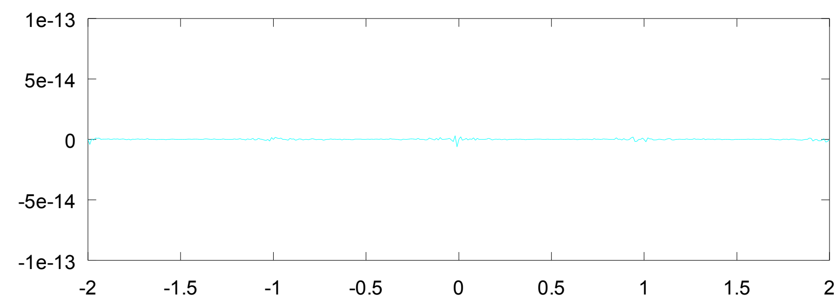




$$\sum_{k=-40}^{40} \cos 2\pi \cdot k \cdot t$$

$\sin 2\pi \cdot 1 \cdot t$





$$\sum_{k=-40}^{40} \sin 2\pi \cdot k \cdot t$$

Inner Product Space

Hilbert Space　　real / complex inner product space

$$\langle f, g \rangle = \int_a^b f(t)\overline{g(t)}\,dt$$

complex conjugate

$$\langle y, x \rangle = \overline{\langle x, y \rangle}$$

linear

$$\langle a x_1 + b x_2, y \rangle = a \langle x_1, y \rangle + b \langle x_2, y \rangle$$

positive semidefinite

$$\langle x, x \rangle \geq 0$$

Norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Cauchy-Schwartz Inequality

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|$$

Orthogonality

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots,\ -2, -1, 0, +1, +2, \ldots$$

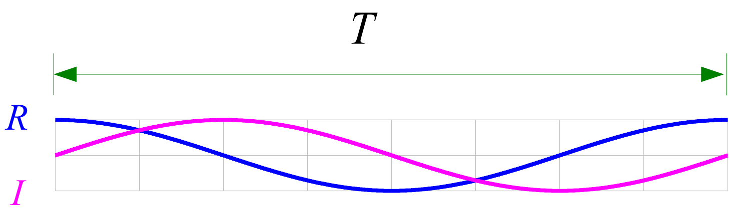
fundamental frequency $\quad f_0 = \frac{1}{T} \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

n-th harmonic frequency $\quad f_n = n\, f_0 \qquad \omega_n = 2\pi f_n = \frac{2\pi n}{T}$

$$\langle e^{jm\omega_0 t},\ e^{jn\omega_0 t} \rangle = \int_0^T e^{+j(m-n)\omega_0 t}\, dt = \begin{cases} 0 & (m \neq n) \\ T & (m = n) \end{cases}$$

m, n : integer

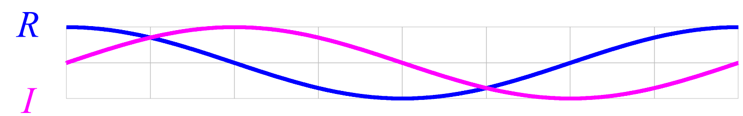
Inner Product Examples



$f_0 = 1/T$ $\qquad \omega_0 = 2\pi/T$

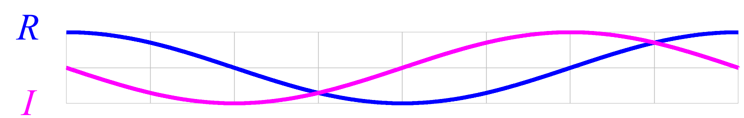
$e^{j1\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j1\omega_0 t}\rangle = \int_0^T e^{+j(1-1)\omega_0 t}\, dt = T$$



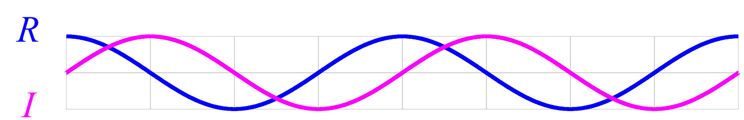
$e^{j1\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j-1\omega_0 t}\rangle = \int_0^T e^{+j(1+1)\omega_0 t}\, dt = 0$$



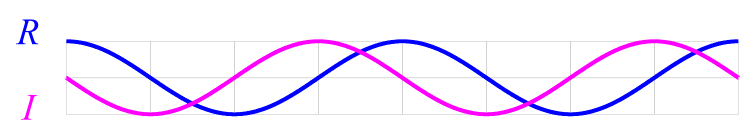
$e^{j-1\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j2\omega_0 t}\rangle = \int_0^T e^{+j(1-2)\omega_0 t}\, dt = 0$$



$e^{j2\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j-2\omega_0 t}\rangle = \int_0^T e^{+j(1+2)\omega_0 t}\, dt = 0$$



$e^{j-2\omega_0 t}$

Cauchy-Schwartz Inequality

For all vectors ***x*** and ***y*** of an inner product space

$$\left|\langle x,\ y\rangle\right|^2 \le \langle x,\ x\rangle \cdot \langle y,\ y\rangle$$

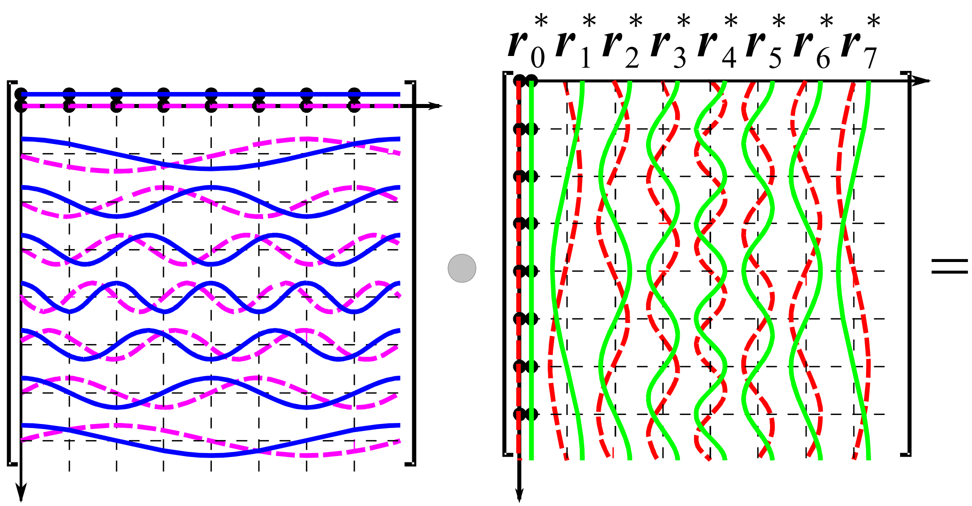
$$\left|\langle x,\ y\rangle\right| \le \|x\| \cdot \|y\|$$

The equality holds if and only if ***x*** and ***y*** are linearly dependent ⇨ maximum

$$\left|\int_a^b x(t)\overline{y(t)}\,dt\right| \le \sqrt{\int_a^b x(t)\overline{x(t)}\,dt}\ \sqrt{\int_a^b y(t)\overline{y(t)}\,dt}$$

Inner product is maximum
when $y = k\,x$

Orthogonality



$r_0 \cdot r_0^*$	$r_0 \cdot r_1^*$	$r_0 \cdot r_2^*$	$r_0 \cdot r_3^*$	$r_0 \cdot r_4^*$	$r_0 \cdot r_5^*$	$r_0 \cdot r_6^*$	$r_0 \cdot r_7^*$
$r_1 \cdot r_0^*$	$r_1 \cdot r_1^*$	$r_1 \cdot r_2^*$	$r_1 \cdot r_3^*$	$r_1 \cdot r_4^*$	$r_1 \cdot r_5^*$	$r_1 \cdot r_6^*$	$r_1 \cdot r_7^*$
$r_2 \cdot r_0^*$	$r_2 \cdot r_1^*$	$r_2 \cdot r_2^*$	$r_2 \cdot r_3^*$	$r_2 \cdot r_4^*$	$r_2 \cdot r_5^*$	$r_2 \cdot r_6^*$	$r_2 \cdot r_7^*$
$r_3 \cdot r_0^*$	$r_3 \cdot r_1^*$	$r_3 \cdot r_2^*$	$r_3 \cdot r_3^*$	$r_3 \cdot r_4^*$	$r_3 \cdot r_5^*$	$r_3 \cdot r_6^*$	$r_3 \cdot r_7^*$
$r_4 \cdot r_0^*$	$r_4 \cdot r_1^*$	$r_4 \cdot r_2^*$	$r_4 \cdot r_3^*$	$r_4 \cdot r_4^*$	$r_4 \cdot r_5^*$	$r_4 \cdot r_6^*$	$r_4 \cdot r_7^*$
$r_5 \cdot r_0^*$	$r_5 \cdot r_1^*$	$r_5 \cdot r_2^*$	$r_5 \cdot r_3^*$	$r_5 \cdot r_4^*$	$r_5 \cdot r_5^*$	$r_5 \cdot r_6^*$	$r_5 \cdot r_7^*$
$r_6 \cdot r_0^*$	$r_6 \cdot r_1^*$	$r_6 \cdot r_2^*$	$r_6 \cdot r_3^*$	$r_6 \cdot r_4^*$	$r_6 \cdot r_5^*$	$r_6 \cdot r_6^*$	$r_6 \cdot r_7^*$
$r_7 \cdot r_0^*$	$r_7 \cdot r_1^*$	$r_7 \cdot r_2^*$	$r_7 \cdot r_3^*$	$r_7 \cdot r_4^*$	$r_7 \cdot r_5^*$	$r_7 \cdot r_6^*$	$r_7 \cdot r_7^*$

$$\langle \boldsymbol{r}_i^H , \boldsymbol{r}_i^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_i^* = N$$

$$\langle \boldsymbol{r}_i^H , \boldsymbol{r}_j^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_j^* = 0 \quad (i \neq j)$$

Complex Vector Inner Product

Hermitian inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} = \sum x_i^* y_i$$

$\boldsymbol{x}^H$: conjugate transpose

Norm of Hermitian inner products

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\boldsymbol{x}^H \cdot \boldsymbol{x}} = \sqrt{\sum x_i^* x_i}$$

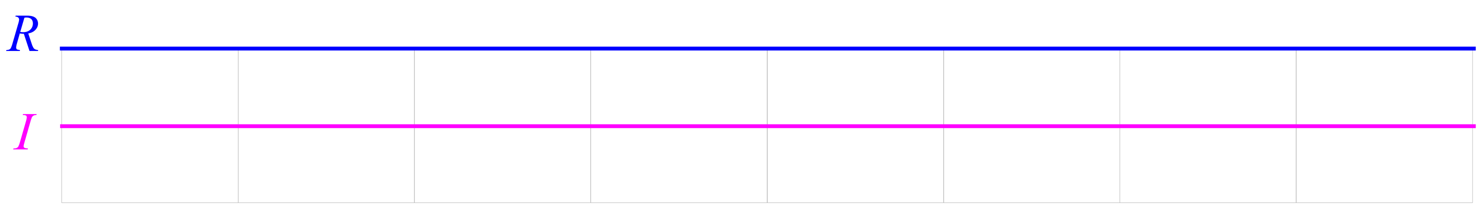
the length of a vector

$$\boldsymbol{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - j b_1 & a_2 - j b_2 & \cdots & a_n - j b_n \end{pmatrix} \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} = \sum_{i=1}^{n} a_i^2 + b_i^2$$

The 1st Row of the DFT Matrix



0 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 0, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *samples of* $\sin(-\omega t) = -\sin(\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi \cdot \left(\tfrac{0}{8}\right) \cdot f_s \cdot t$$

$\boldsymbol{X}[0]$ *measures how much of the* $\boldsymbol{+0\cdot\omega}$ *component is present in* $\boldsymbol{x}$.

The 3rd Row of the DFT Matrix

R

I

2 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 2, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-2\omega t) = \cos(2\omega t)$

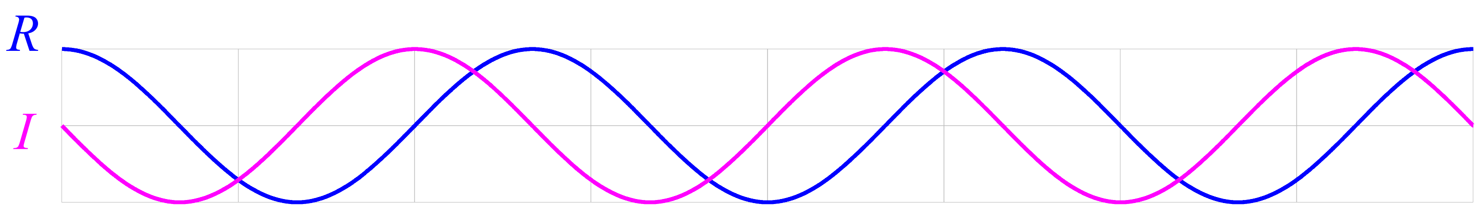
I ➡ *samples of* $\sin(-2\omega t) = -\sin(2\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi \cdot (\tfrac{2}{8}) \cdot f_s \cdot t$$

$X[2]$ *measures how much of the* $+2\cdot\omega$ *component is present in* $\boldsymbol{x}$.

The 4th Row of the DFT Matrix



3 cycles

$$W_8^{k\,n} = e^{-j(\frac{2\pi}{8})k\,n} \qquad k = 3, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-3\,\omega\, t) = \cos(3\,\omega\, t)$

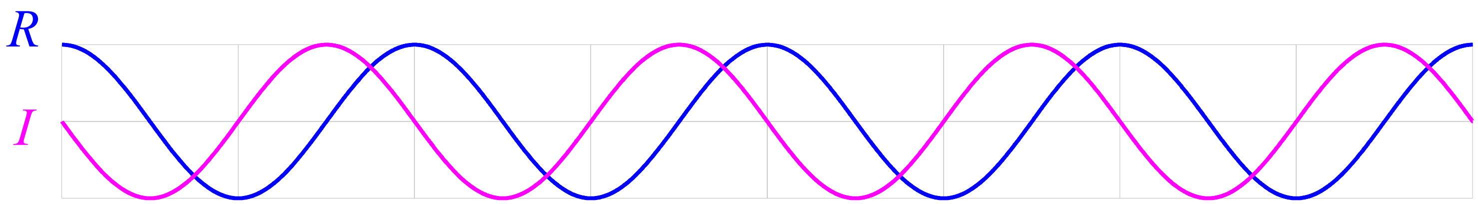
I ➡ *samples of* $\sin(-3\,\omega\, t) = -\sin(3\,\omega\, t)$

measure →

$$\omega t = 2\pi f\, t$$
$$2\pi\cdot(\tfrac{3}{8})\cdot f_s\cdot t$$

$X[3]$ *measures how much of the* $+3\cdot\omega$ *component is present in* x.

The 5th Row of the DFT Matrix



$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 4, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-4\omega t) = \cos(4\omega t)$

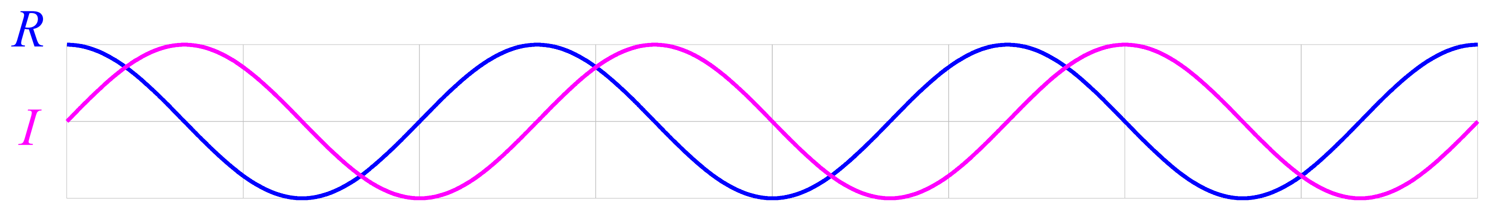
I ➡ *samples of* $\sin(-4\omega t) = -\sin(4\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi \cdot \left(\tfrac{4}{8}\right) \cdot f_s \cdot t$$

X[4] *measures how much of the* ***+4·ω*** *component is present in* ***x***.

The 6th Row of the DFT Matrix



3 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 5, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-(-3\omega)t) = \cos(3\omega t)$

I ➡ *samples of* $\sin(-(-3\omega)t) = \sin(3\omega t)$

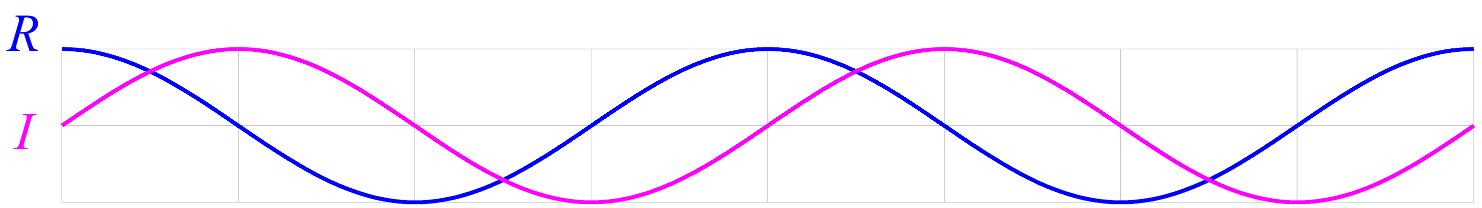
measure →

$$-\omega t = -2\pi f t$$

$$2\pi \cdot \left(\tfrac{-3}{8}\right) \cdot f_s \cdot t$$

X[5] *measures how much of the* ***–3·ω*** *component is present in* ***x***.

The 7th Row of the DFT Matrix



2 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 2, \quad n = 0, 1, \dots, 7$$

R ➡ *samples of* $\cos(-(-2\omega)t) = \cos(2\omega t)$

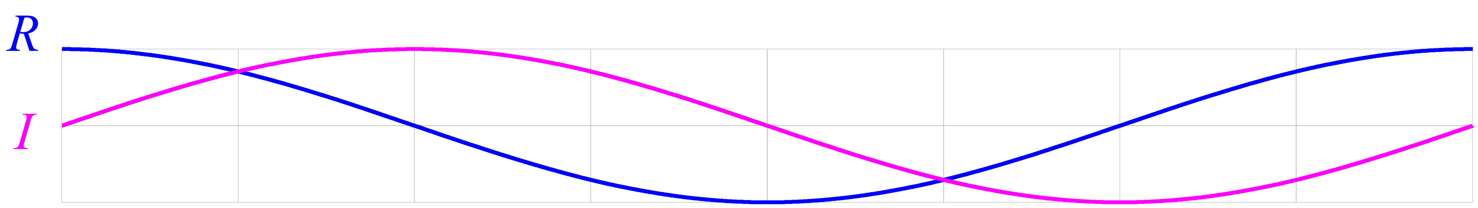
I ➡ *samples of* $\sin(-(-2\omega)t) = \sin(2\omega t)$

measure → $-\omega t = -2\pi f t$

$2\pi \cdot (\frac{-2}{8}) \cdot f_s \cdot t$

X[6] *measures how much of the* $-2\cdot\omega$ *component is present in* ***x***.

The 8th Row of the DFT Matrix



1 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 7, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-(-\omega)t) = \cos(\omega t)$

I ➡ *samples of* $\sin(-(-\omega)t) = \sin(\omega t)$

measure →

$$-\omega t = -2\pi f t$$

$$2\pi \cdot \left(\frac{-1}{8}\right) \cdot f_s \cdot t$$

X[7] *measures how much of the* $-1 \cdot \omega$ *component is present in* ***x***.

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] G. Beale, http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html
[4] C. Langton, http://www.complextoreal.com/chapters/fft1.pdf