

Matrices (3A)

Copyright (c) 2012 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

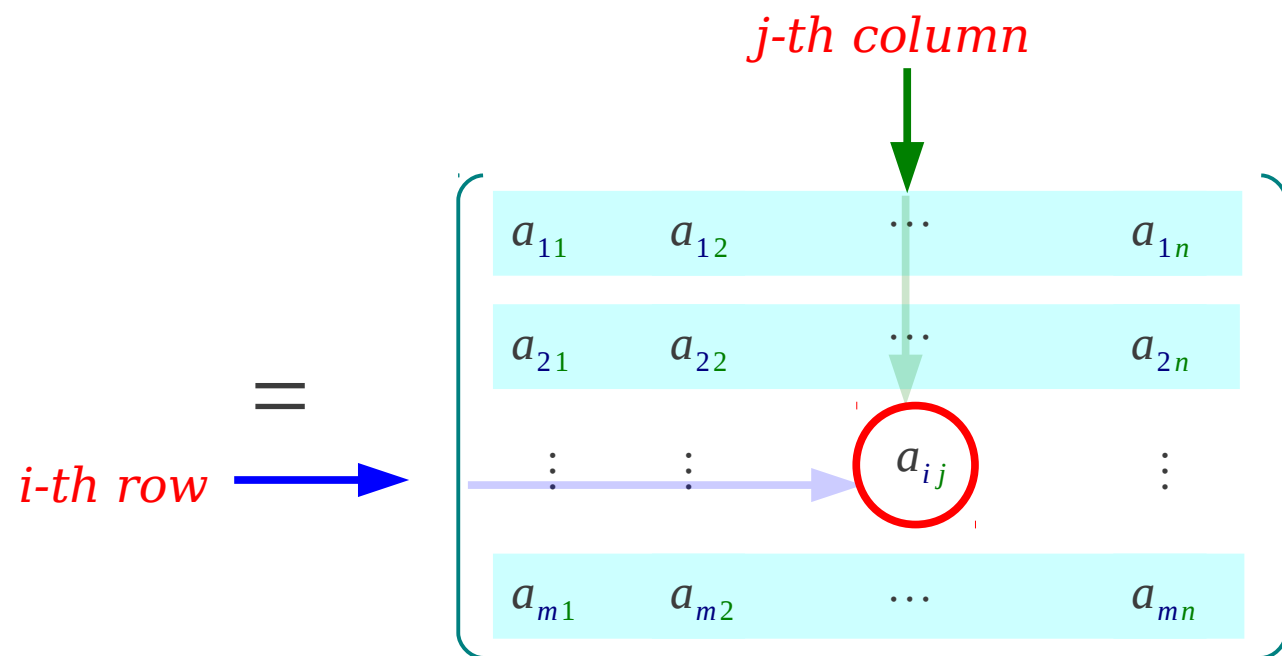
A General $m \times n$ Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A Element of a Matrix

$$A = [a_{ij}]_{m \times n} = [a_{ij}]$$



$$(A)_{m \times n} = a_{ij}$$

Matrix Multiplication (1)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{x} = [x_{ij}]_{n \times 1}$$

$$\mathbf{b} = [b_{ij}]_{m \times 1}$$

$$\mathbf{A} \quad \mathbf{x} = \mathbf{b}$$

$$(m) \times n \quad n \times (1) \quad (m \times 1)$$

Matrix Multiplication (2)

1st row \rightarrow

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

\downarrow 1st column

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{B} = [b_{ij}]_{n \times 2}$$

$$\mathbf{C} = [c_{ij}]_{m \times 2}$$

2nd row \rightarrow

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

\downarrow 1st column

$$\mathbf{A} \quad \mathbf{B} = \mathbf{C}$$

$m \times n \quad n \times 2 \quad m \times 2$

Matrix Multiplication (3)

1st row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \\ \vdots \\ \vdots \\ b_{n1} \\ b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

2nd row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \\ \vdots \\ \vdots \\ b_{n1} \\ b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$A \quad B = C$$

$$m \times n \quad n \times 2 \quad m \times 2$$

Matrix Multiplication (4)

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

1st column

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

2nd column

$$A \quad B = C$$

$$m \times n \quad n \times 2 \quad m \times 2$$

Matrix Multiplication (4)

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

1st column

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

m-th row →

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

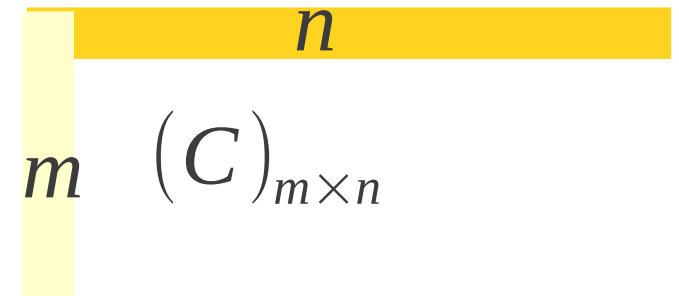
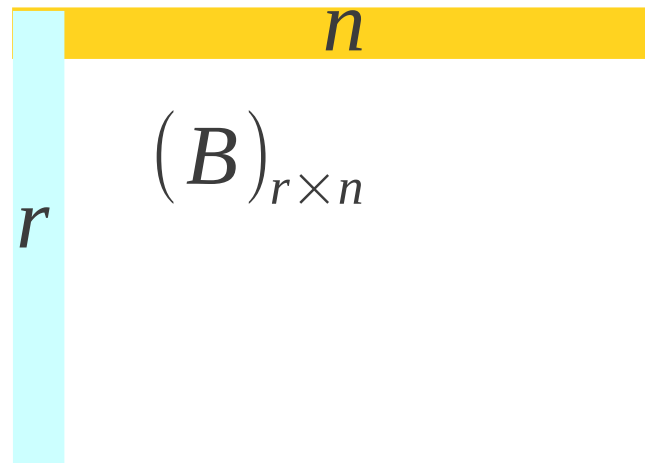
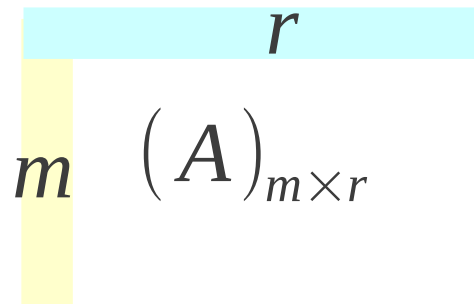
2nd column

$$A \quad B = C$$

$m \times n \quad n \times 2 \quad m \times 2$

Multiplication of Matrices (1)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ii}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$



Multiplication of Matrices (2)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \boxed{c_{ij}} & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$

$$(AB)_{ij} = c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} \cdots + a_{ir}b_{rj}$$

$$= \sum_{k=1}^r a_{ik}b_{kj}$$

Partitioned Matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ \vdots & \vdots \\ c_{m1} & c_{m2} \end{pmatrix}$$

$$A = [a_{ij}]_{m \times n}$$

$$B = [b_{ij}]_{n \times 2}$$

$$C = [c_{ij}]_{m \times 2}$$

$$\begin{matrix} B \\ n \times 2 \end{matrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{matrix} n \times 1 & n \times 1 \end{matrix}$$

$$\begin{matrix} AB \\ m \times 2 \end{matrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} \begin{matrix} m \times 1 & m \times 1 \end{matrix}$$

$$\begin{matrix} m \times n & n \times 2 & m \times n & n \times 1 & m \times n & n \times 1 \end{matrix}$$

Partitioned Matrix

$$\begin{array}{l}
 \mathbf{a}_1 \rightarrow \\
 \mathbf{a}_2 \rightarrow \\
 \mathbf{a}_m \rightarrow
 \end{array}
 \begin{pmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
 \end{pmatrix}
 \begin{pmatrix}
 b_{11} & b_{12} \\
 b_{21} & b_{22} \\
 \vdots & \vdots \\
 b_{n1} & b_{n2}
 \end{pmatrix}
 =
 \begin{pmatrix}
 c_{11} & c_{12} \\
 c_{21} & c_{22} \\
 \vdots & \vdots \\
 c_{m1} & c_{m2}
 \end{pmatrix}$$

$$\mathbf{A} = [a_{ij}]_{m \times n}$$

$$\mathbf{B} = [b_{ij}]_{n \times 2}$$

$$\mathbf{C} = [c_{ij}]_{m \times 2}$$

$$\begin{array}{l}
 \mathbf{A} \\
 m \times n
 \end{array}
 =
 \begin{bmatrix}
 \mathbf{a}_1 \\
 \mathbf{a}_2 \\
 \vdots \\
 \mathbf{a}_m
 \end{bmatrix}
 \begin{array}{l}
 1 \times n \\
 1 \times n \\
 \\
 1 \times n
 \end{array}
 \quad
 \begin{array}{l}
 \mathbf{A} \mathbf{B} \\
 m \times 2 \\
 m \times n \quad n \times 2
 \end{array}
 =
 \begin{bmatrix}
 \mathbf{a}_1 \\
 \mathbf{a}_2 \\
 \vdots \\
 \mathbf{a}_m
 \end{bmatrix}
 \begin{array}{l}
 1 \times 2 \quad 1 \times n \quad n \times 2 \\
 1 \times 2 \quad 1 \times n \quad n \times 2 \\
 \\
 1 \times 2 \quad 1 \times n \quad n \times 2
 \end{array}$$

Determinant

Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

Determinant

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

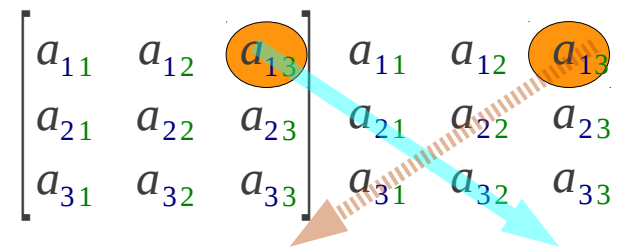
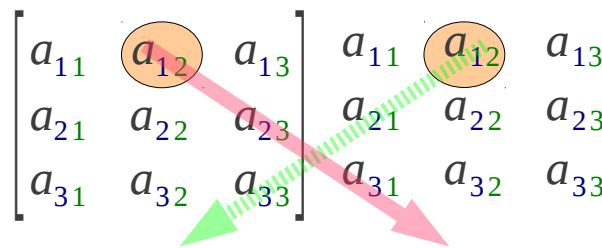
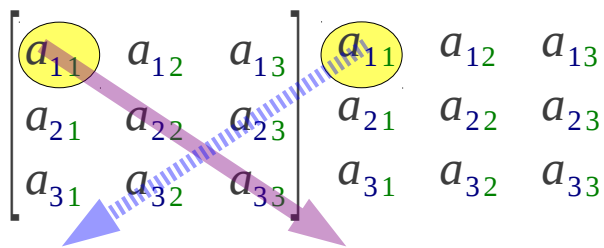
$$\begin{bmatrix} a_1 & & \\ & b_2 & b_3 \\ & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Determinant

Determinant of order 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
 = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$



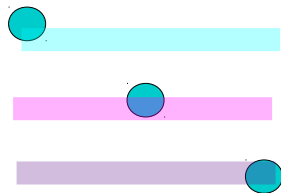
Matrix Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

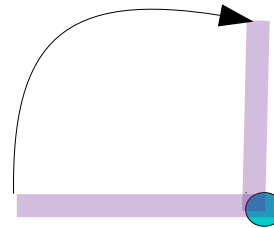
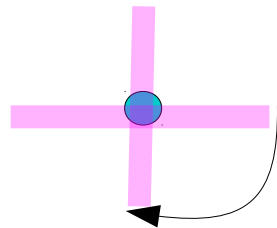
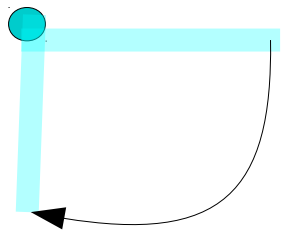
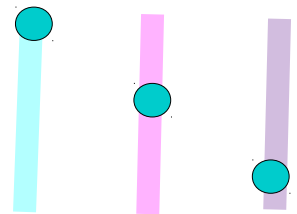
$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\det(A) = \det(A^T)$$

$[a_{ij}]$



$[a_{ji}]$



References

- [1] <http://en.wikipedia.org/>
- [2] Anton & Busby, "Contemporary Linear Algebra"
- [3] Anton & Rorres, "Elementary Linear Algebra"