# CTFS (1B)

- Continuous Time Fourier Series

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# Fourier Series

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx \, dx$$

$k = 1, 2, 3, \ldots$ → **one-sided spectrum**

only positive frequencies

# Trigonometric Identities

$$\cos\theta \cos\phi = \frac{1}{2}\left(\cos(\theta-\phi) + \cos(\theta+\phi)\right)$$

$$\sin\theta \sin\phi = \frac{1}{2}\left(\cos(\theta-\phi) - \cos(\theta+\phi)\right)$$

$$\sin\theta \cos\phi = \frac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta-\phi)\right)$$

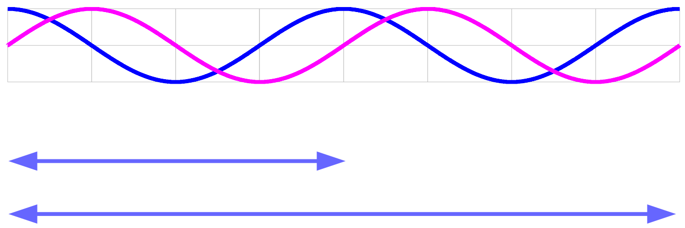
$$\cos\theta \sin\phi = \frac{1}{2}\left(\sin(\theta+\phi) - \sin(\theta-\phi)\right)$$

$$\frac{1}{2}\left(1 + \cos(\theta+\phi)\right) \quad (\theta=\phi)$$

$$\frac{1}{2}\left(1 - \cos(\theta+\phi)\right) \quad (\theta=\phi)$$

$$\frac{1}{2}\left(\sin(\theta+\phi)\right) \quad (\theta=\phi)$$

$$\frac{1}{2}\left(\sin(\theta+\phi)\right) \quad (\theta=\phi)$$



Integration Area : zero
positive area = negative area

Assumed integration interval :
Integer multiples of a period

# Integration of the trigonometric identities

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \cos \boldsymbol{m}x \, dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \sin \boldsymbol{m}x \, dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \cos \boldsymbol{m}x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \sin \boldsymbol{m}x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \cos \boldsymbol{m}x \, dx = \pi \qquad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \sin \boldsymbol{m}x \, dx = \pi \qquad (n = m)$$

***n*, *m*** : integer

| | |
|---|---|
| $\cos((n-m)x)$ | $\pm\cos((n+m)x)$ |
| $\pm\cos((n-m)x)$ | $\sin((n+m)x)$ |

$$\int_{-\pi}^{+\pi} \square \, dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \frac{1}{2} \, dx = \pi \qquad (n = m)$$

# Correlation : zero or non-zero

$$\cos\theta \cos\phi = \frac{1}{2}\left(\cos(\theta - \phi) + \cos(\theta + \phi)\right)$$

$$\sin\theta \sin\phi = \frac{1}{2}\left(\cos(\theta - \phi) - \cos(\theta + \phi)\right)$$

$$\sin\theta \cos\phi = \frac{1}{2}\left(\sin(\theta + \phi) + \sin(\theta - \phi)\right)$$

$$\cos\theta \sin\phi = \frac{1}{2}\left(\sin(\theta + \phi) - \sin(\theta - \phi)\right)$$

$$\frac{1}{2}\left(1 + \cos(\theta + \phi)\right) \quad (\theta = \phi)$$

$$\frac{1}{2}\left(1 - \cos(\theta + \phi)\right) \quad (\theta = \phi)$$

$$\frac{1}{2}\left(\sin(\theta + \phi)\right) \quad (\theta = \phi)$$

$$\frac{1}{2}\left(\sin(\theta + \phi)\right) \quad (\theta = \phi)$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \cos \boldsymbol{m}x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \sin \boldsymbol{m}x \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \cos \boldsymbol{m}x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \sin \boldsymbol{m}x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \cos \boldsymbol{m}x \, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \sin \boldsymbol{m}x \, dx = \pi \quad (n = m)$$

***n*, *m*** : integer

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx$$

$$a_k \leftarrow f(x) \cdot \cos kx = a_0 \cdot \cos kx + \sum_{m=1}^{\infty} (a_m \cos mx \cdot \cos kx + b_m \sin mx \cdot \cos kx)$$

$$m = k$$

$$\int_{-\pi}^{+\pi} \quad dx = 0$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

# Fourier Series Coefficients $b_k$

$$f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx$$

$$b_k \leftarrow f(x) \cdot \sin kx = a_0 \cdot \sin kx + \sum_{m=1}^{\infty} (a_m \cos mx \cdot \sin kx + b_m \sin mx \cdot \sin kx)$$

$$m = k$$

$$\int_{-\pi}^{+\pi} \quad dx = 0$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

# Computing Fourier Coefficients

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n}x \cos \boldsymbol{m}x\, dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n}x \sin \boldsymbol{m}x\, dx = \pi \quad (n = m)$$

$\boldsymbol{n}, \boldsymbol{m}$ : integer

$$a_0 \leftarrow \quad f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m}x + b_m \sin \boldsymbol{m}x)$$

$$a_k \leftarrow \quad f(x) \cdot \cos \boldsymbol{k}x = a_0 \cdot \cos \boldsymbol{k}x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m}x \cdot \cos \boldsymbol{k}x + b_m \sin \boldsymbol{m}x \cdot \cos \boldsymbol{k}x)$$

$$b_k \leftarrow \quad f(x) \cdot \sin \boldsymbol{k}x = a_0 \cdot \sin \boldsymbol{k}x + \sum_{m=1}^{\infty} (a_m \cos \boldsymbol{m}x \cdot \sin \boldsymbol{n}x + b_m \sin \boldsymbol{m}x \cdot \sin \boldsymbol{k}x)$$

# Any Period $p = 2L$

$$g(v) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kv + b_k \sin kv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v)\, dv$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos kv\, dv$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin kv\, dv$$

$$k = 1,\ 2,\ \ldots$$

$$v:\ [-\pi,\ +\pi]$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L}\, dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L}\, dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x:\ [-L,\ +L]$$

$$v = \frac{\pi}{L} x$$

$$dv = \frac{\pi}{L} dx$$

$$f(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi}{L}x + b_k \sin\frac{k\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{+L} f(x)\,dx$$

$$a_k = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{k\pi x}{L}\,dx$$

$$b_k = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\frac{k\pi x}{L}\,dx$$

$$k = 1,\ 2,\ 3,\ \ldots$$

$$x : [-L,\ +L]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2\pi k}{T}t + b_k \sin\frac{2\pi k}{T}t\right)$$

$$a_0 = \frac{1}{T}\int_{0}^{T} x(t)\,dt$$

$$a_k = \frac{2}{T}\int_{0}^{T} x(t)\cos\frac{2\pi k t}{T}\,dt$$

$$b_k = \frac{2}{T}\int_{0}^{T} x(t)\sin\frac{2\pi k t}{T}\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$t : [0,\ T]$$

$$2L = T$$

*Continuous Time Periodic Signal* $x(t)$

# Harmonic Frequency

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos\frac{2\pi k}{T} t + b_k \sin\frac{2\pi k}{T} t \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi k t}{T}\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi k t}{T}\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(2\pi \boldsymbol{k} f_0 t) + b_k \sin(2\pi \boldsymbol{k} f_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(2\pi \boldsymbol{k} f_0 t)\, dt \qquad k = 1,\ 2,\ \ldots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(2\pi \boldsymbol{k} f_0 t)\, dt \qquad k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

resolution frequency $f_0 = \frac{1}{T}$

n-th harmonic frequency $f_n = \boldsymbol{n} f_0 = \boldsymbol{n} \frac{1}{T}$

# Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_k \cos(k 2\pi f_0 t) + b_k \sin(k 2\pi f_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k 2\pi f_0 t)\, dt \qquad k = 1,\ 2,\ \ldots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k 2\pi f_0 t)\, dt \qquad k = 1,\ 2,\ \ldots$$

$$t : [0,\ T]$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left( a_k \cos(\boldsymbol{k} \omega_0 t) + b_k \sin(\boldsymbol{k} \omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(\boldsymbol{k} \omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(\boldsymbol{k} \omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$t : [0,\ T]$$

linear frequency $f$

angular (radial) frequency $\omega = 2\pi f$

# Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\, dt$$

$$k = 1,\ 2,\ \ldots$$

$$t:\ [0,\ T]$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left( A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$t:\ [0,\ T]$$

**Real** coefficients

$$a_0,\ a_k,\ b_k,\quad k = 1,\ 2,\ \ldots$$

→ **one-sided spectrum**
only positive frequencies

**Complex** coefficients

$$A_0,\ A_k,\ B_k,\quad k = 1,\ 2,\ \ldots$$

→ **two-sided spectrum**
Both pos and neg frequencies

# Applying the Euler Formula

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$= \frac{a_k}{2} \left( e^{jk\omega_0 t} + e^{-jk\omega_0 t} \right) + \frac{b_k}{2j} \left( e^{jk\omega_0 t} - e^{-jk\omega_0 t} \right)$$

$$= \frac{(a_k - j b_k)}{2} e^{jk\omega_0 t} + \frac{(a_k + j b_k)}{2} e^{-jk\omega_0 t}$$

$$= A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}$$

$$e^{+j\omega t} = \cos\omega t + j \sin\omega t$$

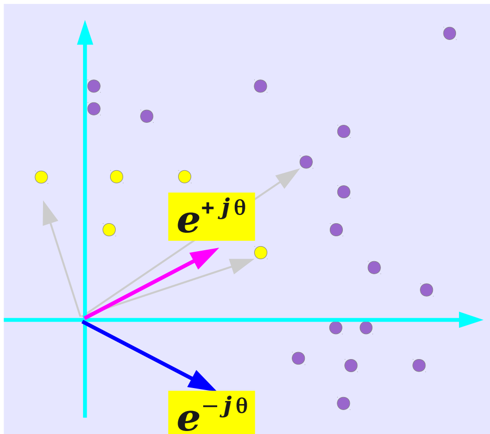
$$e^{-j\omega t} = \cos\omega t - j \sin\omega t$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left( A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

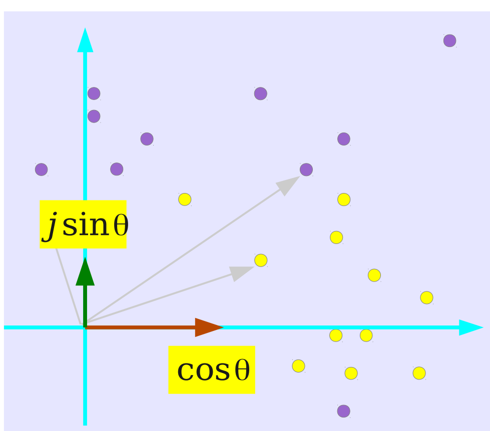
**Basis : a set of linear independent spanning vectors**




every complex number can be represented by

$$k_1 \, e^{+j\theta} + k_2 \, e^{+j\theta}$$

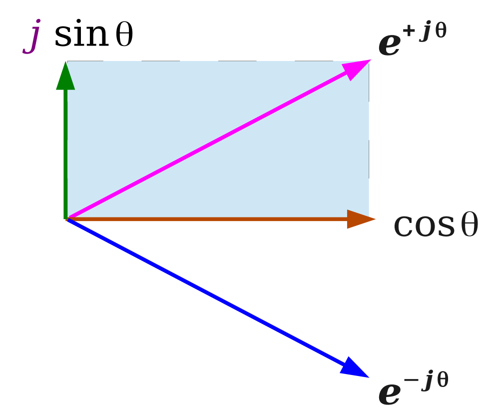
linear combination of $e^{+j\theta}$ and $e^{+j\theta}$
which are one set of linear independent
two vectors




every complex number can also be represented by

$$l_1 \cos\theta + l_2 \, j \sin\theta$$

$$l_1 \cos\theta + l_2 \; j\sin\theta$$



**Basis** : a set of linear independent spanning vectors

$e^{+j\theta}$ $e^{+j\theta}$

$l_1$ $l_2 j$

$\cos\theta$ $j\sin\theta$

every complex number can be represented by

$$k_1 e^{+j\theta} + k_2 e^{+j\theta}$$

$C^1$ over **R**

$k_1, k_2 \in \mathbf{R}$

every complex number can also be represented by

$$l_1 \cos\theta + l_2 j \sin\theta$$

$C^1$ over **R**

$\cos\theta, \sin\theta \in \mathbf{R}$

$$l_1 \cos\theta + l_2 j \sin\theta$$

$C^1$ over **R**

$l_1, l_2 \in \mathbf{R}$

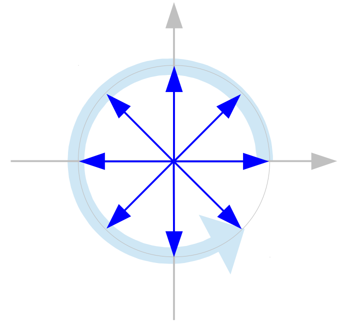
$$c_1 e^{+i\omega} + c_2 e^{-i\omega} \quad \longleftrightarrow \quad c_3 \cos(\omega) + c_4 \sin(\omega)$$

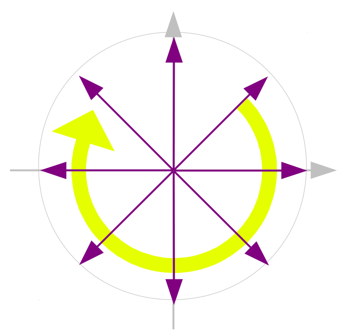
| | |
|---|---|
| *real number* | $c_1 = (c_3 - c_4 i)/2$ |
| *real number* | $c_2 = (c_3 + c_4 i)/2$ |

$\mathbf{C}^1$ over $\mathbf{R}$

| | |
|---|---|
| $c_3 = (c_1 + c_2)$ | *real number* |
| $c_4 = i(c_1 - c_2)$ | *imaginary number* |

$(c_1, \ c_2)$





$(\Re(c_3), \ \Im(c_4))$

| $c_1 e^{+i\omega} + c_2 e^{-i\omega}$ | $c_3 \cos(\omega) + c_4 \sin(\omega)$ |
|---|---|
| $1 \cdot e^{+i\omega} + 0 \cdot e^{-i\omega}$ | $1 \cdot \cos(\omega) + 1 i \cdot \sin(\omega)$ |
| $\frac{1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega}$ | $\sqrt{2} \cdot \cos(\omega) + 0 i \cdot \sin(\omega)$ |
| $0 \cdot e^{+i\omega} + 1 \cdot e^{-i\omega}$ | $1 \cdot \cos(\omega) - 1 i \cdot \sin(\omega)$ |
| $\frac{-1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega}$ | $0 \cdot \cos(\omega) - \sqrt{2} i \cdot \sin(\omega)$ |
| $-1 \cdot e^{+i\omega} + 0 \cdot e^{-i\omega}$ | $-1 \cdot \cos(\omega) - 1 i \cdot \sin(\omega)$ |
| $\frac{-1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega}$ | $-\sqrt{2} \cdot \cos(\omega) - 0 i \cdot \sin(\omega)$ |
| $0 \cdot e^{+i\omega} - 1 \cdot e^{-i\omega}$ | $-1 \cdot \cos(\omega) + 1 i \cdot \sin(\omega)$ |
| $\frac{1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega}$ | $0 \cdot \cos(\omega) + \sqrt{2} i \cdot \sin(\omega)$ |
| ↑ $c_1$ ↑ $c_2$ | ↑ $c_3$ ↑ $c_4$ |

# Real Coefficients $k_3$ & $k_4$

$$c_1 e^{+i\omega} + c_2 e^{-i\omega} \quad \Longleftrightarrow \quad c_3 \cos(\omega) + c_4 \sin(\omega)$$

$C^1$ over $R$

| | | | |
|---|---|---|---|
| *real number* | $c_1 = (c_3 - c_4 i)/2$ | $c_3 = (c_1 + c_2)$ | *real number* |
| *real number* | $c_2 = (c_3 + c_4 i)/2$ | $c_4 = i(c_1 - c_2)$ | *imaginary number* |

$c_1 \in R \qquad c_1 : real$

$c_2 \in R \qquad c_2 : real$

$c_3 \in C \qquad c_3 : real$

$c_4 \in C \qquad c_4 : imag$

$$m_1 e^{+i\omega} + m_2 e^{-i\omega} \quad \Longleftrightarrow \quad k_3 \cos(\omega) + k_4 \sin(\omega)$$

$R^1$ over $R$

| | | | | |
|---|---|---|---|---|
| $m_1 = (k_3 - k_4 i)/2$ | *conjugate* | $+2*real\ part$ | $k_3 = (m_1 + m_2)$ | *real number* |
| $m_2 = (k_3 + k_4 i)/2$ | *complex number* | $-2*imag\ part$ | $k_4 = i(m_1 - m_2)$ | *real number* |

$m_1 \in C \qquad (m_1 + m_2) : real$

$m_2 \in C \qquad i(m_1 - m_2) : real$

$k_3 \in R \qquad k_3 : real$

$k_4 \in R \qquad k_4 : real$

# Subspace : Real Line

$$m_1 e^{+i\omega} + m_2 e^{-i\omega} \quad \Longleftrightarrow \quad k_3 \cos(\omega) + k_4 \sin(\omega)$$

$$m_1 = (k_3 - k_4 i)/2$$
$$m_2 = (k_3 + k_4 i)/2$$

*conjugate complex number*

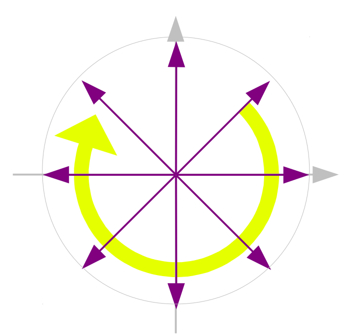
**$R^1$ over R**

$+2*real\ part$: $k_3 = (m_1 + m_2)$ — *real number*

$-2*imag\ part$: $k_4 = i(m_1 - m_2)$ — *real number*

| | |
|---|---|
| $\frac{(+1-0i)}{2} \cdot e^{+i\omega} + \frac{(+1+0i)}{2} \cdot e^{-i\omega}$ | $1 \cdot \cos(\omega) + 0 \cdot \sin(\omega)$ |
| $\frac{(+1-i)}{2\sqrt{2}} \cdot e^{+i\omega} + \frac{(+1+i)}{2\sqrt{2}} \cdot e^{-i\omega}$ | $\frac{1}{\sqrt{2}} \cdot \cos(\omega) + \frac{1}{\sqrt{2}} \cdot \sin(\omega)$ |
| $\frac{(0-i)}{2} \cdot e^{+i\omega} + \frac{(0+i)}{2} \cdot e^{-i\omega}$ | $0 \cdot \cos(\omega) + 1 \cdot \sin(\omega)$ |
| $\frac{(-1-i)}{2\sqrt{2}} \cdot e^{+i\omega} + \frac{(-1+i)}{2\sqrt{2}} \cdot e^{-i\omega}$ | $\frac{-1}{\sqrt{2}} \cdot \cos(\omega) + \frac{1}{\sqrt{2}} \cdot \sin(\omega)$ |
| $\frac{(-1-0i)}{2} \cdot e^{+i\omega} + \frac{(-1+0i)}{2} \cdot e^{-i\omega}$ | $-1 \cdot \cos(\omega) + 0 \cdot \sin(\omega)$ |
| $\frac{(-1+i)}{2\sqrt{2}} \cdot e^{+i\omega} + \frac{(-1-i)}{2\sqrt{2}} \cdot e^{-i\omega}$ | $\frac{-1}{\sqrt{2}} \cdot \cos(\omega) + \frac{-1}{\sqrt{2}} \cdot \sin(\omega)$ |
| $\frac{(0+i)}{2} \cdot e^{+i\omega} + \frac{(0-i)}{2} \cdot e^{-i\omega}$ | $0 \cdot \cos(\omega) - 1 \cdot \sin(\omega)$ |
| $\frac{(+1+i)}{2\sqrt{2}} \cdot e^{+i\omega} + \frac{(+1-i)}{2\sqrt{2}} \cdot e^{-i\omega}$ | $\frac{1}{\sqrt{2}} \cdot \cos(\omega) - \frac{1}{\sqrt{2}} \cdot \sin(\omega)$ |

*real line*



$(k_3,\ k_4)$

$\uparrow m_1 \quad \uparrow m_2 \quad \uparrow k_3 \quad \uparrow k_4$

# Trigonometric Relationship

$m_1 e^{+i\omega} + m_2 e^{-i\omega}$ ⟷ $k_3 \cos(\omega) + k_4 \sin(\omega)$

$m_1 = (k_3 - k_4 i)/2$
$m_2 = (k_3 + k_4 i)/2$

*conjugate complex number*

$\mathbf{R^1}$ over $\mathbf{R}$

$+2 * real\ part$ $\quad k_3 = (m_1 + m_2)$ *real number*

$-2 * imag\ part$ $\quad k_4 = i(m_1 - m_2)$ *real number*

$A\cos(\omega t - \phi)$ ⟷ $k_3 \cos(\omega) + k_4 \sin(\omega)$

$$\sqrt{k_3^2 + k_4^2} = A$$

$$\frac{k_3}{\sqrt{k_3^2 + k_4^2}} = \cos(\phi)$$

$$\frac{k_4}{\sqrt{k_3^2 + k_4^2}} = \sin(\phi)$$

$$A \cdot [\cos(\phi)\cos(\omega) + \sin(\phi)\sin(\omega)]$$

# Linear combination of $\cos(\omega t)$, $\sin(\omega t)$

$$k_3 \cos(\omega) + k_4 \sin(\omega)$$

$$m_1 = (k_3 - k_4 i)/2$$
$$m_2 = (k_3 + k_4 i)/2$$

*conjugate complex number*

$\mathbf{R^1}$ **over R**

$$m_1 e^{+i\omega} + m_2 e^{-i\omega}$$

$+2*real\ part$ $\quad k_3 = (m_1 + m_2)$ $\quad$ *real number*

$-2*imag\ part$ $\quad k_4 = i(m_1 - m_2)$ $\quad$ *real number*

$$A\cos(\omega t - \phi)$$

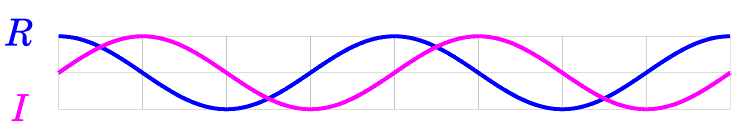
$$\sqrt{k_3^2 + k_4^2} = A$$

$$k_4 / k_3 = \tan(\phi)$$

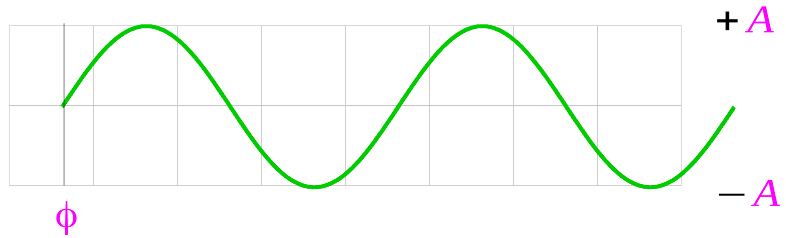
$$\frac{c_3}{\sqrt{c_3^2 + c_4^2}} = \cos(\phi)$$

$$\frac{c_4}{\sqrt{c_3^2 + c_4^2}} = \sin(\phi)$$

$\cos(\omega)$ , $\sin(\omega)$




$k_3, k_3$



# Real & Complex Fourier Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left( A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

| | |
|---|---|
| zero freq | $a_0 = A_0$ |
| pos freq $(+k\omega_0)$ | $a_k = (A_k + B_k)$ |
| pos freq $(+k\omega_0)$ | $b_k = j(A_k - B_k)$ |

| | |
|---|---|
| zero freq | $A_0 = a_0$ |
| pos freq $(+k\omega_0)$ | $A_k = \frac{1}{2}(a_k - j b_k)$ |
| neg freq $(-k\omega_0)$ | $B_k = \frac{1}{2}(a_k + j b_k)$ |

$a_k,\ b_k$ real number

$A_k,\ B_k$ complex conjugate

# Real & Complex Fourier Coefficients

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$
$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$
$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$
$$k = 1,\ 2,\ \ldots$$

$$A_0 = \frac{1}{T}\int_0^T x(t)\,dt$$
$$A_k = \frac{1}{T}\int_0^T x(t)\,e^{-jk\omega_0 t}\,dt$$
$$B_k = \frac{1}{T}\int_0^T x(t)\,e^{+jk\omega_0 t}\,dt$$

$$A_0 = a_0$$
$$A_k = \frac{1}{2}(a_k - j b_k)$$
$$B_k = \frac{1}{2}(a_k + j b_k)$$

$$A_k = \frac{1}{T}\int_0^T x(t)\,(\cos(k\omega_0 t) - j\sin(k\omega_0 t))\,dt$$
$$B_k = \frac{1}{T}\int_0^T x(t)\,(\cos(k\omega_0 t) + j\sin(k\omega_0 t))\,dt$$

$$A_k = \frac{1}{T}\int_0^T x(t)\,e^{-jk\omega_0 t}\,dt$$ pos freq $(+k\omega_0)$

$$B_k = \frac{1}{T}\int_0^T x(t)\,e^{+jk\omega_0 t}\,dt$$ neg freq $(-k\omega_0)$

# Complex Fourier Series $A_k, \; B_k$

**1**

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T}\int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T}\int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$$k = 1, 2, \ldots$$

**2**

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left( A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left( A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$x(t) = \sum_{k=0}^{\infty} \left( A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

**3**

$$x(t) = \sum_{k=0}^{\infty} \left( A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = 0, 1, 2, \ldots$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\, dt$$

$$k = 1, 2, \ldots$$

# Complex Fourier Series $C_k$

**3**

$$x(t) = \sum_{k=0}^{\infty} \left( A_k\, e^{+jk\omega_0 t} + B_k\, e^{-jk\omega_0 t} \right)$$

$$A_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad k = 0,\ 1,\ 2,\ \ldots$$

$$B_k = \frac{1}{T} \int_0^T x(t)\, e^{+jk\omega_0 t}\, dt \qquad k = 1,\ 2,\ \ldots$$

$$x(t) = \sum_{k=0}^{\infty} \left( C_k\, e^{+jk\omega_0 t} + C_{-k}\, e^{-jk\omega_0 t} \right)$$

$$C_0 = A_0 \qquad a_0$$
$$C_{+k} = A_k \quad (k > 0) \qquad \tfrac{1}{2}(a_k - j b_k)$$
$$C_{-k} = B_k \quad (k > 0) \qquad \tfrac{1}{2}(a_k + j b_k)$$

$$x(t) = \sum_{k=-\infty}^{\infty} \left( C_k\, e^{+jk\omega_0 t} \right)$$

$$C_0 = A_0 \qquad a_0$$
$$C_k = A_k \quad (k > 0) \qquad \tfrac{1}{2}(a_k - j b_k)$$
$$C_k = B_{-k} \quad (k < 0) \qquad \tfrac{1}{2}(a_k + j b_k)$$

**4**

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_{-k} & (k < 0) \end{cases} \qquad \begin{matrix} a_0 \\ \tfrac{1}{2}(a_k - j b_k) \\ \tfrac{1}{2}(a_k + j b_k) \end{matrix}$$

# Complex Fourier Series

**1**

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$k = 1,\ 2,\ \ldots$$

**2**

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left( A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2}(a_k - j b_k)$$

$$B_k = \frac{1}{2}(a_k + j b_k)$$

$$k = 1,\ 2,\ \ldots$$

**3**

$$x(t) = \sum_{k=0}^{\infty} \left( A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$A_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\,dt$$

$$k = 0,\ 1,\ 2,\ \ldots$$

$$B_k = \frac{1}{T}\int_0^T x(t)\, e^{+jk\omega_0 t}\,dt$$

$$k = 1,\ 2,\ \ldots$$

**4**

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\,dt$$

$$k = -2,\ -1,\ 0,\ +1,\ +2,\ \ldots$$

$$C_k = \begin{cases} A_0 & (k = 0) & a_0 \\ A_k & (k > 0) & \frac{1}{2}(a_k - j b_k) \\ B_{-k} & (k < 0) & \frac{1}{2}(a_k + j b_k) \end{cases}$$

# Phasor Representation $X_k$ via $g_k$, $\phi_k$

**1**

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$k = 1,\ 2,\ \ldots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k\, \Re\left\{ e^{+j(k\omega_0 t + \phi_k)} \right\}$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \Re\left\{ g_k \cdot e^{+j\phi_k} \cdot e^{+jk\omega_0 t} \right\}$$

**a**

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left( -\frac{b_k}{a_k} \right)$$

$$k = 1,\ 2,\ \ldots$$

**b**

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re\left\{ X_k e^{+jk\omega_0 t} \right\}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

$$k = 1,\ 2,\ \ldots$$


Young Won Lim
8/12/16

# Phasor Representation $C_k$ via $g_k$, $\phi_k$

**4**

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{2} g_{+k} \, e^{+j\phi_k} \quad (k > 0)$$

$$C_k = \frac{1}{2} g_{-k} \, e^{-j\phi_k} \quad (k < 0)$$

$$k = -2, \; -1, \; 0, \; +1, \; +2, \; \ldots$$

$$x(t) = g_0 + \frac{1}{2} \sum_{k=1}^{\infty} g_k \cdot \left( e^{+j(k\omega_0 t + \phi_k)} + e^{-j(k\omega_0 t + \phi_k)} \right)$$

$$= g_0 + \sum_{k=1}^{\infty} \left( \frac{1}{2} g_k \, e^{+j\phi_k} e^{+jk\omega_0 t} + \frac{1}{2} g_k \, e^{-j\phi_k} e^{-jk\omega_0 t} \right)$$

**a**

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left( -\frac{b_k}{a_k} \right)$$

$$k = 1, \; 2, \; \ldots$$

**b**

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re\{ X_k e^{+jk\omega_0 t} \}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

$$k = 1, \; 2, \; \ldots$$

# Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left( a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t)\, dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t)\, dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t)\, dt$$

$\boldsymbol{k} = +1, +2, \ldots$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

$\boldsymbol{k} = +1, +2, \ldots$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = g_k \cos(\phi_k)\cos(k\omega_0 t) - g_k \sin(\phi_k)\sin(k\omega_0 t)$$

$$a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

# Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$|C_k|^2 = |C_{-k}|^2 = \frac{1}{2}|g_k|^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2} g_{+k}\, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_{-k}\, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

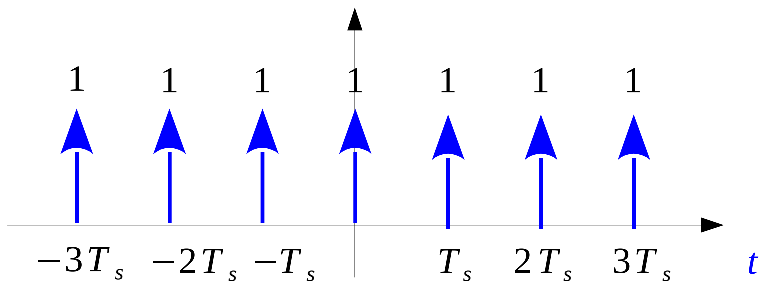
$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = |g_k| = \sqrt{a_k^2 + b_k^2}$$

# CTFS of Impulse Train (1)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$




**Fourier Series Expansion of Impulse Train**

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

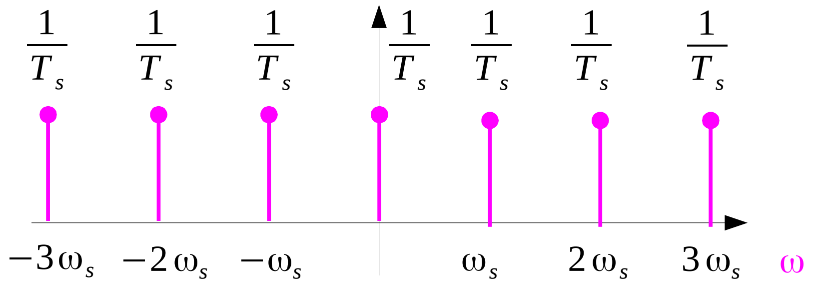
$$\omega_s = \frac{2\pi}{T_s}$$

**Fourier Series Coefficients**

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) \, e^{-jk\omega_s t} \, dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) \, e^{-jk\omega_s 0} \, dt$$

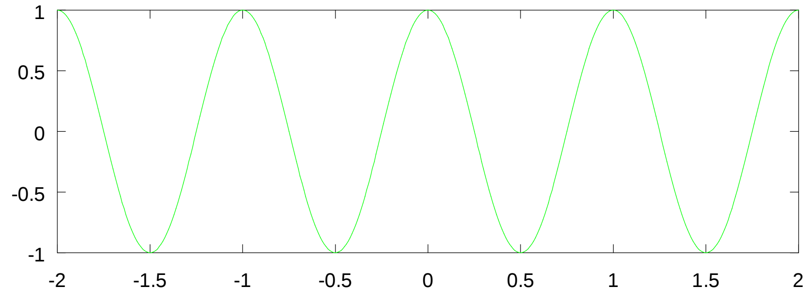
$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) \, dt = \frac{1}{T_s}$$

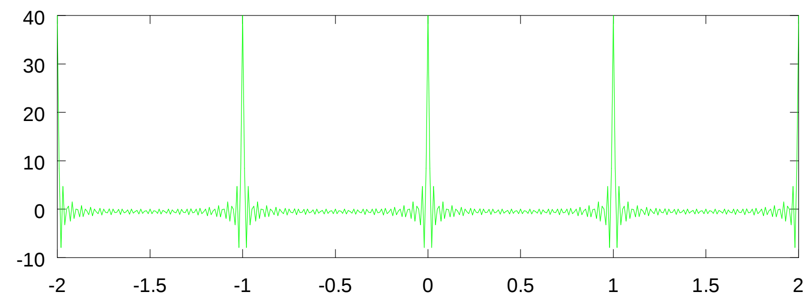


# CTFS of Impulse Train (2)

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

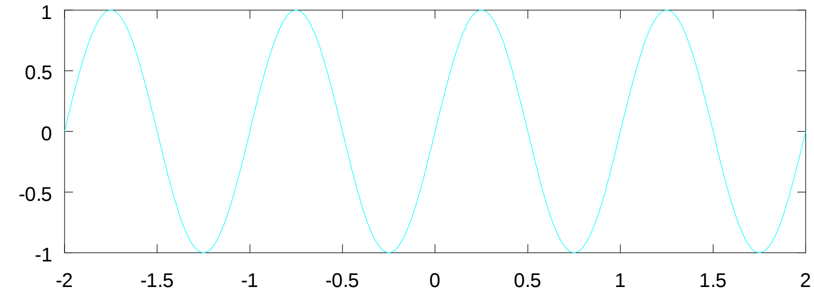
$\cos 2\pi \cdot 1 \cdot t$

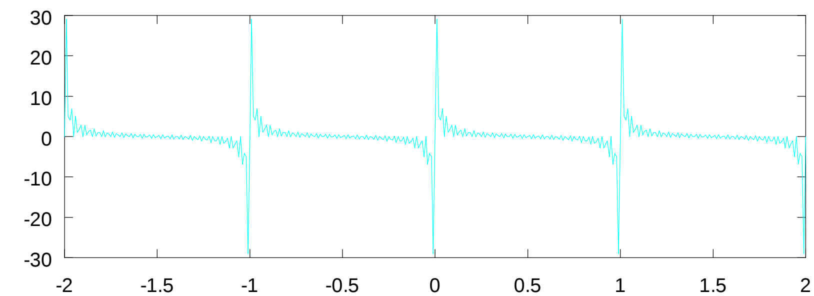



$$\sum_{k=1}^{40} \cos(2\pi \cdot k \cdot t)$$

$\sin 2\pi \cdot 1 \cdot t$



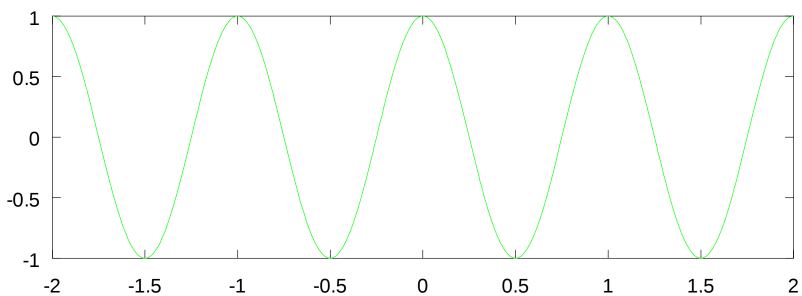

$$\sum_{k=1}^{40} \sin(2\pi \cdot k \cdot t)$$

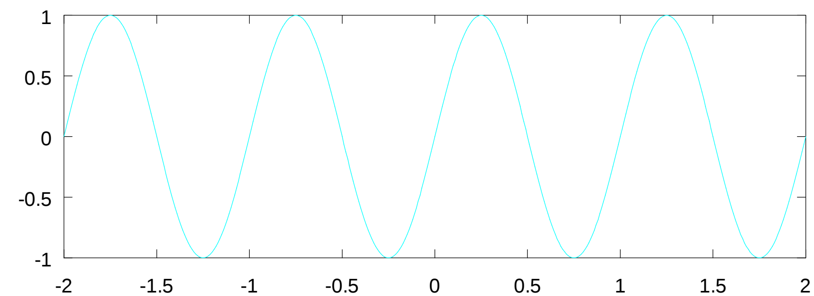
# CTFS of Impulse Train (3)

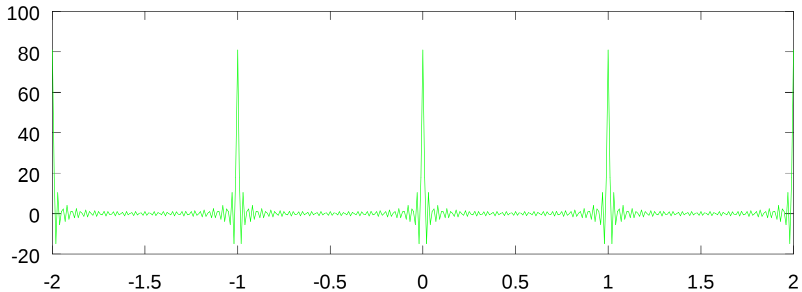
$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

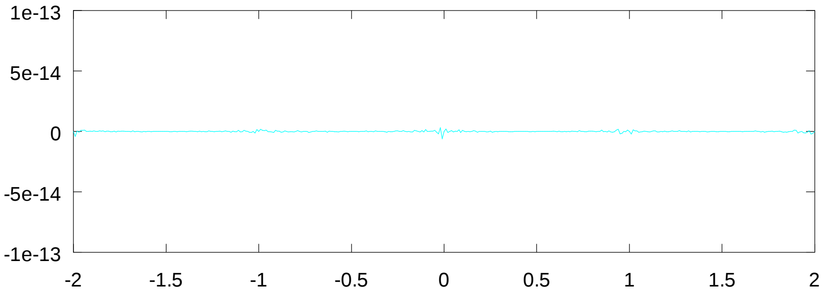
$\cos 2\pi \cdot 1 \cdot t$

$\sin 2\pi \cdot 1 \cdot t$




$$\sum_{k=-40}^{40} \cos(2\pi \cdot k \cdot t)$$

$$\sum_{k=-40}^{40} \sin(2\pi \cdot k \cdot t)$$

# Inner Product Space

Hilbert Space real / complex inner product space

$$\langle f , g \rangle = \int_a^b f(t) \overline{g(t)} \, dt$$

complex conjugate

$$\langle y , x \rangle = \overline{\langle x , y \rangle}$$

linear

$$\langle a x_1 + b x_{2,} \ y \rangle = a \langle x_{1,} \ y \rangle + b \langle x_{2,} \ y \rangle$$

positive semidefinite

$$\langle x , x \rangle \geq 0$$

Norm

$$\|x\| = \sqrt{\langle x , x \rangle}$$

Cauchy-Schwartz Inequality

$$|\langle x , y \rangle| \leq \|x\| \, \|y\|$$

# Cauchy-Schwartz Inequality

For all vectors ***x*** and ***y*** of an inner product space

$$|\langle x, y\rangle|^2 \le \langle x, x\rangle \cdot \langle y, y\rangle$$

$$|\langle x, y\rangle| \le \|x\| \cdot \|y\|$$

The equality holds if and only if ***x*** and ***y*** are linearly dependent ⇨ maximum

$$\left|\int_a^b x(t)\overline{y(t)}\,dt\right| \le \sqrt{\int_a^b x(t)\overline{x(t)}\,dt}\ \sqrt{\int_a^b y(t)\overline{y(t)}\,dt}$$

Inner product is maximum
when $y = k\,x$

# Complex Orthogonality

$$\langle e^{j m \omega_0 t},\ e^{j n \omega_0 t} \rangle = \int_0^T e^{+j(m-n)\omega_0 t}\, dt = \begin{cases} 0 & (m \neq n) \\ T & (m = n) \end{cases}$$

$m, n$ : integer

$$\langle e^{j(+1)\omega_0 t},\ e^{j(+1)\omega_0 t} \rangle = \int_0^T e^{+j(1)\omega_0 t} \cdot \overline{e^{+j(1)\omega_0 t}}\, dt = \int_0^T e^{+j(1-1)\omega_0 t}\, dt = T$$

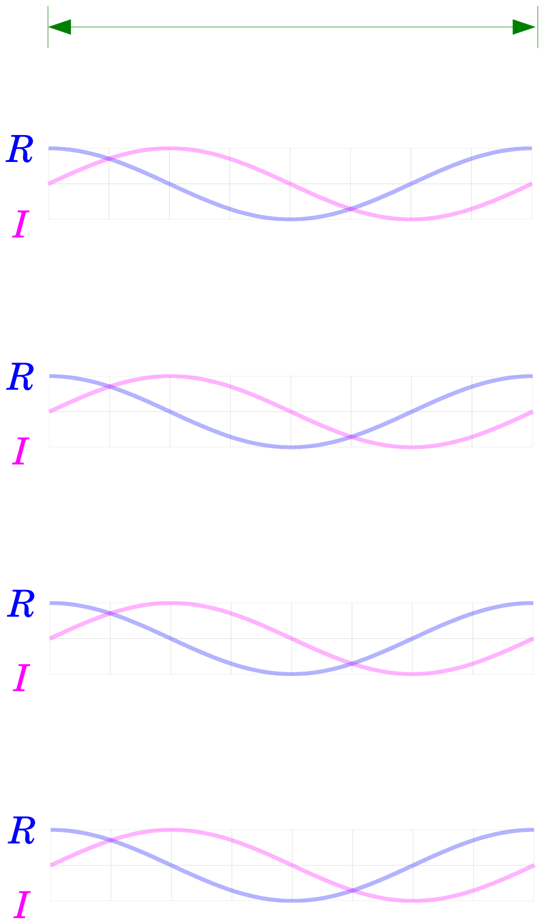
$$\langle e^{j(+1)\omega_0 t},\ e^{j(-1)\omega_0 t} \rangle = \int_0^T e^{+j(1)\omega_0 t} \cdot \overline{e^{-j(1)\omega_0 t}}\, dt = \int_0^T e^{+j(1+1)\omega_0 t}\, dt = 0$$

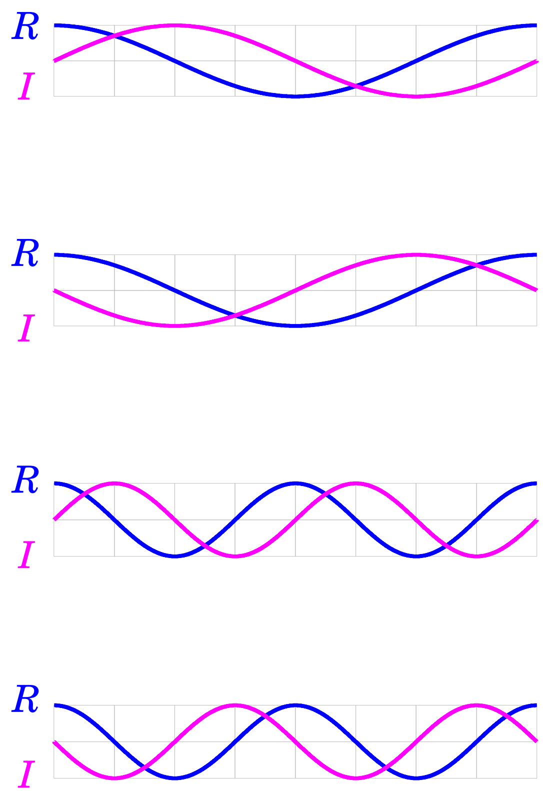
$$\langle e^{j(+1)\omega_0 t},\ e^{j(+2)\omega_0 t} \rangle = \int_0^T e^{+j(1)\omega_0 t} \cdot \overline{e^{+j(2)\omega_0 t}}\, dt = \int_0^T e^{+j(1-2)\omega_0 t}\, dt = 0$$

$$\langle e^{j(+1)\omega_0 t},\ e^{j(-2)\omega_0 t} \rangle = \int_0^T e^{+j(1)\omega_0 t} \cdot \overline{e^{-j(2)\omega_0 t}}\, dt = \int_0^T e^{+j(1+2)\omega_0 t}\, dt = 0$$

# Inner Product Examples

$T \qquad f_0 = 1/T \qquad \omega_0 = 2\pi/T$




$$\langle e^{j1\omega_0 t},\ e^{j1\omega_0 t}\rangle = \int_0^T e^{+j(1-1)\omega_0 t}\,dt = T$$

$$\langle e^{j1\omega_0 t},\ e^{j-1\omega_0 t}\rangle = \int_0^T e^{+j(1+1)\omega_0 t}\,dt = 0$$

$$\langle e^{j1\omega_0 t},\ e^{j2\omega_0 t}\rangle = \int_0^T e^{+j(1-2)\omega_0 t}\,dt = 0$$

$$\langle e^{j1\omega_0 t},\ e^{j-2\omega_0 t}\rangle = \int_0^T e^{+j(1+2)\omega_0 t}\,dt = 0$$

# Complex Fourier Coefficients and Inner Product

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

*fundamental frequency* $f_0 = \frac{1}{T}$ $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

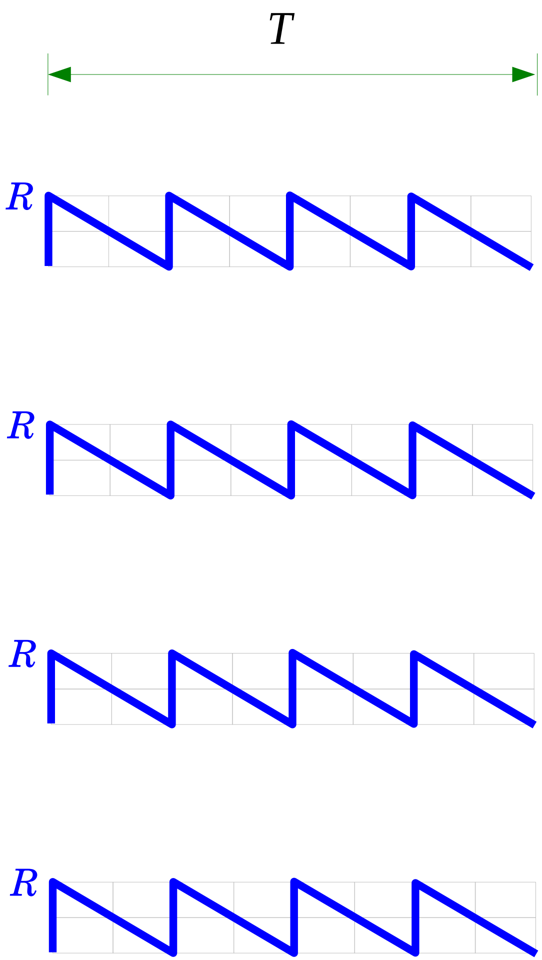
*n-th harmonic frequency* $f_n = n f_0$ $\omega_n = 2\pi f_n = \frac{2\pi n}{T}$

$$\langle f, g \rangle = \int_0^T f(t)\overline{g(t)}\, dt$$

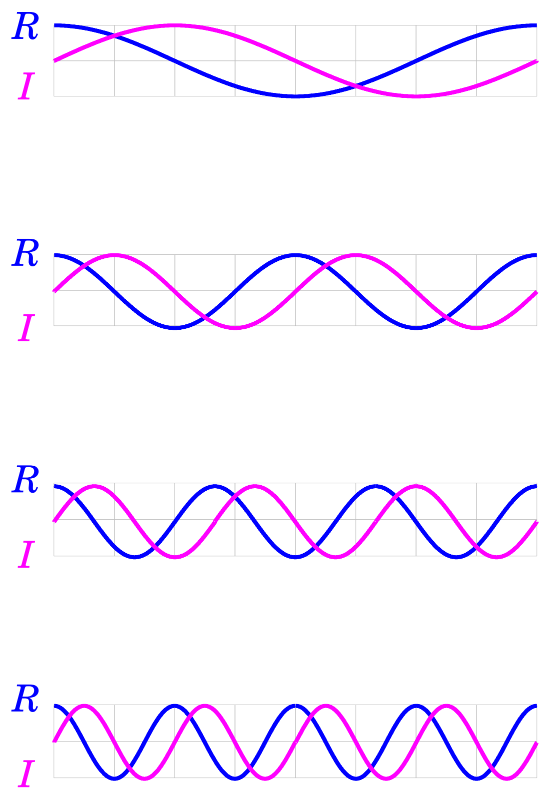
$$C_k = \frac{1}{T}\langle x(t), e^{-jk\omega_0 t} \rangle$$

$$k = \ldots, -2, -1, 0, +1, +2, \ldots$$

# Finding Complex Fourier Coefficients




$$f_0 = 1/T \qquad \omega_0 = 2\pi/T$$



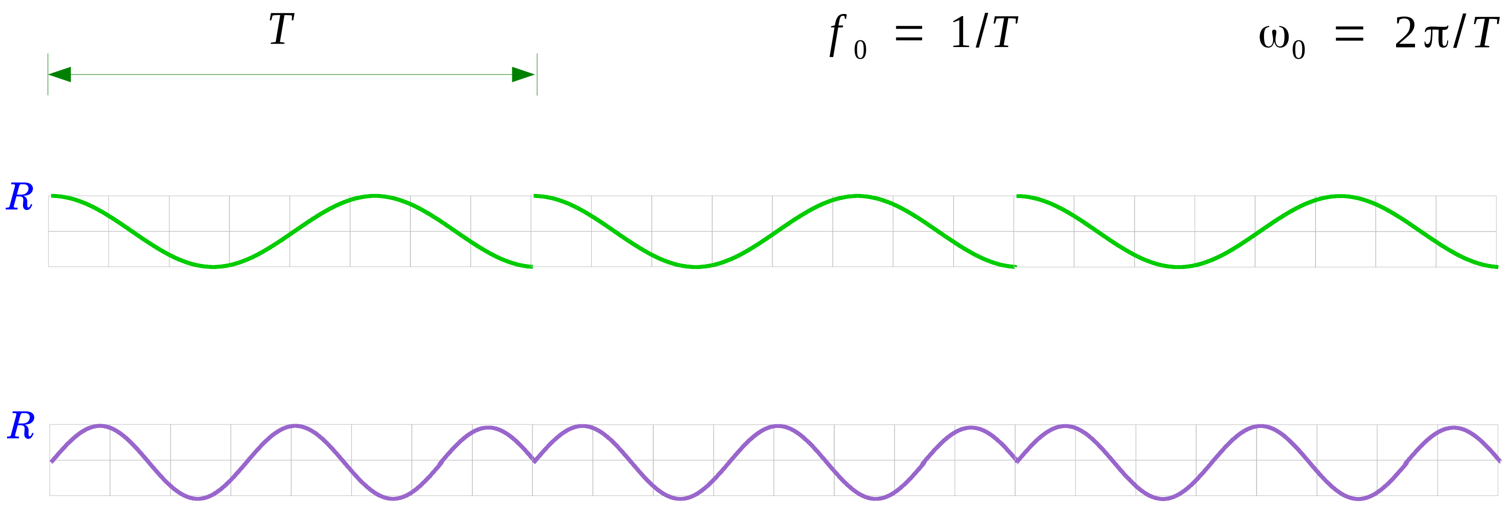

$$\frac{1}{T}\langle x(t),\ e^{+j(1)\omega_0 t}\rangle = C_1$$

$$\frac{1}{T}\langle x(t),\ e^{+j(2)\omega_0 t}\rangle = C_2$$

$$\frac{1}{T}\langle x(t),\ e^{+j(3)\omega_0 t}\rangle = C_3$$

$$\frac{1}{T}\langle x(t),\ e^{+j(4)\omega_0 t}\rangle = C_4$$

# Spectral Leakage




$f_0 = 1/T$ $\omega_0 = 2\pi/T$

## References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[4] C. Langton, http://www.complextoreal.com/chapters/fft1.pdf