1 Matrix : Guass-Jordan Elimination

1. Solving Simultaneous Equations
   Gauss-Jordan Elimination으로 다음 연립방정식 풀기

$$\begin{array}{rrrcl} 3x & +2y & -z & = & 4 \\ x & +3y & +2z & = & 7 \\ 2x & -y & +z & = & 5 \end{array}$$

   (a) 위의 연립 방정식을 $\mathbf{A}\ \mathbf{x} = \mathbf{b}$의 형태로 표현할 때 행렬 $\mathbf{A}$와 열벡터 $\mathbf{x}$와 $\mathbf{b}$를 구하시오.

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix}$$

   (b) 행렬 $\mathbf{A}$와 열벡터 $\mathbf{b}$를 사용한 augmented matrix $[\mathbf{A}|\mathbf{b}]$에 대하여 Gauss-Jordan 소거법을 적용하시오.

$$\left(\begin{array}{rrr|r} 3 & 2 & -1 & 4 \\ 1 & 3 & 2 & 7 \\ 2 & -1 & 1 & 5 \end{array}\right), \left(\begin{array}{rrr|r} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ 1 & 3 & 2 & 7 \\ 2 & -1 & 1 & 5 \end{array}\right),$$

$$\left(\begin{array}{rrr|r} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & \frac{7}{3} & \frac{7}{3} & \frac{17}{3} \\ 2 & -1 & 1 & 5 \end{array}\right), \left(\begin{array}{rrr|r} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & \frac{7}{3} & \frac{7}{3} & \frac{17}{3} \\ 0 & -\frac{7}{3} & \frac{5}{3} & \frac{7}{3} \end{array}\right),$$

$$\left(\begin{array}{rrr|r} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 1 & 1 & \frac{17}{7} \\ 0 & -\frac{7}{3} & \frac{5}{3} & \frac{7}{3} \end{array}\right), \left(\begin{array}{rrr|r} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 1 & 1 & \frac{17}{7} \\ 0 & 0 & 4 & 8 \end{array}\right),$$

$$\left(\begin{array}{rrr|r} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 1 & 1 & \frac{17}{7} \\ 0 & 0 & 1 & 2 \end{array}\right), \left(\begin{array}{rrr|r} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & 2 \end{array}\right),$$

$$\left(\begin{array}{ccc|c} 1 & \frac{2}{3} & 0 & 2 \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & 2 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{12}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & 2 \end{array}\right)$$

wxMaxima results

```
(%i1) l1 : [3, 2, -1, 4];
```

(%o1) $[3, 2, -1, 4]$

```
(%i2) l2 : [1, 3, 2, 7];
```

(%o2) $[1, 3, 2, 7]$

```
(%i3) l3 : [2, -1, 1, 5];
```

(%o3) $[2, -1, 1, 5]$

```
(%i4) l1 : l1 / 3;
```

(%o4) $[1, \frac{2}{3}, -\frac{1}{3}, \frac{4}{3}]$

```
(%i5) l2 : l2 - l1;
```

(%o5) $[0, \frac{7}{3}, \frac{7}{3}, \frac{17}{3}]$

```
(%i6) l3 : l3 - l1 * 2;
```

(%o6) $[0, -\frac{7}{3}, \frac{5}{3}, \frac{7}{3}]$

```
(%i7) l2 : l2 * 3 / 7;
```

(%o7) $[0, 1, 1, \frac{17}{7}]$

```
(%i8) l3 : l3 + l2 * 7/3;
```

(%o8) $[0, 0, 4, 8]$

```
(%i9) l3 : l3 /4;
```

$$(\%o9) \quad [0,0,1,2]$$

```
(%i10) l2 : l2 - l3;
```

$$(\%o10) \quad [0,1,0,\frac{3}{7}]$$

```
(%i11) l1 : l1 + l3 / 3;
```

$$(\%o11) \quad [1,\frac{2}{3},0,2]$$

```
(%i12) l1 : l1 - l2 *2/3;
```

$$(\%o12) \quad [1,0,0,\frac{12}{7}]$$

```
-->
(%i13) linsolve([3*x+2*y-z=4, x+3*y+2*z=7, 2*x-y+z=5], [x, y, z]);
```

$$(\%o13) \quad [x=\frac{12}{7},y=\frac{3}{7},z=2]$$

2. Finding an inverse matrix
   Gauss-Jordan Elimination으로 역행렬 구하기

   (a) 행렬 **A**와 단위 행렬 **I**를 사용한 augmented matrix [**A**|**I**]에 대하여 Gauss-Jordan 소거법을 적용하시오.

$$\left(\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{7}{3} & \frac{7}{3} & -\frac{1}{3} & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{7}{3} & \frac{7}{3} & -\frac{1}{3} & 1 & 0 \\ 0 & -\frac{7}{3} & \frac{5}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{7} & \frac{3}{7} & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 4 & -1 & 1 & 1 \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right), \left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{28} & \frac{5}{28} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right),$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 0 & \frac{1}{4} & \frac{1}{12} & \frac{1}{12} \\ 0 & 1 & 0 & \frac{3}{28} & \frac{5}{28} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right), \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{28} & -\frac{1}{28} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{3}{28} & \frac{5}{28} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right)$$

wxMaxima results

```
(%i14) l1 : [3, 2, -1, 1, 0, 0];
```

$$(\%o14) \quad [3, 2, -1, 1, 0, 0]$$

```
(%i15) l2 : [1, 3, 2, 0, 1, 0];
```

(%o15) $[1,3,2,0,1,0]$

```
(%i16) l3 : [2, -1, 1, 0, 0, 1];
```

(%o16) $[2,-1,1,0,0,1]$

```
(%i17) l1 : l1 / 3;
```

(%o17) $[1,\frac{2}{3},-\frac{1}{3},\frac{1}{3},0,0]$

```
(%i18) l2 : l2 - l1;
```

(%o18) $[0,\frac{7}{3},\frac{7}{3},-\frac{1}{3},1,0]$

```
(%i19) l3 : l3 - l1 * 2;
```

(%o19) $[0,-\frac{7}{3},\frac{5}{3},-\frac{2}{3},0,1]$

```
(%i20) l2 : l2 * 3 / 7;
```

(%o20) $[0,1,1,-\frac{1}{7},\frac{3}{7},0]$

```
(%i21) l3 : l3 + l2 * 7/3;
```

(%o21) $[0,0,4,-1,1,1]$

```
(%i22) l3 : l3 /4;
```

(%o22) $[0,0,1,-\frac{1}{4},\frac{1}{4},\frac{1}{4}]$

```
(%i23) l2 : l2 - l3;
```

(%o23) $[0,1,0,\frac{3}{28},\frac{5}{28},-\frac{1}{4}]$

```
(%i24) l1 : l1 + l3 / 3;
```

(%o24) $[1,\frac{2}{3},0,\frac{1}{4},\frac{1}{12},\frac{1}{12}]$

```
(%i25) l1 : l1 - l2 *2/3;
```

$$(\%o25) \quad [1,0,0,\frac{5}{28},-\frac{1}{28},\frac{1}{4}]$$

```
(%i28) invert(matrix(
        [3,2,-1],
        [1,3,2],
        [2,-1,1]
       ));
```

$$(\%o28) \quad \begin{pmatrix} \frac{5}{28} & -\frac{1}{28} & \frac{1}{4} \\ \frac{3}{28} & \frac{5}{28} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

(b) 위의 결과로부터 역행렬 $\mathbf{A}^{-1}$을 구하고 $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ 임을 보이시오.

$[\mathbf{A}|\mathbf{I}] \longrightarrow [\mathbf{I}|\mathbf{A}^{-1}]$

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{5}{28} & -\frac{1}{28} & \frac{1}{4} \\ \frac{3}{28} & \frac{5}{28} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$\begin{pmatrix} \frac{5}{28} & -\frac{1}{28} & \frac{1}{4} \\ \frac{3}{28} & \frac{5}{28} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 3 & 2 & -1 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{28} & -\frac{1}{28} & \frac{1}{4} \\ \frac{3}{28} & \frac{5}{28} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 Relation

1. Types of Relations
   집합 $\{1, 2, 3, 4\}$에서 $x + y <= 4$인 관계를 R로 정의한다. 즉 $(x, y) \in R$

$$\begin{pmatrix} (1,1) & (1,2) & (1,3) & \\ (2,1) & (2,2) & & \\ (3,1) & & & \\ & & & \end{pmatrix}$$

   (a) R은 반사적인가?
   Not a reflexive relation ($(3, 3)$ and $(4, 4)$ are missing)

   (b) R은 대칭적인가?
   a symmetric relation

   (c) R은 반대칭적인가?
   Not an anti-symmetric relation ($(3, 1)$ and $(1, 3)$: a counter example)

   (d) R을 4×4 행렬 $\mathbf{A}$로 나타내시오.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   (e) $\mathbf{A}^2$, $\mathbf{A}^3$, $\mathbf{A}^4$를 구하시오.

$$\mathbf{A}^2 = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} 6 & 5 & 3 & 0 \\ 5 & 4 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}^4 = \begin{pmatrix} 14 & 11 & 6 & 0 \\ 11 & 9 & 5 & 0 \\ 6 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

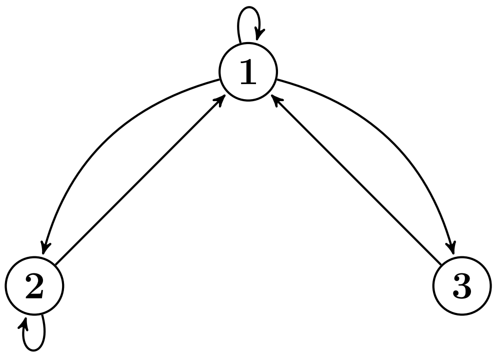
   after replacing all the nonzero elements with 1

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}^4 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(f) 위의 결과로서 R은 추이적인가?

Not a transitive relation ($\mathbf{A} \neq \mathbf{A} \vee \mathbf{A}^2 \vee \mathbf{A}^3 \vee \mathbf{A}^4$)

(g) R을 방향 그래프로 그리시오.



2. Closure
$X = \{1, 2, 3\}$ 위의 관계 $R = \{(1, 1), (1, 2), (2, 3)\}$에 관한 문제이다.

(a) R의 reflexive closure를 구하시오.

{ (1,1), (1,2), (2,3), (2,2), (3,3) }

(b) R의 symmetric closure를 구하시오.

{ (1,1), (1,2), (2,3), (2,1), (3,2) }

(c) R의 transitive closure를 구하시오.

{ (1,1), (1,2), (2,3), (1,3) }

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3 Algorithms

1. $\Theta(g(n))$을 각각 구하시오.

   (a) $f(n) = 6n^3 + 12n^2 + 1$

   $f(n) = \Theta(n^3)$

   (b) $f(n) = 3n^2 + 2n \log n$

   $f(n) = \Theta(n^2)$

   (c) $f(n) = 2 + 4 + 6 + \cdots + 2n$

   $f(n) = \Theta(n^2)$

   (d) $f(n) = (6n + 4)(1 + \log n)$

   $f(n) = \Theta(n \log n)$