DTFS (3B)

- DTFS
-

Young Won Lim
8/20/16

Copyright (c) 2013 - 2016 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

CTFS with Complex Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2}(a_k - j b_k)$$

$$B_k = \frac{1}{2}(a_k + j b_k)$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t}\, dt$$

$$k = 0, 1, 2, \ldots$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk\omega_0 t}\, dt$$

$$k = 1, 2, \ldots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t}\, dt$$

$$k = -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} A_0 = a_0 & (k = 0) \\ A_k = \frac{1}{2}(a_k - j b_k) & (k > 0) \\ B_k = \frac{1}{2}(a_k + j b_k) & (k < 0) \end{cases}$$

CTFS and DTFS

Continuous Time $x(t)$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$0 \le t \le T$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

CTFS

Discrete Time $x[n]$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$n = 0,\ 1,\ 2,\ \ldots,\ N-1,$$

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$k = -M,\ \ldots,\ 0,\ \ldots,\ +M$$

DTFS

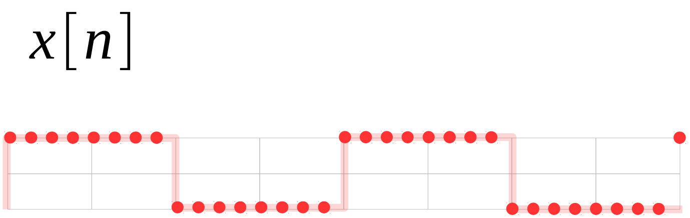
$$\sum_{k=-M}^{+M} = \sum_{k=0}^{N-1} = \sum_{k=k_0}^{k_0+N-1} = \sum_{k=\langle N\rangle}$$

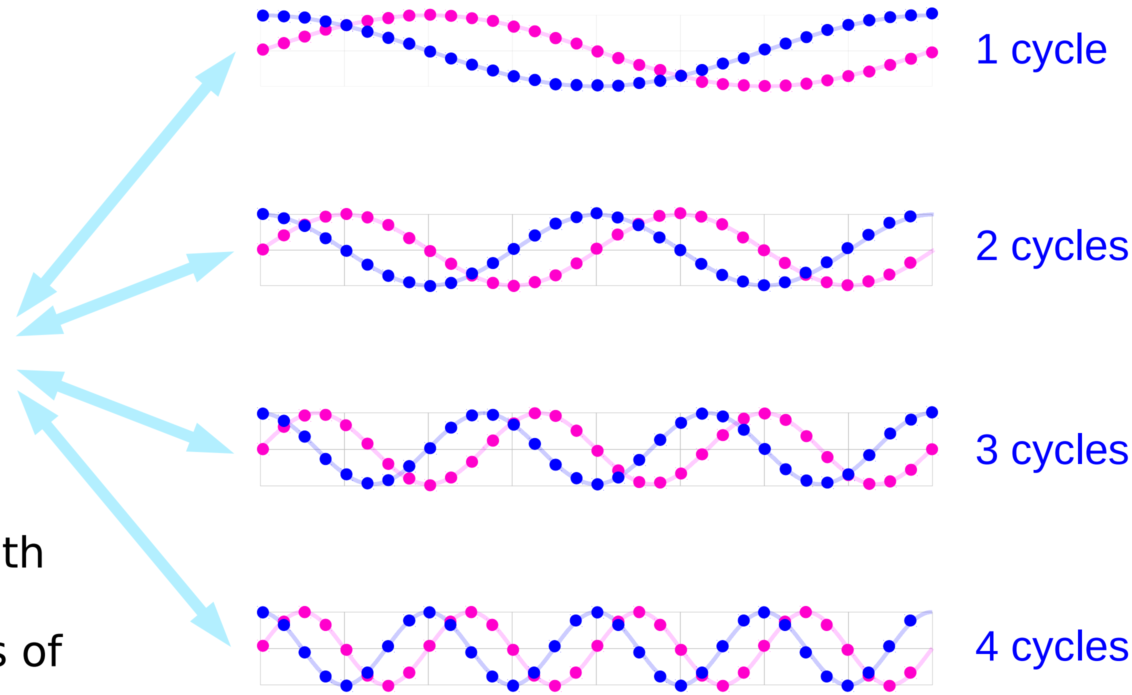
$$N = 2M + 1$$

C_k **Infinite set of *k*'s**

γ_k **Finite set of *k*'s**

DTFS Correlation Process





Measure the degree of correlation with these cosine and sine waves whose frequencies are the integer multiples of the fundamental frequency

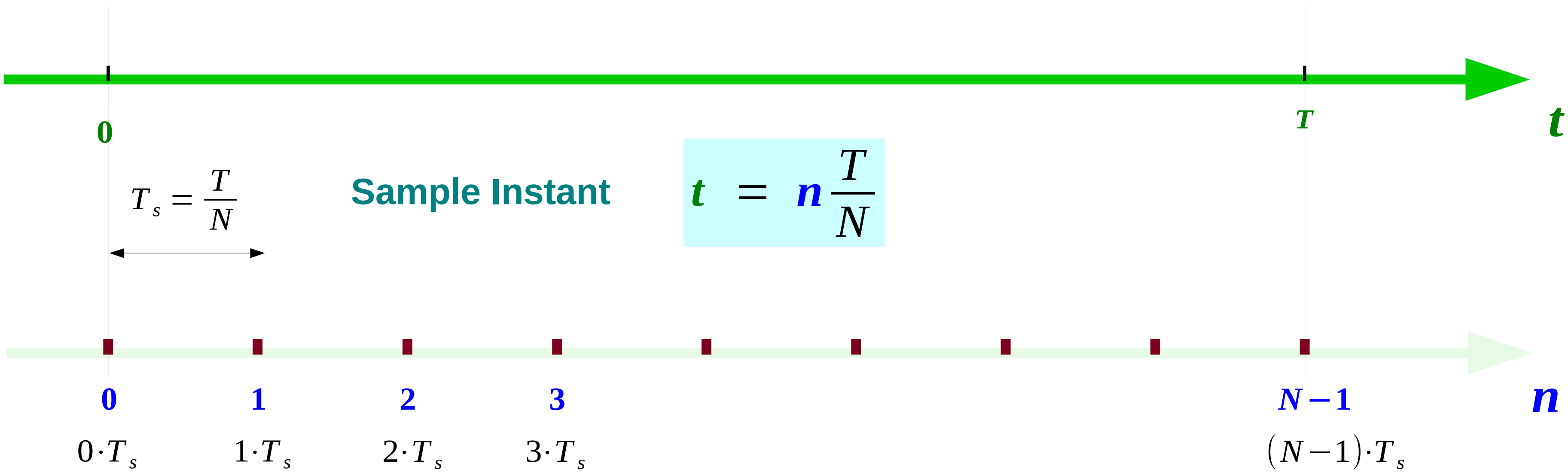
$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \Longleftrightarrow \quad x[n] = \sum_{k=-M}^{+M} \gamma_k\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

N Sampled Instants

$$k\,\omega_0\,t = \left(\frac{2\pi}{T}\right) k\,t$$

$$e^{+j\left(\frac{2\pi}{T}\right)k\,t}$$

$$\left(\frac{2\pi}{T}\right)$$



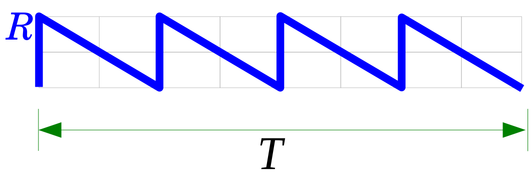
$$k\left(\frac{2\pi}{T}\right) n\left(\frac{T}{N}\right) = \left(\frac{2\pi}{N}\right) k\,n$$

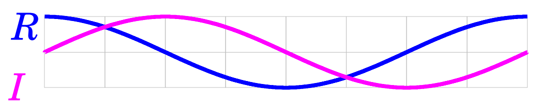
$$e^{+j\left(\frac{2\pi}{N}\right)k\,n}$$

$$\left(\frac{2\pi}{N}\right)$$

Two Types of Inner Product

$x(t)$ | $e^{+jk\omega_0 t}$ | $\frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt = C_k$





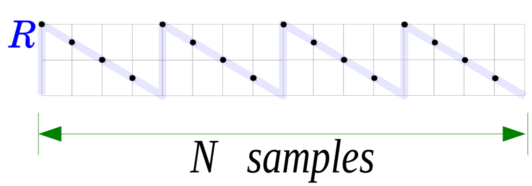
$$f_0 = 1/T$$

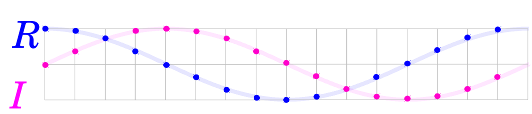
$$\omega_0 = 2\pi/T$$

$$\frac{1}{T}\langle x(t),\ e^{+j(1)\omega_0 t}\rangle = C_1$$

$$\langle e^{+j(1)\omega_0 t},\ e^{+j(1)\omega_0 t}\rangle = T$$

$x[n]$ | $e^{+j\left(\frac{2\pi}{N}\right)nk}$ | $\frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)nk} = \gamma_k$





$$\frac{1}{N}\langle x[n],\ e^{+j\left(\frac{2\pi}{N}\right)n(1)}\rangle = \gamma_1$$

$$\langle e^{+j\left(\frac{2\pi}{N}\right)n(1)},\ e^{+j\left(\frac{2\pi}{N}\right)n(1)}\rangle = N$$

CTFS and DTFS Inner Product Representations

Continuous Time **CTFS**

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$0 \le t \le T$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$\frac{\langle x(t)\ ,\ e^{+j(2\pi/T)kt} \rangle}{\langle e^{+j(2\pi/T)kt},\ e^{+j(2\pi/T)kt} \rangle} = C_k$$

$x(t)$ $\longleftrightarrow$ **invertible** C_k

Discrete Time **DTFS**

$$x[n] = \sum_{k=-M}^{+M} \gamma_k\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$n = 0,\ 1,\ 2,\ \ldots,\ N-1,$$

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$k = -M,\ \ldots,\ 0,\ \ldots,\ +M$$

$$\frac{\langle x(t)\ ,\ e^{+j(2\pi/N)kn} \rangle}{\langle e^{+j(2\pi/N)kn},\ e^{+j(2\pi/N)kn} \rangle} = \gamma_k$$

$x[n]$ $\longleftrightarrow$ **invertible** γ_k

Truncate CTFS Coefficients

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$0 \le t \le T$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

Infinite set of k's

CTFS

$$x(t) \approx \sum_{k=-M}^{+M} C_k \, e^{+jk\omega_0 t} \qquad N = 2M + 1$$

synthesis with truncated coefficients

Use a finite subset of N coefficients

Finite set of k's

Approximated Coefficients

CTFS DTFS DFT

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)k\,n}$$

$$n = 0,\ 1,\ 2,\ \ldots\ ,\ N-1,$$

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \, e^{-j\left(\frac{2\pi}{N}\right)k\,n}$$

$$k = -M,\ \ldots\ ,\ 0,\ \ldots\ ,\ +M$$

DTFS

Approximated Synthesis

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k \, e^{+jk\omega_0 t}$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$n = 0, 1, 2, \ldots, N-1,$$

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$k = -M, \ldots, 0, \ldots, +M$$

DTFS

Approximated CTFS and DTFS Synthesis

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+j\left(\frac{2\pi}{T}\right)k t}$$

$$0 \le t \le T$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)k n}$$

$$n = 0,\ 1,\ 2,\ \dots,\ N-1,$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-j\left(\frac{2\pi}{T}\right)k t} \, dt$$

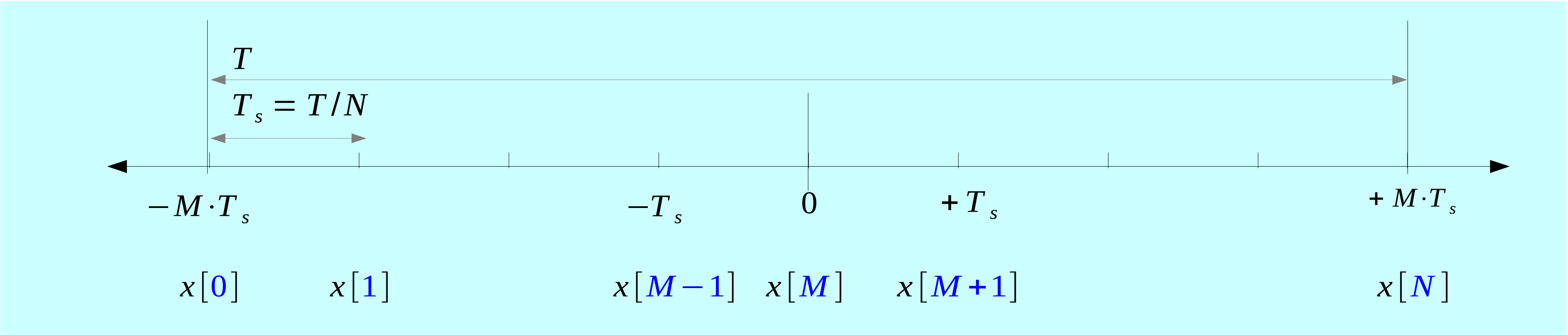
$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \dots$$

$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)k n}$$

$$k = -M,\ \dots,\ 0,\ \dots,\ +M$$

$$x(t) \approx \sum_{k=-M}^{+M} C_k \, e^{+jk\omega_0 t} \qquad N = 2M+1$$

$$x(t) \approx x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k \, e^{+jk\omega_0 t}$$



CTFS, DTFS, and DFT

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$0 \le t \le T$$

$$C_k = \frac{1}{T} \int_0^T x(t) \, e^{-jk\omega_0 t} \, dt$$

$$k = -2, -1, 0, +1, +2, \ldots$$

CTFS

$$x[n] = \sum_{k=0}^{N} \gamma_k \, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$n = 0, 1, 2, \ldots, N-1$$

$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$k = 0, 1, 2, \ldots, N-1$$

DTFS

$$x(t) \approx \sum_{k=-M}^{+M} \gamma_k \, e^{+jk\omega_0 t}$$

Approximated Synthesis

$$0 \le t \le T$$

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$

Approximated Coefficients

$$k = -2, -1, 0, +1, +2, \ldots$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$n = 0, 1, 2, \ldots, N-1$$

$$X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

$$k = 0, 1, 2, \ldots, N-1$$

DFT

DTFS and DFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

Discrete Time Fourier Series

DTFS

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$DTFS(x[n]) = \frac{1}{N} DFT(x[n])$$

Fourier Analysis Types

Continuous Time Fourier Series CTFS

$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

Continuous Time Fourier Transform CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier Transform DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}$$

Discrete Fourier Transform DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

CTFT and DFT

Continuous Time Fourier <u>Transform</u> CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

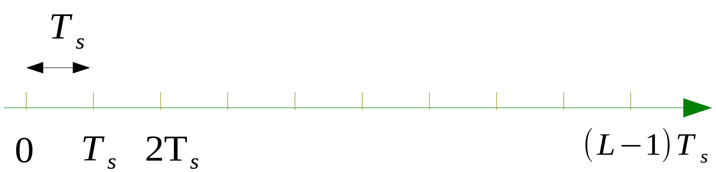
Discrete Fourier <u>Transform</u> DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Time Samples

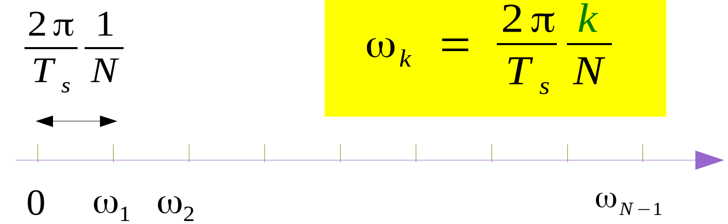


$$t \rightarrow nT_s \qquad dt \rightarrow T_s \qquad \int \rightarrow \sum$$

$$0 \le n < L \qquad T_s \rightarrow 0$$

Frequency Samples

$$\omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$



$$\omega \rightarrow \omega_k \qquad d\omega \rightarrow \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \rightarrow \sum$$

$$0 \le k < N \qquad 0 \le \omega_k < \frac{2\pi}{T_s}$$

Young Won Lim
8/20/16

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t} d\omega$$

$$t \to nT_s \qquad dt \to T_s \qquad \int \to \sum \qquad 0 \le n < L$$

$$\hat{X}(j\omega) = \sum_{n=-\infty}^{+\infty} x(nT_s)\, e^{-j\omega n T_s} \cdot T_s \quad \Leftrightarrow \quad x(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{X}(j\omega)\, e^{+j\omega n T_s} d\omega$$

$$\omega \to \omega_k \qquad d\omega \to \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \to \sum \qquad 0 \le k < N$$

$$\omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n]\, e^{-j\omega_k n T_s} \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{X}(j\omega_k)\, e^{+j\omega_k n T_s} \, \frac{2\pi}{T_s}\frac{1}{N}$$

CTFT → DFT (3)

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Time Samples

$$t \to nT_s \qquad dt \to T_s \qquad \int \to \sum$$

Frequency Samples

$$\omega \to \omega_k \qquad d\omega \to \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \to \sum$$

$$\hat{X}(j\omega_k) = T_s \sum_{n=0}^{L-1} x[n]\, e^{-j\omega_k n T_s} \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi}\sum_{k=0}^{N-1} \hat{X}(j\omega_k)\, e^{+j\omega_k n T_s}\, \frac{2\pi}{T_s}\frac{1}{N}$$

$$\omega_k T_s \to \frac{2\pi}{N}k \quad \Rightarrow \quad \omega_k n T_s \to \frac{2\pi}{N}kn$$

$$\omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$

$$\frac{1}{T_s}\hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1}\frac{1}{T_s}\hat{X}(j\omega_k)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t} d\omega$$

Time Samples

$$t \rightarrow nT_s \qquad dt \rightarrow T_s \qquad \int \rightarrow \sum$$

Frequency Samples

$$\omega \rightarrow \omega_k \qquad d\omega \rightarrow \frac{2\pi}{T_s}\frac{1}{N} \qquad \int \rightarrow \sum$$

$$\omega_k T_s \rightarrow \frac{2\pi}{N} k \quad \Rightarrow \quad \omega_k n T_s \rightarrow \frac{2\pi}{N} k n \qquad \omega_k = \frac{2\pi}{T_s}\frac{k}{N}$$

$$\frac{1}{T_s}\hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{T_s}\hat{X}(j\omega_k)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$\frac{1}{T_s}\hat{X}(j\omega_k) = \sum_{n=0}^{L-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{T_s}\hat{X}(j\omega_k)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

Discrete Fourier Transform

$$L = N$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

CTFT of a Sampled Signal

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x_s(t) = x_c(t)\sum_{n=-\infty}^{+\infty}\delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

CTFS ||

DTFT ||

$$x_s(t) = x_c(t)\sum_{k=-\infty}^{+\infty}\frac{1}{T_s}\, e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

Continuous Time Fourier Transform CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier Transform DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$=$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$$=$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$X(e^{j\hat{\omega}}) \qquad \hat{\omega} = \omega T_s$$

$$= \quad X(e^{j\omega T_s}) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c\left(j\left(\omega - \frac{2\pi k}{T_s}\right)\right)$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

DTFT of a sampled signal

DTFT and DFT

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

$$\hat{\omega} = \omega T_s$$

Frequency Samples

$\hat{\omega} \rightarrow \hat{\omega}_k \qquad 0 \le \hat{\omega}_k < 2\pi \qquad 0 \le k < N \qquad 0 \le n < L$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = \left(\frac{2\pi}{N}\right) k$$

DFT of a sampled signal

$$X[k] = X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j(2\pi/N)kn}$$

DTFT *sampled in frequency*

CTFT and DFT

DFT of a sampled signal

$$X[k] = X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j(2\pi/N)kn}$$

DTFT *sampled in frequency*

$$X(e^{j\omega T_s})\Big|_{\omega = \frac{2\pi k}{N T_s}}$$

CTFT evaluated at $\omega = \frac{2\pi k}{N T_s}$

$$= \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\omega_s))\Big|_{\omega = \frac{2\pi k}{N T_s}}$$

$$= \frac{1}{T_s} \sum_{l=-\infty}^{+\infty} X_c(j(\omega - l\frac{2\pi}{T_s}))\Big|_{\omega = \frac{2\pi k}{N T_s}}$$

From DTFT to DFT (1)

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega}n}$$

$$\hat{\omega} = \omega T_s$$

Frequency Samples

$$\hat{\omega} \rightarrow \hat{\omega}_k \qquad 0 \le \hat{\omega}_k < 2\pi \qquad 0 \le k < N \qquad 0 \le n < L$$

$$X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = \left(\frac{2\pi}{N}\right) k$$

DFT of a sampled signal

$$X[k] = \quad X(e^{j(2\pi/N)k}) = \sum_{n=0}^{L-1} x[n]\, e^{-j(2\pi/N)kn}$$

DTFT *sampled in frequency*

From FT to DFT (1)

$f(t)$

$x(k\tau) = f(t)\delta(t-k\tau)w(t)$

sampling time τ

window function $0 \le t \le T$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform

discrete time $t \Rightarrow k\tau$

discrete freq $\omega \Rightarrow \omega_k = 2\pi f_k$

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n\tau)\, e^{-j2\pi f_k n\tau}$$

$$e^{-j2\pi f_k n\tau} = e^{-j2\pi k\left(\frac{1}{N\tau}\right)n\tau}$$

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n\tau)\, e^{-j\frac{2\pi}{N}kn}$$

fundamental frequency

$$f_0 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$$

k-th harmonics

$$f_k = k\cdot\frac{1}{T} = k\cdot\frac{1}{N\tau} = k\cdot\frac{f_s}{N}$$

From FT to DFT (2)

$$f(t)$$

$$x(k\tau) = f(t)\delta(t - k\tau)w(t)$$

sampling time τ

window function $0 \le t \le T$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform

discrete time $t \Rightarrow k\tau$

discrete freq $\omega \Rightarrow \omega_k = 2\pi f_k$

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n\tau)\, e^{-j\frac{2\pi}{N}kn}$$

$$f_k = k \cdot \frac{f_s}{N}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

fundamental frequency

$$f_0 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$$

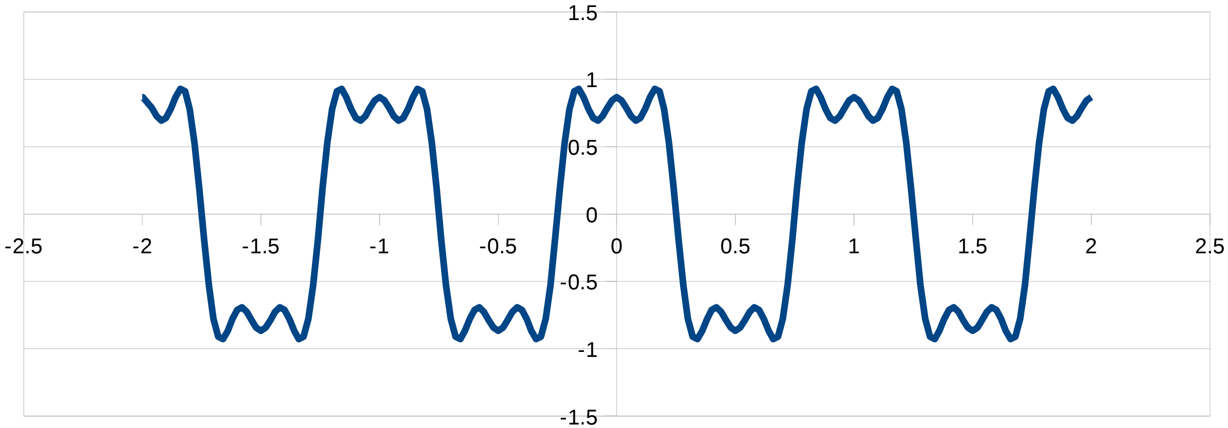
k-th harmonics

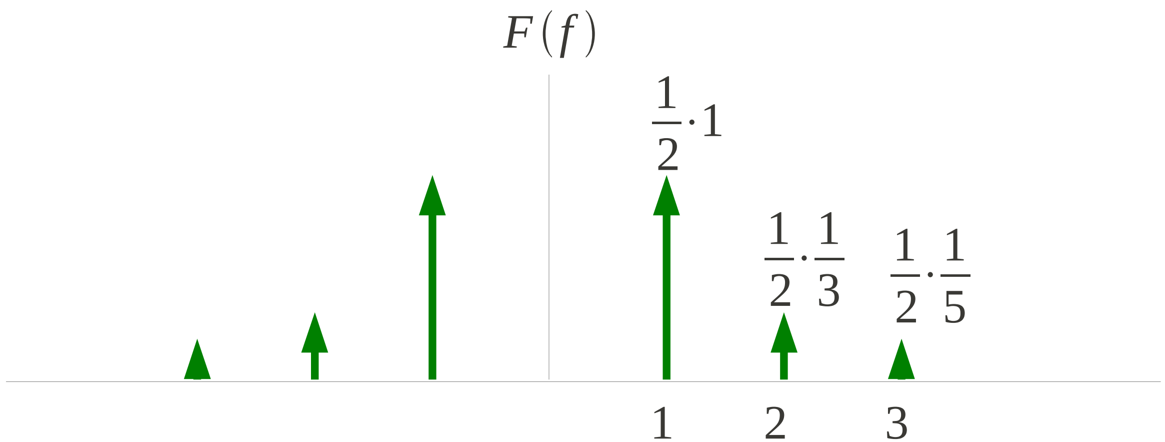
$$f_k = k \cdot \frac{1}{T} = k \cdot \frac{1}{N\tau} = k \cdot \frac{f_s}{N}$$

Discrete Fourier Transform

FT → DFT Example (1)

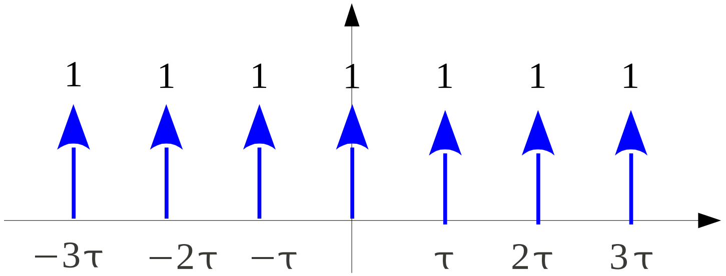
$$f(t) = \cos(2\pi\cdot 1\cdot t) - \frac{1}{3}\cos(2\pi\cdot 3\cdot t) + \frac{1}{5}\cos(2\pi\cdot 5\cdot t)$$





FT → DFT Example (2)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$



Fourier Series Expansion of Impulse Train

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{\tau}$$

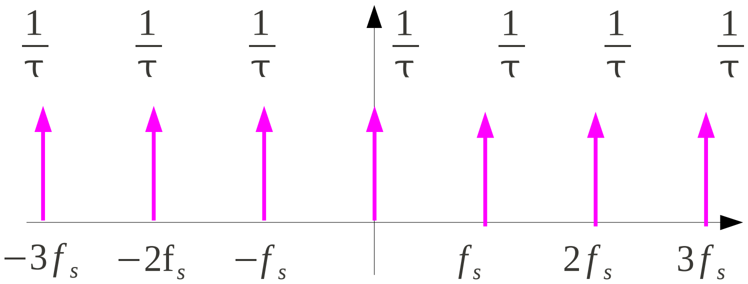
$$f_s = \frac{1}{\tau}$$

Fourier Series Coefficients

$$a_k = \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2} \delta(t)\, e^{-jk\omega_s t}\, dt$$

$$= \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2} \delta(t)\, e^{-jk\omega_s 0}\, dt$$

$$= \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2} \delta(t)\, dt = \frac{1}{T_s}$$



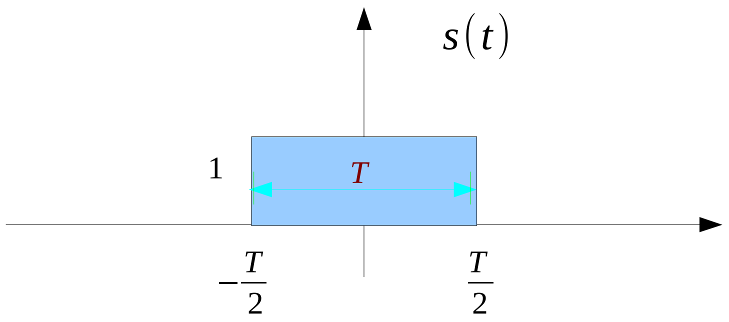
FT → DFT Example (3)

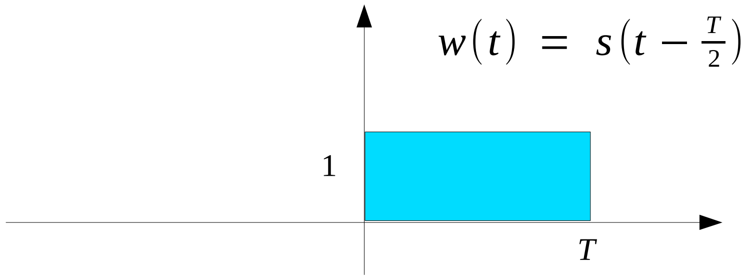
$$S(\omega) = \int_0^T e^{-j\omega t}\,dt = \left[\frac{e^{-j\omega t}}{-j\omega} \right]_0^T$$

$$= \frac{e^{-j\omega T} - 1}{-j\omega}$$

$$= \frac{e^{-j\omega T/2}\left(e^{-j\omega T/2} - e^{+j\omega T/2}\right)}{-j\omega}$$

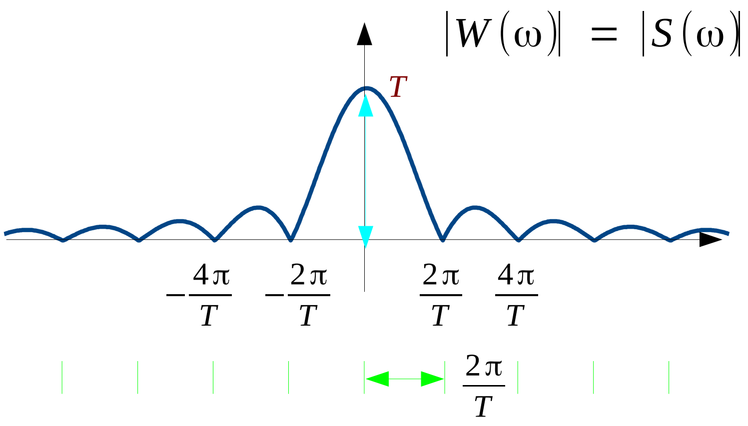
$$= \frac{\sin(\omega T/2)}{\omega T/2} \cdot e^{-j\omega T/2}$$



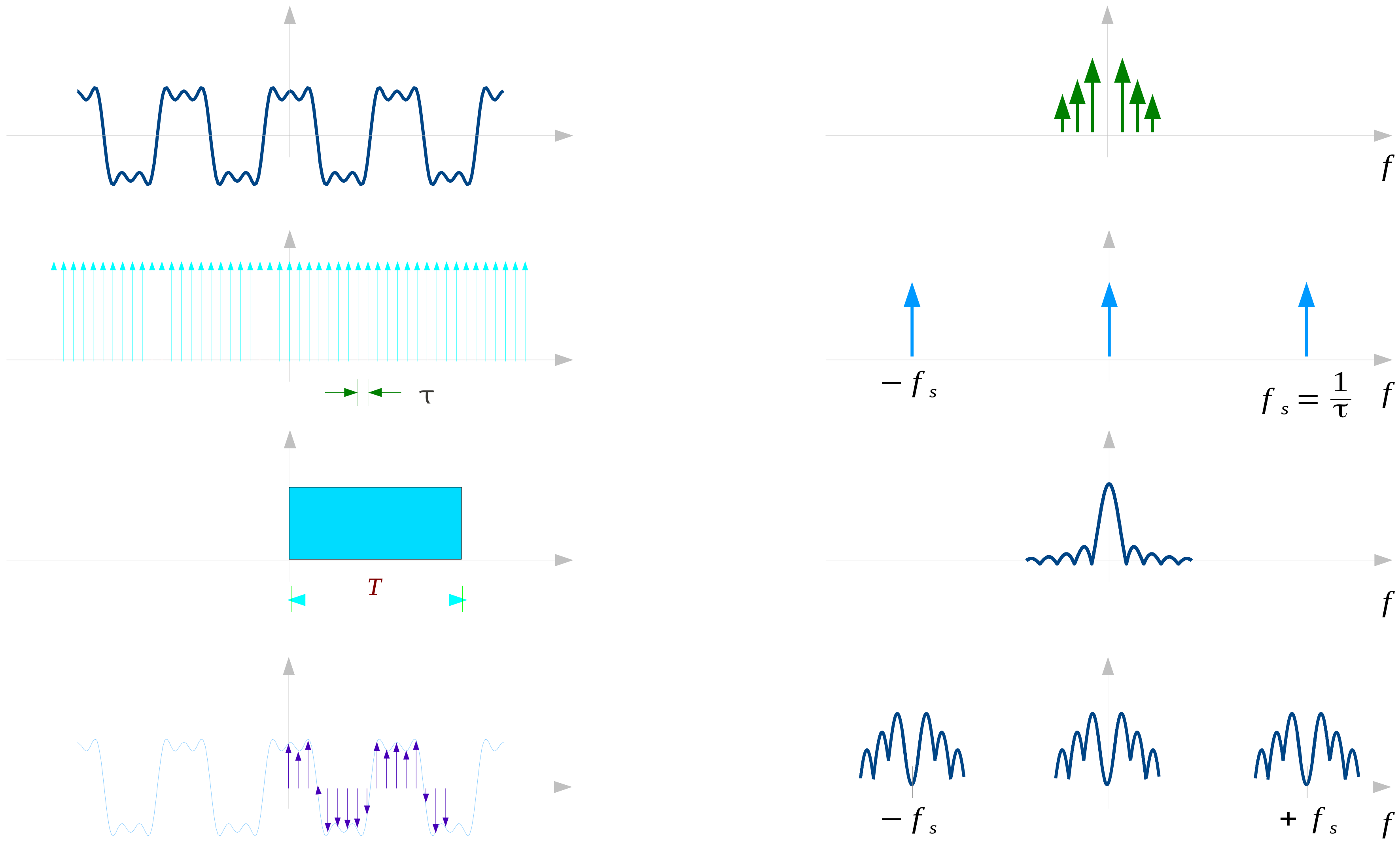


$$W(\omega) = e^{-j\omega T/2} \cdot S(\omega)$$

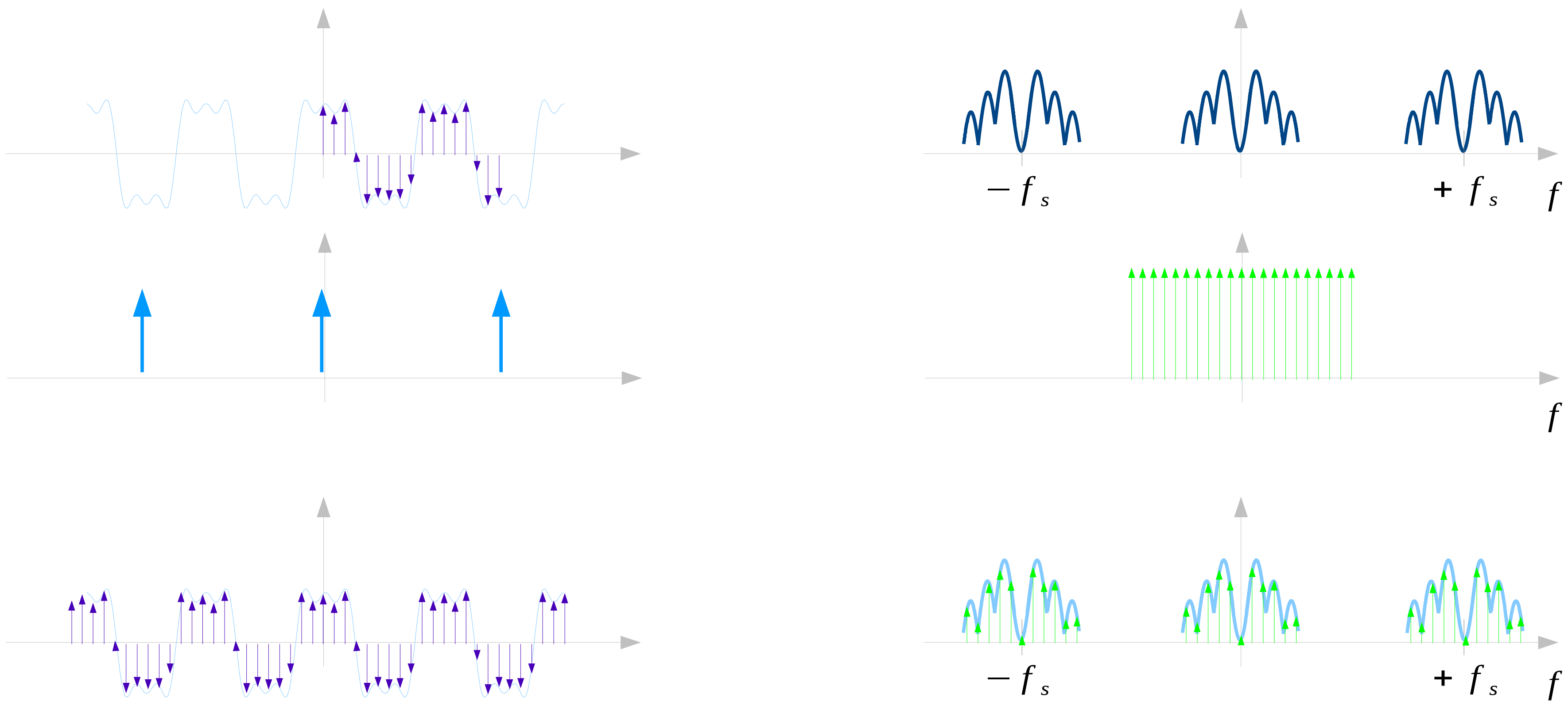
$$= \frac{\sin(\omega T/2)}{\omega T/2} \cdot e^{-j\omega T}$$



From DTFT to DFT (4)



From DTFT to DFT (5)



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] M.J. Roberts, Fundamentals of Signals and Systems
[4] S.J. Orfanidis, Introduction to Signal Processing
[5] K. Shin, et al., Fundamentals of Signal Processing for Sound and Vibration Engineerings