CTFS (1A)

- Continuous Time Fourier Series

Young Won Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Fourier Series

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx\, dx$$

$k = 1, 2, 3, \ldots$ ➡ **one-sided spectrum**

only positive frequencies

Fourier series with real coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \qquad t \in [-\pi, +\pi]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} t + b_k \sin \frac{k\pi}{L} t\right) \qquad t \in [-L, +L]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi k}{T} t + b_k \sin \frac{2\pi k}{T} t\right) \qquad t \in [0, +T]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t)) \qquad t \in [0, +T]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)) \qquad t \in [0, +T]$$

Complex Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

Real coefficients

$$a_0,\ a_k,\ b_k, \quad k = 1,\ 2,\ \ldots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

Complex coefficients

$$A_0,\ A_k,\ B_k, \quad k = 1,\ 2,\ \ldots$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t} \right)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

two-sided spectrum

Both pos and neg frequencies

Trigonometric Orthogonality

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0 \qquad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0 \qquad (n \neq m)$$

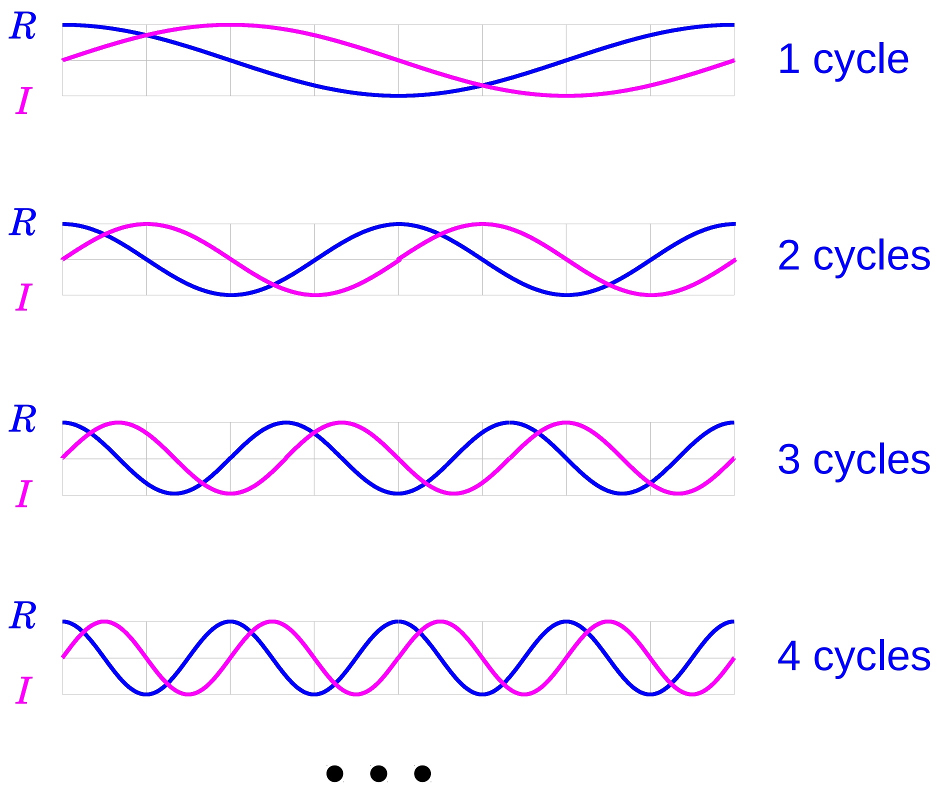
$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = 0$$

$$\int_{-\pi}^{+\pi} \cos \boldsymbol{n} x \cos \boldsymbol{m} x \, dx = \pi \qquad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin \boldsymbol{n} x \sin \boldsymbol{m} x \, dx = \pi \qquad (n = m)$$

$\boldsymbol{n}, \boldsymbol{m}$: integer



The **correction** of the following waves are zero

two of these sine waves

two of these cosine waves

these sine and cosine waves

Fundamental and Harmonic Frequencies

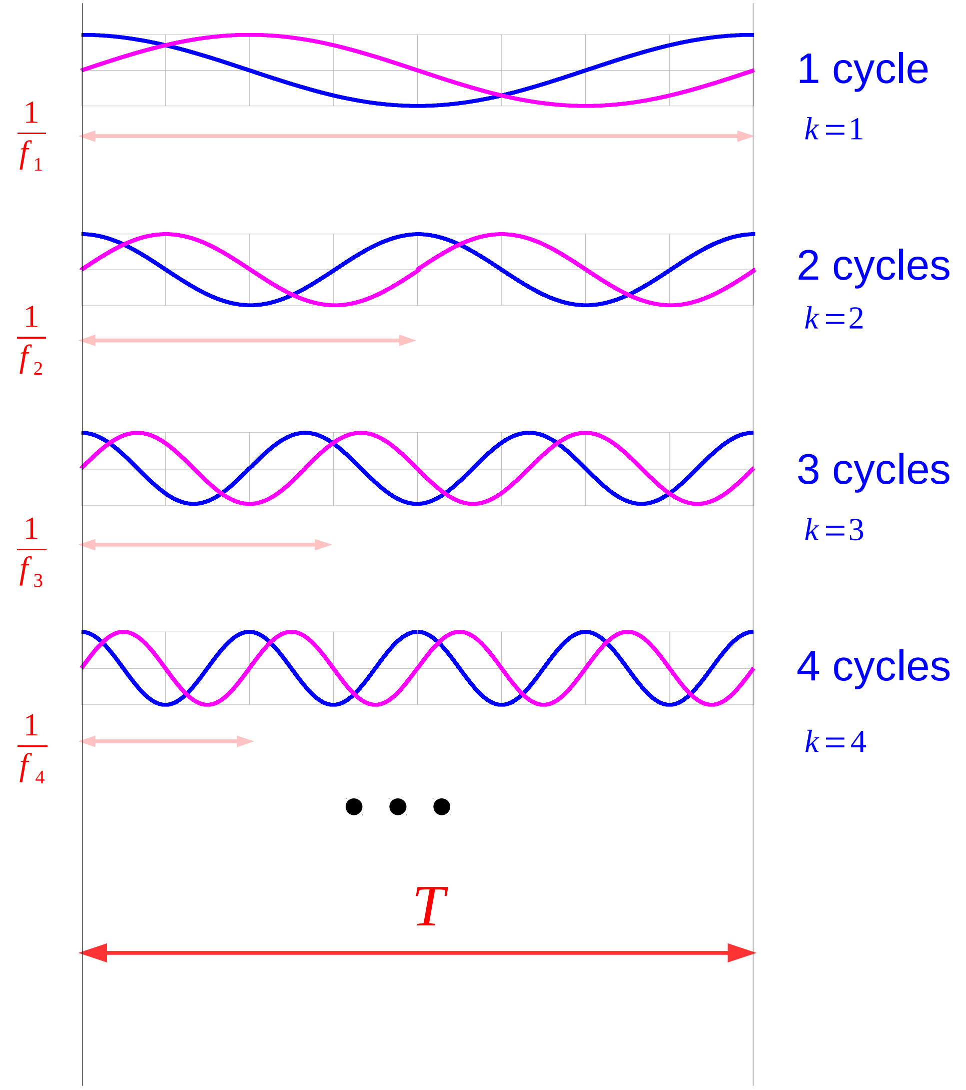
$$f_0 = \frac{2\pi}{T}$$

f_0 : *fundamental frequency*

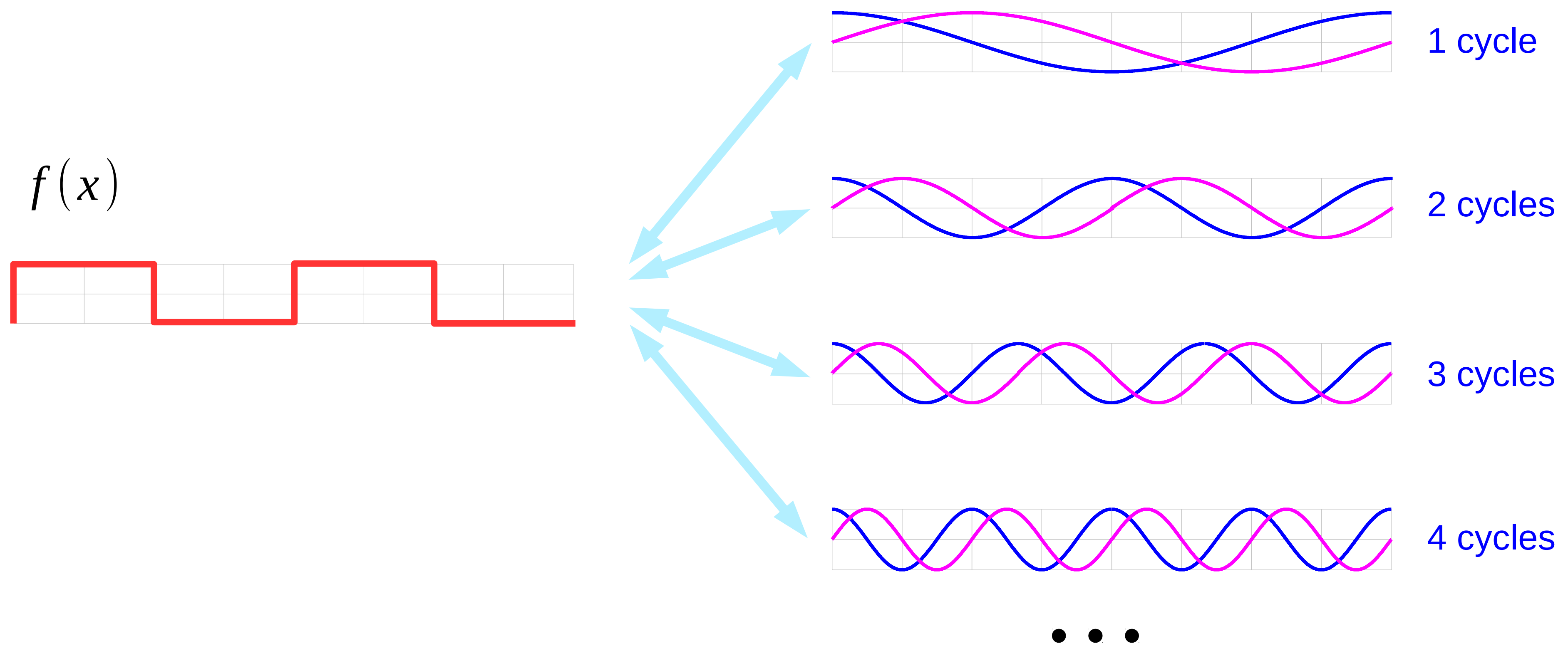
$$f_k = k \cdot f_0 = k \cdot \frac{2\pi}{T}$$

k : integer

f_k : *harmonic frequency*



Correlation Process



Measure the degree of correlation with these cosine and sine waves whose frequencies are the integer multiples of the fundamental frequency

Fourier Series Coefficients

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos 1x\,dx$$
$$b_1 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin 1x\,dx$$

$f(x) \Leftrightarrow \cos(1x),\ \sin(1x)$ $\quad 1\cdot f_0 \quad$ 1 cycle

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos 2x\,dx$$
$$b_2 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin 2x\,dx$$

$f(x) \Leftrightarrow \cos(2x),\ \sin(2x)$ $\quad 2\cdot f_0 \quad$ 2 cycles

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos 3x\,dx$$
$$b_3 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin 3x\,dx$$

$f(x) \Leftrightarrow \cos(3x),\ \sin(3x)$ $\quad 3\cdot f_0 \quad$ 3 cycles

$$a_4 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos 4x\,dx$$
$$b_4 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin 4x\,dx$$

$f(x) \Leftrightarrow \cos(4x),\ \sin(4x)$ $\quad 4\cdot f_0 \quad$ 4 cycles

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Real & Complex Fourier Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$x(t) = A_0 + \sum_{k=1}^{\infty}\left(A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}\right)$$

$$a_0 = A_0$$

$$a_k = (A_k + B_k)$$

$$b_k = j(A_k - B_k)$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2}(a_k - jb_k)$$

$$B_k = \frac{1}{2}(a_k + jb_k)$$

Orthogonality

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$k = \ldots, -2, -1, 0, +1, +2, \ldots$

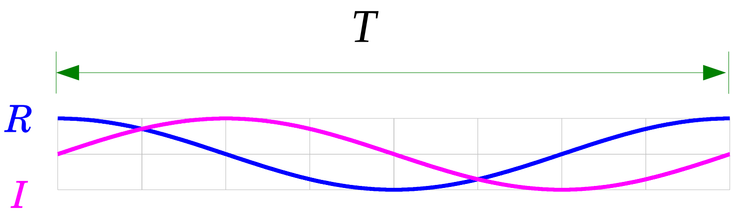
fundamental frequency $f_0 = \frac{1}{T}$ $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

n-th harmonic frequency $f_n = n f_0$ $\omega_n = 2\pi f_n = \frac{2\pi n}{T}$

$$\langle e^{jm\omega_0 t}, \; e^{jn\omega_0 t} \rangle = \int_0^T e^{+j(m-n)\omega_0 t}\, dt = \begin{cases} 0 & (m \neq n) \\ T & (m = n) \end{cases}$$

m, n : integer

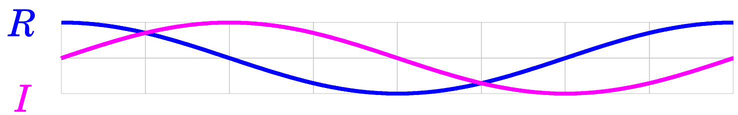
Inner Product Examples



$f_0 = 1/T \qquad \omega_0 = 2\pi/T$

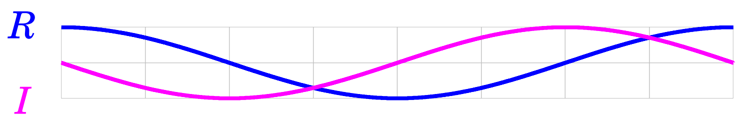
$\leftarrow e^{j1\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j1\omega_0 t}\rangle = \int_0^T e^{+j(1-1)\omega_0 t}\,dt = T$$



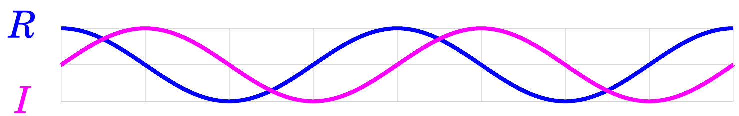
$\leftarrow e^{j1\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j-1\omega_0 t}\rangle = \int_0^T e^{+j(1+1)\omega_0 t}\,dt = 0$$



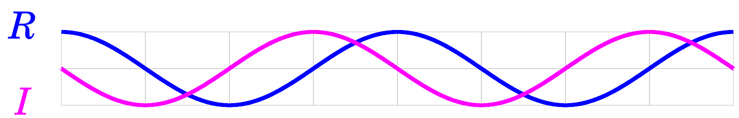
$\leftarrow e^{j-1\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j2\omega_0 t}\rangle = \int_0^T e^{+j(1-2)\omega_0 t}\,dt = 0$$



$\leftarrow e^{j2\omega_0 t}$

$$\langle e^{j1\omega_0 t},\ e^{j-2\omega_0 t}\rangle = \int_0^T e^{+j(1+2)\omega_0 t}\,dt = 0$$



$\leftarrow e^{j-2\omega_0 t}$

References

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[3] G. Beale, http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html
[4] C. Langton, http://www.complextoreal.com/chapters/fft1.pdf