

Elementary Matrix

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Gauss-Jordan Elimination

Forward Phase - Gaussian Elimination

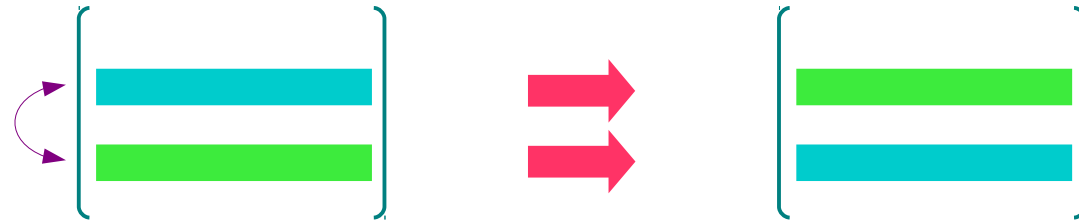
$$\begin{array}{c}
 \left(\begin{array}{ccc|c}
 \textcircled{+2} & +1 & -1 & +8 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 \textcircled{+1} & +1/2 & -1/2 & +4 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 \boxed{0} & +1/2 & +1/2 & +1 \\
 \boxed{0} & +2 & +1 & +5
 \end{array} \right) \\
 \\
 \left(\begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & \textcircled{+1} & +1 & +2 \\
 0 & +2 & +1 & +5
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & \boxed{0} & -1 & +1
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & 0 & \textcircled{+1} & -1
 \end{array} \right)
 \end{array}$$

Backward Phase

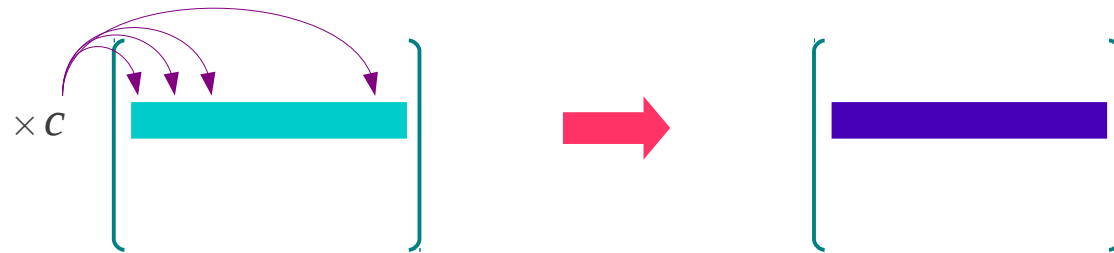
$$\begin{array}{c}
 \left(\begin{array}{ccc|c}
 +1 & +1/2 & \boxed{-1/2} & +4 \\
 0 & +1 & \boxed{+1} & +2 \\
 0 & 0 & +1 & -1
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 +1 & +1/2 & \boxed{0} & +7/2 \\
 0 & +1 & \boxed{0} & +3 \\
 0 & 0 & +1 & -1
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 +1 & \boxed{0} & \boxed{0} & +2 \\
 0 & +1 & \boxed{0} & +3 \\
 0 & 0 & +1 & -1
 \end{array} \right)
 \end{array}$$

Elementary Row Operation

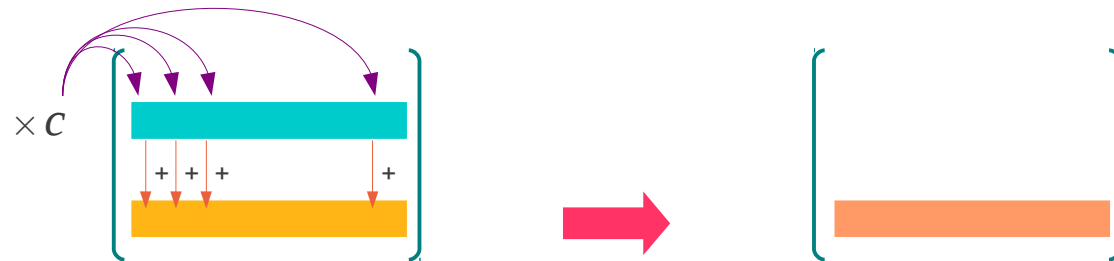
Interchange two rows



Multiply a row by a nonzero constant



Add a multiple of one row to another



Elementary Matrix

Identity Matrix

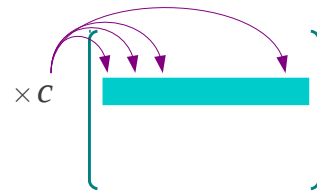
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Interchange two rows



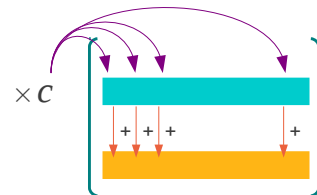
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiply a row by a nonzero constant



$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add a multiple of one row to another



$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplication by an Elementary Matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

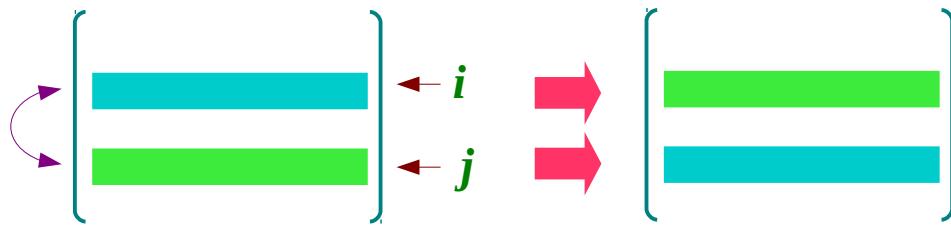
$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

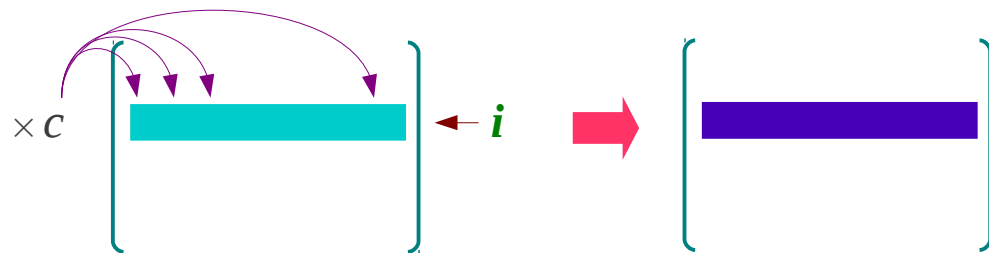
$$\begin{bmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 7 & 8 & 9 \end{bmatrix}$$

Elementary Matrix

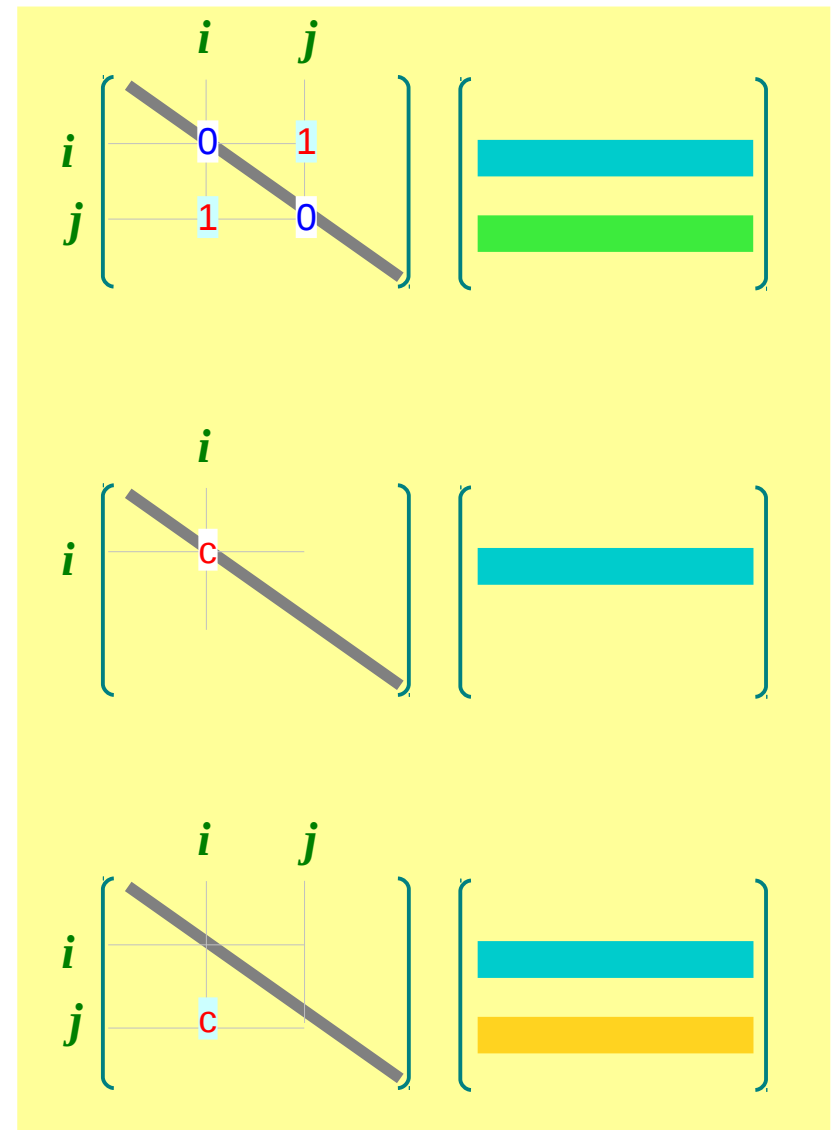
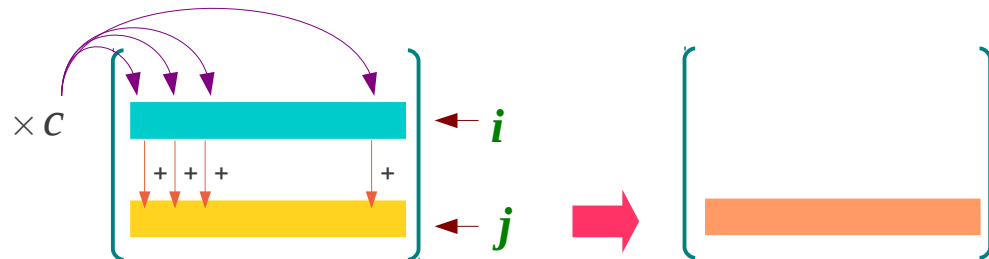
Interchange two rows



Multiply a row by a nonzero constant



Add a multiple of one row to another



Gauss-Jordan Elimination – Step 1

$$\begin{array}{rcl} +2x_1 + x_2 - x_3 = 8 & (L_1) & \\ -3x_1 - x_2 + 2x_3 = -11 & (L_2) & \\ -2x_1 + x_2 + 2x_3 = -3 & (L_3) & \end{array} \quad \left[\begin{array}{ccc|c} \textcircled{+2} & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$

$$E_1 \quad \left[\begin{array}{ccc} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \quad \left[\begin{array}{ccc|c} \textcircled{+2} & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$

$$\begin{array}{rcl} +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = 4 & (\frac{1}{2} \times L_1) & \\ -3x_1 - x_2 + 2x_3 = -11 & (L_2) & \\ -2x_1 + x_2 + 2x_3 = -3 & (L_3) & \end{array} \quad \left[\begin{array}{ccc|c} \textcircled{+1} & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$

Gauss-Jordan Elimination - Step 2

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$-3x_1 - x_2 + 2x_3 = -11 \quad (L_2)$$

$$-2x_1 + x_2 + 2x_3 = -3 \quad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$

E_3

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{array} \right]$$

E_2

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right]$$

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = +1 \quad (3 \times L_1 + L_2)$$

$$0x_1 + 2x_2 + 1x_3 = +5 \quad (2 \times L_1 + L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array} \right]$$

Gauss-Jordan Elimination – Step 3

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = +1 \quad (L_2)$$

$$0x_1 + 2x_2 + 1x_3 = +5 \quad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array} \right]$$

E_4

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array} \right]$$

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (2 \times L_2)$$

$$0x_1 + 2x_2 + 1x_3 = +5 \quad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array} \right]$$

Gauss-Jordan Elimination – Step 4

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

$$0x_1 + 2x_2 + 1x_3 = +5 \quad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array} \right]$$

E_5

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array} \right]$$

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

$$0x_1 + 0x_2 - 1x_3 = +1 \quad (-2 \times L_2 + L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array} \right]$$

Gauss-Jordan Elimination – Step 5

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

$$0x_1 + 0x_2 - 1x_3 = +1 \quad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array} \right]$$

E_6

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array} \right]$$

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

$$0x_1 + 0x_2 + 1x_3 = -1 \quad (-1 \times L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

Forward Phase

$$\begin{array}{c}
 \left(\begin{array}{ccc|c}
 \textcircled{+2} & +1 & -1 & +8 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 \textcircled{+1} & +1/2 & -1/2 & +4 \\
 -3 & -1 & +2 & -11 \\
 -2 & +1 & +2 & -3
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 \boxed{0} & +1/2 & +1/2 & +1 \\
 \boxed{0} & +2 & +1 & +5
 \end{array} \right) \rightarrow \\
 \left(\begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & \textcircled{+1} & +1 & +2 \\
 0 & +2 & +1 & +5
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & \boxed{0} & -1 & +1
 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c}
 +1 & +1/2 & -1/2 & +4 \\
 0 & +1 & +1 & +2 \\
 0 & 0 & \textcircled{+1} & -1
 \end{array} \right)
 \end{array}$$

Forward Phase - Gaussian Elimination

Gauss-Jordan Elimination – Step 6

$$+1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 \quad (L_1)$$

$$0x_1 + 1x_2 + 1x_3 = +2 \quad (L_2)$$

$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

E_8

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array} \right]$$

E_7

$$\left[\begin{array}{ccc} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

$$+1x_1 + \frac{1}{2}x_2 + 0x_3 = +\frac{7}{2} \quad \left(+\frac{1}{2} \times L_3 + L_1\right)$$

$$0x_1 + 1x_2 + 0x_3 = +3 \quad (-1 \times L_3 + L_2)$$

$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

Gauss-Jordan Elimination – Step 7

$$+1x_1 + \frac{1}{2}x_2 + 0x_3 = +\frac{7}{2} \quad (L_1)$$

$$0x_1 + 1x_2 + 0x_3 = +3 \quad (L_2)$$

$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

E_9

$$\left[\begin{array}{ccc} 1 & -1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

$$+1x_1 + 0x_2 - 0x_3 = +2 \quad \left(-\frac{1}{2} \times L_2 + L_1\right)$$

$$0x_1 + 1x_2 + 0x_3 = +3 \quad (L_2)$$

$$0x_1 + 0x_2 + 1x_3 = -1 \quad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array} \right]$$

Backward Phase

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array} \right) \rightarrow$$

Gauss-Jordan Elimination

Forward Phase - Gaussian Elimination

$$\begin{array}{c}
 \left(\begin{array}{ccc|c} \textcircled{+2} & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} \textcircled{+1} & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array} \right) \\
 \\
 \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & \textcircled{+1} & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & \textcircled{+1} & -1 \end{array} \right)
 \end{array}$$

Backward Phase

$$\begin{array}{c}
 \left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{-1/2} & +4 \\ 0 & +1 & \boxed{+1} & +2 \\ 0 & 0 & +1 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{0} & +7/2 \\ 0 & +1 & \boxed{0} & +3 \\ 0 & 0 & +1 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} +1 & \boxed{0} & \boxed{0} & +2 \\ 0 & +1 & \boxed{0} & +3 \\ 0 & 0 & +1 & -1 \end{array} \right)
 \end{array}$$

Product of Elementary Matrices

$$\begin{matrix} E_3 & E_2 & E_1 & \cdot & \left(\begin{array}{ccc|c} \textcircled{+2} & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array} \right) \end{matrix}$$

$$\begin{matrix} E_6 & E_5 & E_4 \end{matrix}$$

$$\begin{matrix} E_9 & E_8 & E_7 \end{matrix}$$

Equivalent Statements

A : invertible

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \begin{bmatrix} A^{-1} \\ \text{green square} \end{bmatrix} = \begin{bmatrix} A^{-1} \\ \text{green square} \end{bmatrix} \begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} = \begin{bmatrix} I_n \\ \text{red diagonal} \end{bmatrix}$$

$Ax = 0$
only the trivial solution

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \begin{bmatrix} x \\ \text{orange bar} \end{bmatrix} = \begin{bmatrix} 0 \\ \text{zero vector} \end{bmatrix}$$

A the RREF is I_n
(Reduced Row Echelon Form)

$$\begin{bmatrix} A \\ \text{cyan square} \end{bmatrix} \xrightarrow{\text{Elem Row Op}} \begin{bmatrix} I_n \\ \text{red diagonal} \end{bmatrix}$$

A can be written as a product of E_k
(Elementary Matrices)

$$\begin{matrix} i & j \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ j \end{matrix} \quad \begin{matrix} i \\ \begin{bmatrix} c \\ \text{row } i \end{bmatrix} \end{matrix} \quad \begin{matrix} i & j \\ \begin{bmatrix} c \\ \text{row } j \end{bmatrix} \\ j \end{matrix}$$

Proof (1)

A : invertible

$$\begin{bmatrix} A \\ \text{[cyan box]} \end{bmatrix} \begin{bmatrix} A^{-1} \\ \text{[green box]} \end{bmatrix} = \begin{bmatrix} A^{-1} \\ \text{[green box]} \end{bmatrix} \begin{bmatrix} A \\ \text{[cyan box]} \end{bmatrix} = \begin{bmatrix} I_n \\ \text{[cyan box]} \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$Ax = 0$$

only the trivial solution

$$\begin{bmatrix} A \\ \text{[cyan box]} \end{bmatrix} \begin{bmatrix} x \\ \text{[orange box]} \end{bmatrix} = \begin{bmatrix} 0 \\ \text{[cyan box]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A : invertible

x_0 a solution of $Ax = 0$

$$Ax_0 = 0$$

$$A^{-1}Ax_0 = A^{-1}0$$

$$I_n x_0 = 0$$

$$x_0 = 0 \quad \text{trivial}$$

Proof (2)

$$Ax = 0$$

only the **trivial** solution

$$A \quad x = 0$$

$$\left[\begin{array}{c} \square \\ \square \\ \square \end{array} \right] \left[\begin{array}{c} \square \\ \square \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

A the RREF is I_n
(Reduced Row Echelon Form)

$$A \xrightarrow{\text{Elem Row Op}} I_n$$

$$\left[\begin{array}{c} \square \\ \square \end{array} \right] \xrightarrow{\text{Elem Row Op}} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

only the **trivial** solution

After the forward and backward phases of Gauss-Jordan Elimination

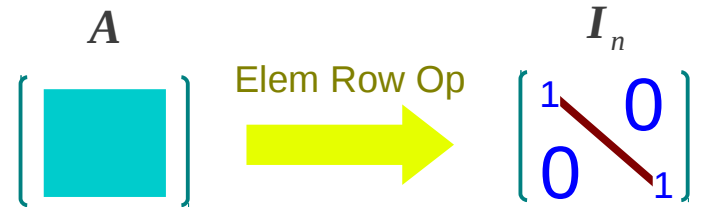
$$\left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{array} \right)$$

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{array}$$

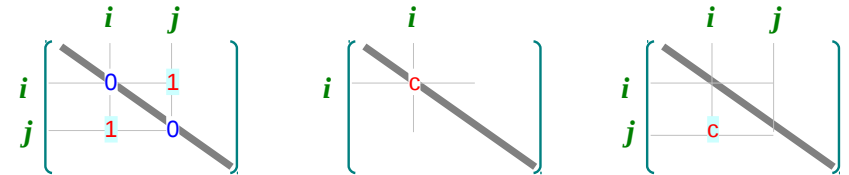
$$\begin{array}{l} 1x_1 = 0 \\ + 1x_2 = 0 \\ \vdots \\ 1x_n = 0 \end{array}$$

Proof (3)

A the RREF is I_n
(Reduced Row Echelon Form)



A can be written as a product of E_k
(Elementary Matrices)



$$E_k \cdots E_2 E_1 A = I_n$$



$$E_k^{-1} E_k E_{k-1} \cdots E_2 E_1 A = E_k^{-1} I_n$$

$$E_{k-1} \cdots E_2 E_1 A = E_k^{-1} A$$



$$E_{k-1}^{-1} E_{k-1} \cdots E_2 E_1 A = E_{k-1}^{-1} E_k^{-1} A$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A$$

(Elementary Matrices)

Proof (4)

A can be written as a product of E_k
(Elementary Matrices)

A : invertible

$$\begin{pmatrix} A \\ \text{[cyan box]} \end{pmatrix} \begin{pmatrix} A^{-1} \\ \text{[green box]} \end{pmatrix} = \begin{pmatrix} A^{-1} \\ \text{[green box]} \end{pmatrix} \begin{pmatrix} A \\ \text{[cyan box]} \end{pmatrix} = \begin{pmatrix} I_n \\ \text{[red diagonal]} \end{pmatrix}$$

$$E_k \cdots E_2 E_1 A = I_n$$

$$A^{-1} A = I_n$$

$$A^{-1} = E_k \cdots E_2 E_1$$

Inversion Algorithm (1)

$$\begin{array}{c}
 \mathbf{A} \\
 \left[\begin{array}{c} \text{cyan square} \end{array} \right]
 \end{array}
 \begin{array}{c}
 \mathbf{A}^{-1} \\
 \left[\begin{array}{c} \text{green square} \\ \text{blue vertical lines} \end{array} \right] \\
 \begin{array}{c} \swarrow \quad \searrow \quad \searrow \\ \mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_n \end{array}
 \end{array}
 =
 \begin{array}{c}
 \left[\begin{array}{c} \text{cyan square} \\ \text{blue vertical lines} \end{array} \right] \\
 \begin{array}{c} \swarrow \quad \searrow \quad \searrow \\ \mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_n \end{array}
 \end{array}$$

$$\begin{array}{c}
 \mathbf{A} \\
 \left[\begin{array}{c} \text{cyan square} \end{array} \right]
 \end{array}
 \begin{array}{c}
 \mathbf{x}_1 \\
 \left[\begin{array}{c} \text{green vertical bar} \end{array} \right]
 \end{array}
 =
 \begin{array}{c}
 \mathbf{b}_1 \\
 \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right]
 \end{array}$$

$$\begin{array}{c}
 \mathbf{A} \\
 \left[\begin{array}{c} \text{cyan square} \end{array} \right]
 \end{array}
 \begin{array}{c}
 \mathbf{x}_n \\
 \left[\begin{array}{c} \text{green vertical bar} \end{array} \right]
 \end{array}
 =
 \begin{array}{c}
 \mathbf{b}_n \\
 \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]
 \end{array}$$

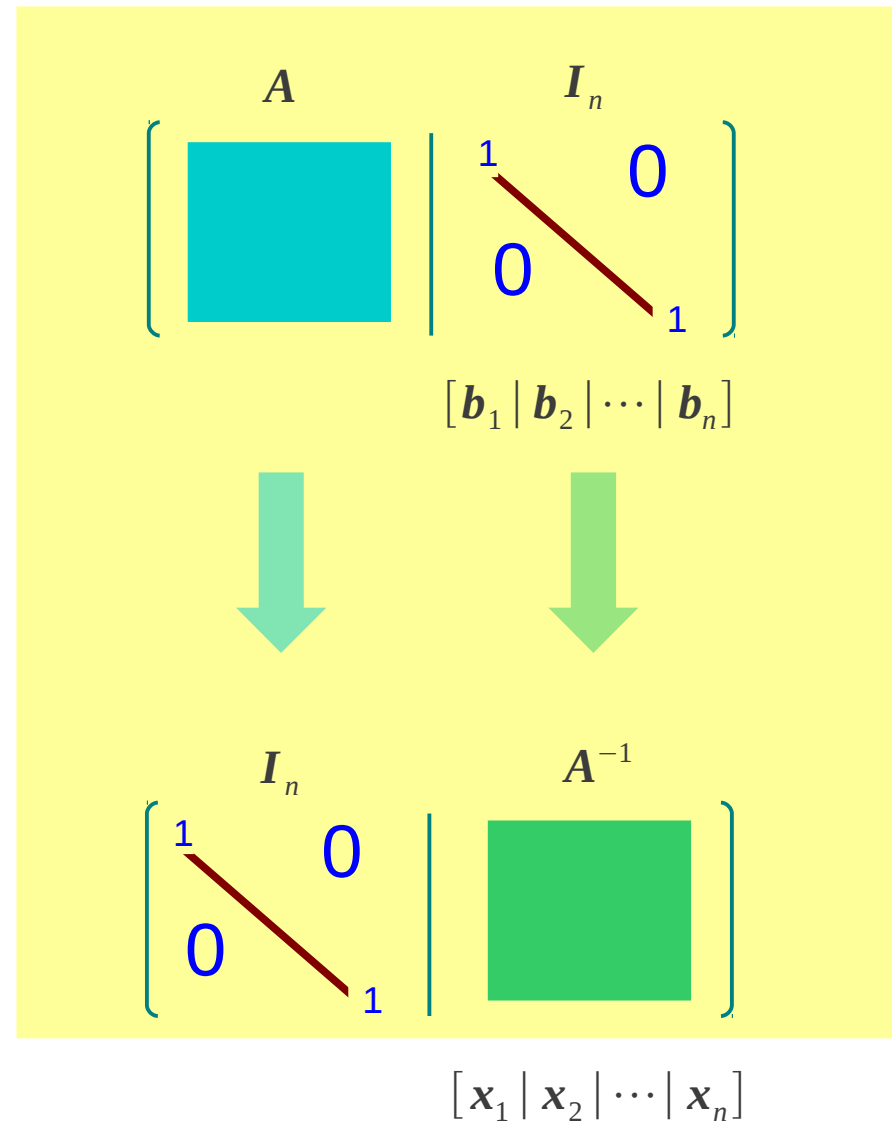
$$\begin{array}{c}
 \mathbf{A} \\
 \left[\begin{array}{c} \text{cyan square} \end{array} \right]
 \end{array}
 \begin{array}{c}
 \mathbf{x}_2 \\
 \left[\begin{array}{c} \text{green vertical bar} \end{array} \right]
 \end{array}
 =
 \begin{array}{c}
 \mathbf{b}_2 \\
 \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right]
 \end{array}$$

Inversion Algorithm (2)

$$\left[\begin{array}{c|c} A & \mathbf{x}_1 \end{array} \right] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{c|c} A & \mathbf{x}_2 \end{array} \right] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{c|c} A & \mathbf{x}_n \end{array} \right] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Homogeneous System

$$\begin{array}{ccccccccccc} a_{11} & x_1 & + & a_{12} & x_2 & + & \cdots & + & a_{1n} & x_n & = & 0 \\ a_{21} & x_1 & + & a_{22} & x_2 & + & \cdots & + & a_{2n} & x_n & = & 0 \\ \vdots & & & \vdots & & & & & \vdots & & & \vdots \\ a_{m1} & x_1 & + & a_{m2} & x_2 & + & \cdots & + & a_{mn} & x_n & = & 0 \end{array}$$

All constant terms are zero

Homogeneous System

All constant terms
are zero

$$\begin{matrix} & & i & j \\ i & & 0 & 1 \\ j & & 1 & 0 \end{matrix} \left[\begin{array}{c} \diagdown \\ \diagup \end{array} \right]$$

$$\begin{matrix} & & i \\ i & & c \end{matrix} \left[\begin{array}{c} \diagdown \\ \diagup \end{array} \right]$$

$$\begin{matrix} & & i & j \\ i & & & \\ j & & c & \end{matrix} \left[\begin{array}{c} \diagdown \\ \diagup \end{array} \right]$$

References

- [1] <http://en.wikipedia.org/>
- [2] Anton & Busby, "Contemporary Linear Algebra"