Fundamental Vector Spaces (H.1)

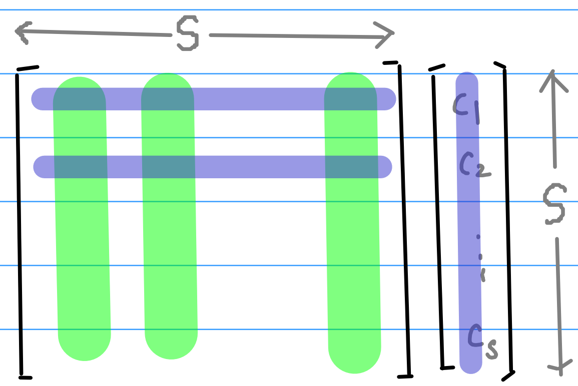
20151211

$\{v_1, v_2, \dots, v_s\}$ linear independent

$c_1 v_1 + c_2 v_2 + \cdots + c_s v_s = 0$ $\quad$ $c_1 = c_2 = \cdots = c_s = 0$

$$\begin{bmatrix} v_1 & v_2 & & v_s \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{bmatrix} = 0$$

only
trivial solution
$c_1 = c_2 = \cdots = c_s = 0$



$$c_1 \begin{bmatrix} v_1 \end{bmatrix} + c_2 \begin{bmatrix} v_2 \end{bmatrix} + \cdots + c_s \begin{bmatrix} v_s \end{bmatrix} = 0$$

Consistent Linear System

- Consistent homogereous linear system

$\boxed{Ax = 0}$ $\quad W$ solution space $\quad x_h$ homogeneous solution

translation

$\boxed{Ax = b}$ $\quad W + x_0$ solution space $\quad x_p$ particular solution

- Consistent non-homogereous linear system

$$
\begin{array}{rl}
 & A x_h = 0 \\
+) & A x_p = b \\
\hline
 & A(x_h + x_p) = b
\end{array}
$$

$\Rightarrow$ $(x_h + x_p)$ is also "general" solution to

$$\boxed{Ax = b}$$

⑥ Consistent homogeneous linear system

homogeneous sol.

$\boxed{Ax = 0}$ W : Null space x_h

always consistent

$\begin{cases} \text{rank}(A) = n \Rightarrow \text{unique solution } x = 0 \ (\because A^{-1}0) \\ \text{rank}(A) < n \Rightarrow \text{infinitely many solution} \\ \qquad (n-r) \text{ parameters} = \text{nullity} \end{cases}$

⑥ Consistent non-homogeneous linear system

$\boxed{Ax = b}$ $x_p + W$: solution space x_p

$\text{rank}(A) = \text{rank}(A|b) \Rightarrow$ Consistent

b is in column space of A

$\begin{cases} \text{rank}(A) = n \Rightarrow \text{unique solution } x = A^{-1}b \\ \text{rank}(A) < n \Rightarrow \text{infinitely many solution} \\ \qquad (n-r) \text{ parameters} = \text{nullity} \end{cases}$

$\text{rank}(A) < \text{rank}(A|b) \Rightarrow$ inconsistent no solution

Translation

Line

$$x = t\,v \xrightarrow{\text{translate}} x = t\,v + x_0$$

plane

$$x = t_1 v_1 + t_2 v_2 \xrightarrow{\text{translate}} x = t_1 v_1 + t_2 v_2 + x_0$$

hyperplane

$$x = t_1 v_1 + t_2 v_2 + \cdots + t_s v_s \xrightarrow{\text{translate}} x = t_1 v_1 + t_2 v_2 + \cdots + t_s v_s + x_0$$

$A\vec{x} = \vec{0}$ — W: solution space

translation

$A\vec{x} = \vec{b}$ — $W + x_0$: solution space

Line, Plane, Hyperplane

Line $\boxed{a_1 x_1 + a_2 x_2 = b}$ $\vec{a} \cdot \vec{x} = b$

$(a_1 \neq 0)$ & $(a_2 \neq 0)$

Plane $\boxed{a_1 x_1 + a_2 x_2 + a_3 x_3 = b}$ $\vec{a} \cdot \vec{x} = b$

$(a_1 \neq 0)$ & $(a_2 \neq 0)$ & $(a_3 \neq 0)$

Hyperplane in R^n $\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$ $\vec{a} \cdot \vec{x} = b$

$(a_1 \neq 0)$ & $(a_2 \neq 0)$ & $\cdots$ & $(a_n \neq 0)$

Hyperplane in R^n that passes through the origin $\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0}$ $\vec{a} \cdot \vec{x} = 0$

$(a_1 \neq 0)$ & $(a_2 \neq 0)$ & $\cdots$ & $(a_n \neq 0)$

Hyperplane : the orthogonal complement of a line

$$\boxed{a \cdot x = b} \iff A \cdot x = b$$

compare

$$(a_1, a_2, \dots, a_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = b$$

$\vec{a}$

normal vector

translate

$$\boxed{a \cdot x = 0}$$

hyperplane through the origin with normal a

Orthogonal complement of a

line

$\vec{a}$

normal vector

$$\{x \mid x \in \mathbb{R}^n,\ x \perp a\}$$

for a given $a \neq 0$

Hyperplane $a^{\perp}$ (a perp)

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ $\quad a \cdot x = b \quad (a \neq 0)$

$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0}$ $\quad a \cdot x = 0 \quad (a \neq 0)$

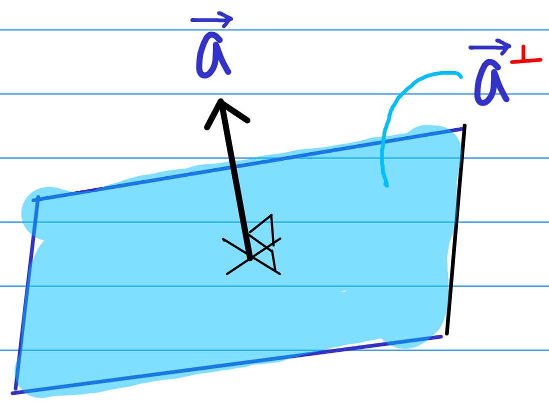
hyperplane that
pass through the origin
with a normal vector a

the normal vector

Orthogonal complement of a

$a^{\perp}$ (a perp)

a normal vector



(a perp)

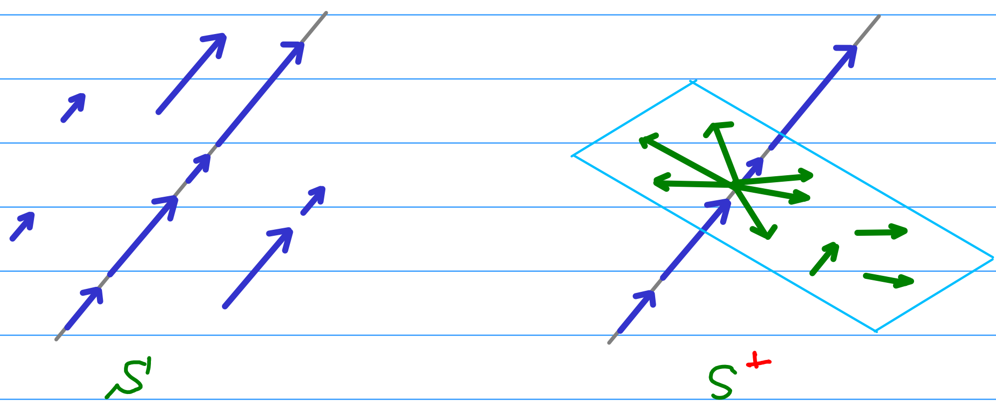
hyperplane
through the origin

Orthogonal Complement

S : a non-empty set in $\mathbb{R}^n$

$S^\perp$: the orthogonal complement of S

$$= \left\{ \begin{array}{l} \text{all vectors in } \mathbb{R}^n \\ \text{that are orthogonal} \\ \text{to every vector in } S \end{array} \right\}$$



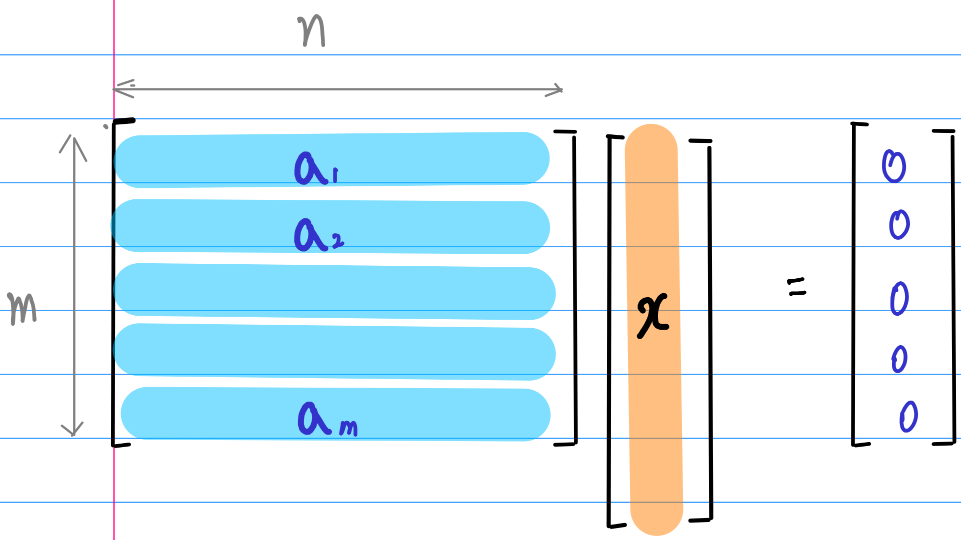
in this case,
also a hyperplane
since S is a set
of vectors in a line

$\dim(S) = 1$ $\qquad\qquad$ $\dim(S^\perp) = 2$

$$\dim(S) + \dim(S^\perp) = n \qquad \mathbb{R}^n$$

Null Space

$$A \cdot x = 0$$



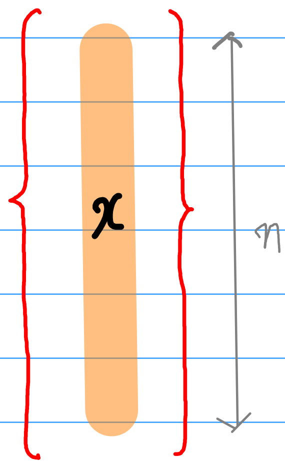
$$\boxed{\begin{aligned} a_1 \cdot x &= 0 \\ a_2 \cdot x &= 0 \\ &\vdots \\ a_m \cdot x &= 0 \end{aligned}}$$

solution space $\Rightarrow$ null (A)

$$= \{ x \mid a_1 \cdot x = 0 \;\&\; a_2 \cdot x = 0 \;\&\; \cdots \;\&\; a_m \cdot x = 0 \}$$

$$= \{ x \mid a_1 \perp x \;\&\; a_2 \perp x \;\&\; \cdots \;\&\; a_m \perp x \}$$

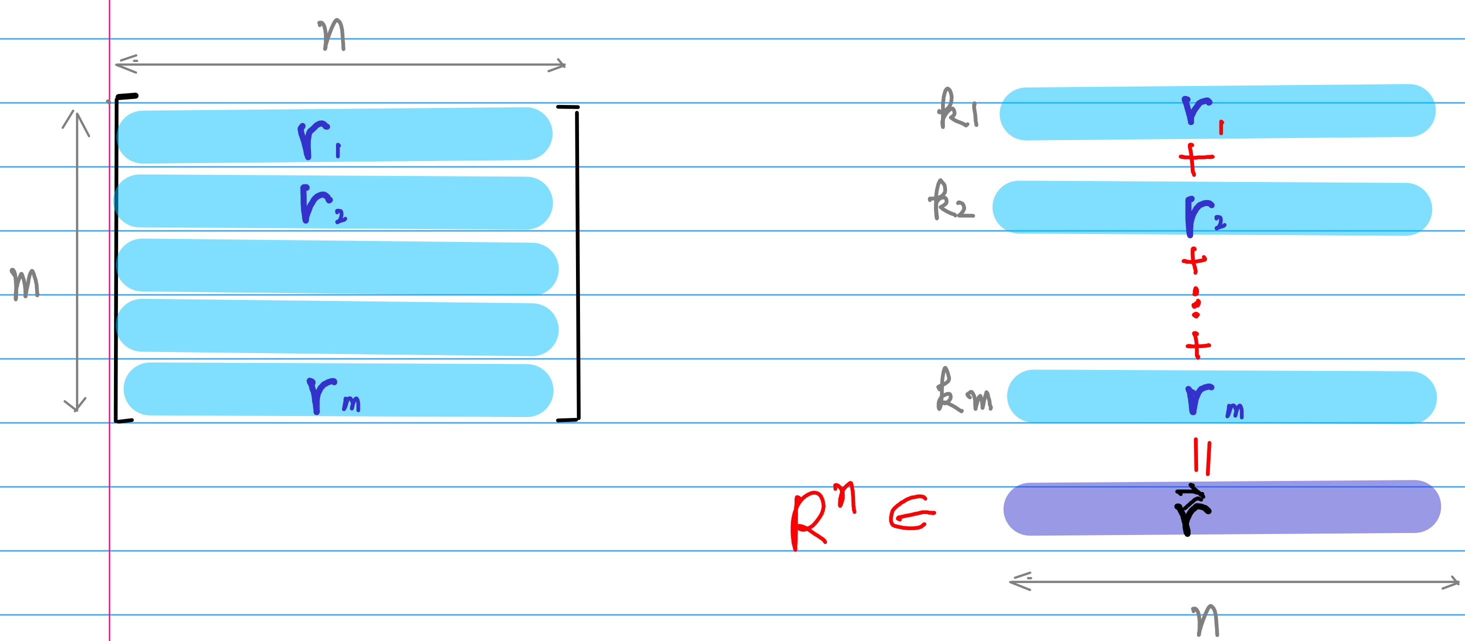
null (A) =



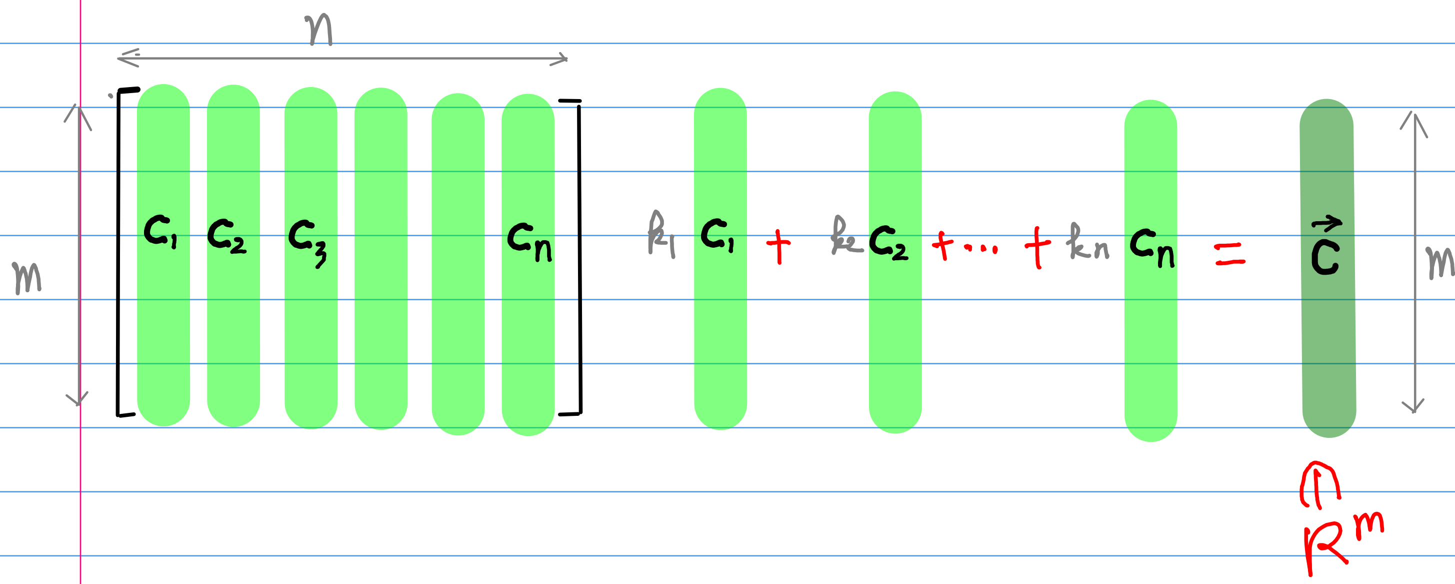
all vectors in R^n that are orthogonal to every row vector of **A**

Row Space & Column Space

the row space of A $\boxed{row(A)}$
the subspace of R^n
that is spanned by the row vectors of A



the column space of A $\boxed{col(A)}$
the subspace of R^m
that is spanned by the column vectors of A



Finding W basis and $W^{\perp}$ basis

Anton's Examples

$$\begin{array}{l}
v_1 = (1, 0, 0, 0, 2) \\
v_2 = (-2, 1, -3, -2, +4) \\
v_3 = (0, 5, -14, -9, 0) \\
v_4 = (2, 10, -28, -18, 4)
\end{array} \Bigg\} \Rightarrow R^5$$

subspace W basis

subspace $W^{\perp}$ basis

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

$W = \text{row}(A)$

row space

$$\Downarrow \text{REF}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(W) = 3$

$$\left.\begin{array}{l}
w_1 = (1, 0, 0, 0, 2) \\
w_2 = (0, 1, -3, -2, 0) \\
w_3 = (0, 0, 1, 1, 0)
\end{array}\right\} W = \text{row}(A)$$

W basis $\quad$ row (A)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

$W = row(A)$

row space

$$\Big\Downarrow \text{RREF}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$dim(W) = 3$

$$\text{RREF} \Rightarrow \left.\begin{array}{l} w_1' = (1, 0, 0, 0, 2) \\ w_2' = (0, 1, 0, 1, 0) \\ w_3' = (0, 0, 1, 1, 0) \end{array}\right\} W = row(A)$$

$$\text{REF} \Rightarrow \left.\begin{array}{l} w_1 = (1, 0, 0, 0, 2) \\ w_2 = (0, 1, -3, -2, 0) \\ w_3 = (0, 0, 1, 1, 0) \end{array}\right\} W = row(A)$$

$W^{\perp}$ basis null (A)

$W^{\perp} \Rightarrow AX = 0$ solution space

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

leading variables

$x_1 + 2x_5 = 0$ $\qquad x_1 = -2x_5$

$x_2 + x_4 = 0$ $\qquad x_2 = -x_4$

$x_3 + x_4 = 0$ $\qquad x_3 = -x_4$

free variables

$$\boxed{\begin{aligned} x_5 &= s \\ x_4 &= t \end{aligned}}$$

$x_1 = -2s$

$x_2 = -t$

$x_3 = -t$

$x_4 = t$

$x_5 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$W^{\perp}$ basis $\begin{cases} u_1 = (-2,\ 0,\ 0,\ 0,\ 1)^T \\ u_2 = (0,\ -1,\ -1,\ 1,\ 0)^T \end{cases}$

basis for Col (A)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

$W = \text{row}(A)$

row space

$\Downarrow$ RREF (Reduced Row Echelon Form)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot column

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

Column space basis

$$A = \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array}$$
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

$C_4 = C_2 + C_3$
$C_5 = 2C_1$

$\Downarrow$ RREF

$$\begin{array}{c} \begin{array}{ccccc} C_1' & C_2' & C_3' & C_4' & C_5' \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

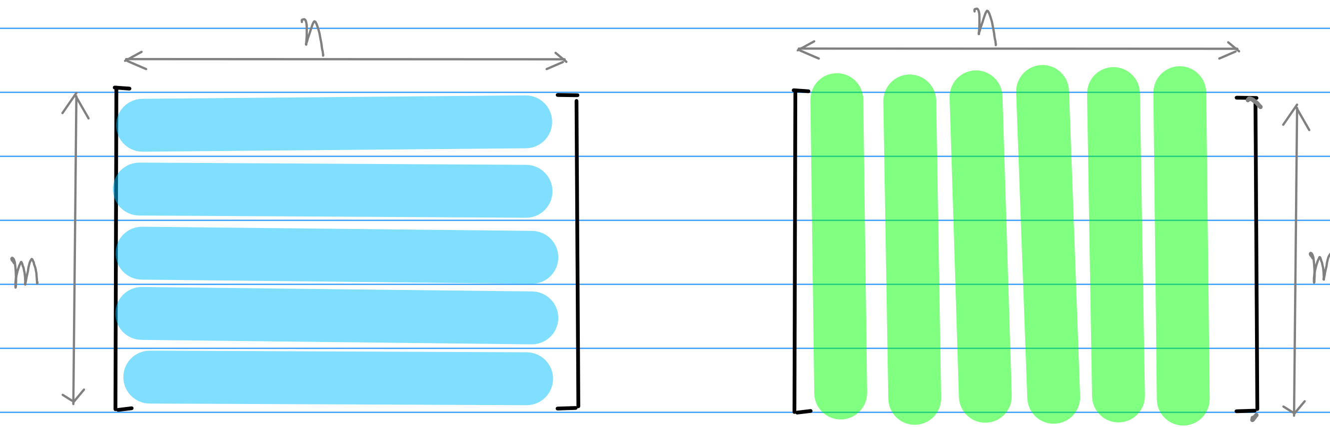
$C_4' = C_2' + C_3'$
$C_5' = 2C_1'$

Pivot column

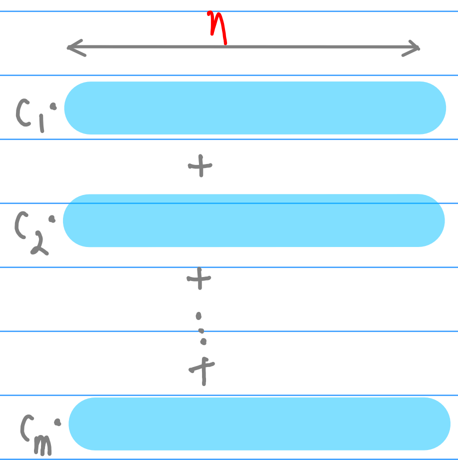
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

Column space basis

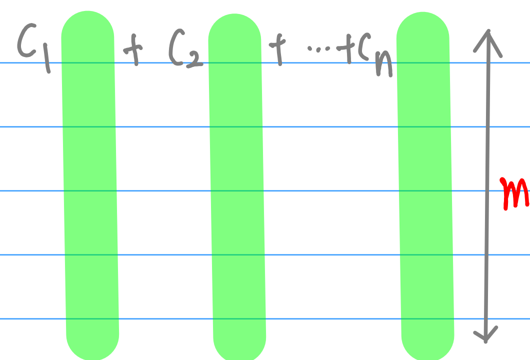
row(A) & col(A)



row(A) =



col(A) =



row(A) $\Rightarrow$ subspace of R^n

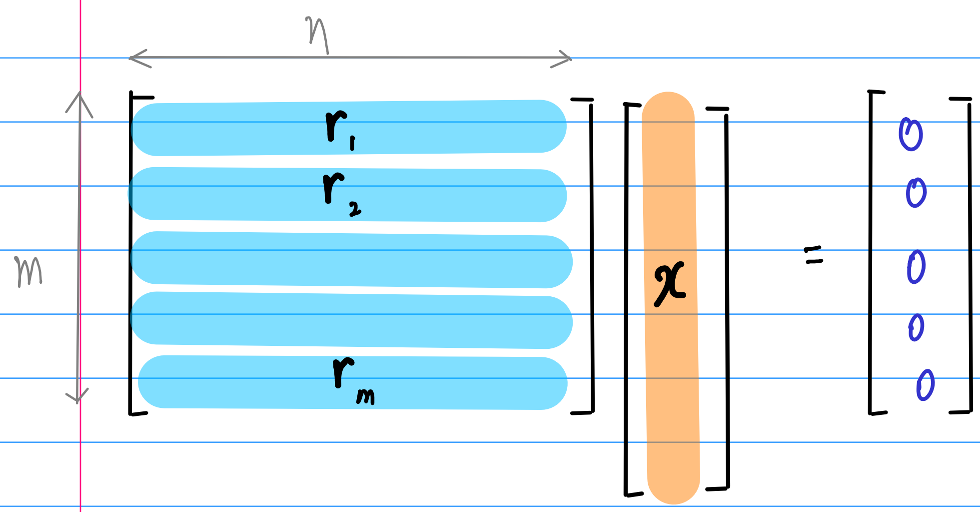
col(A) $\Rightarrow$ subspace of R^m

R^3

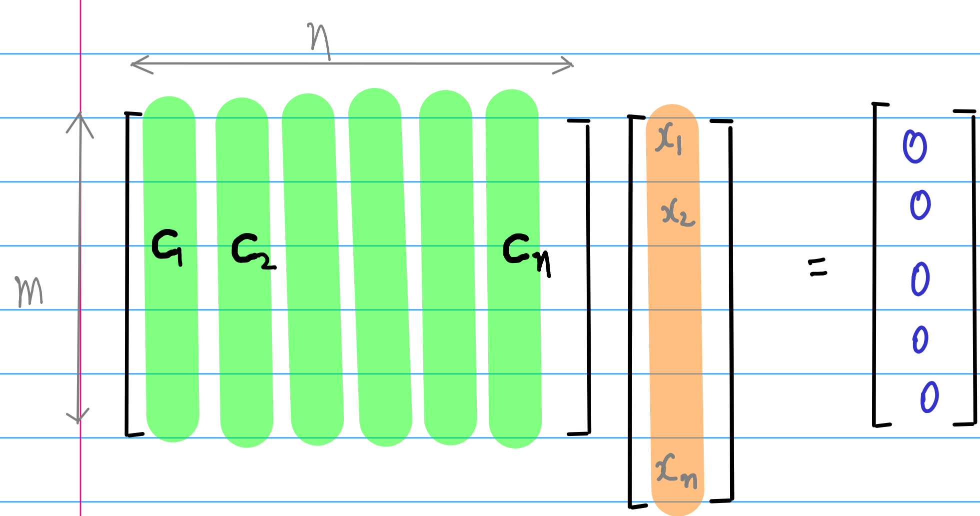
$(x, y, 0)$

R^3 $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

A x = 0



$$\boxed{\begin{aligned} r_1 \cdot x &= 0 \\ r_2 \cdot x &= 0 \\ &\vdots \\ r_m \cdot x &= 0 \end{aligned}}$$



$$\boxed{x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = 0}$$

$\boxed{A\,x = 0}$ & $\boxed{A\,x = b}$

$\boxed{A\,x = 0}$

$$\begin{bmatrix} r_1 \\ r_2 \\ \\ \\ r_m \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{matrix} r_1 \cdot x = 0 \\ r_2 \cdot x = 0 \\ \vdots \\ r_m \cdot x = 0 \end{matrix}}$$

$$(k_1 r_1 + k_2 r_2 + \cdots + k_m r_m) \cdot x = 0$$

$$\boxed{\text{row}(A) \perp \text{null}(A)}$$

$\boxed{A\,x = b}$

$$\begin{bmatrix} c_1 & c_2 & & & & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\boxed{x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = b}$$

$$\boxed{\begin{matrix} A x = b \\ \text{consistent} \end{matrix}} \iff \boxed{b \in \text{col}(A)}$$

A x = 0

$$\boxed{\mathrm{row}(A) \perp \mathrm{null}(A)}$$

A x = b

$$\boxed{\begin{array}{l} Ax = b \\ \quad \text{consistent} \end{array}} \iff \boxed{b \in \mathrm{col}(A)}$$

$$Ax = 0 \qquad A^T y = 0$$

$row(A) \perp null(A)$ $\qquad$ $row(A^T) \perp null(A^T)$

$col(A) \perp null(A^T)$

$$\boxed{\begin{array}{l} row(A) \perp null(A) \\ col(A) \perp null(A^T) \end{array}}$$

$row(A) \cup null(A) = R^n$

$col(A) \cup null(A^T) = R^n$

Orthogonal Complents

$row(A)^{\perp} = null(A)$ $\qquad$ $null(A)^{\perp} = row(A)$

$col(A)^{\perp} = null(A^T)$ $\qquad$ $null(A^T)^{\perp} = col(A)$

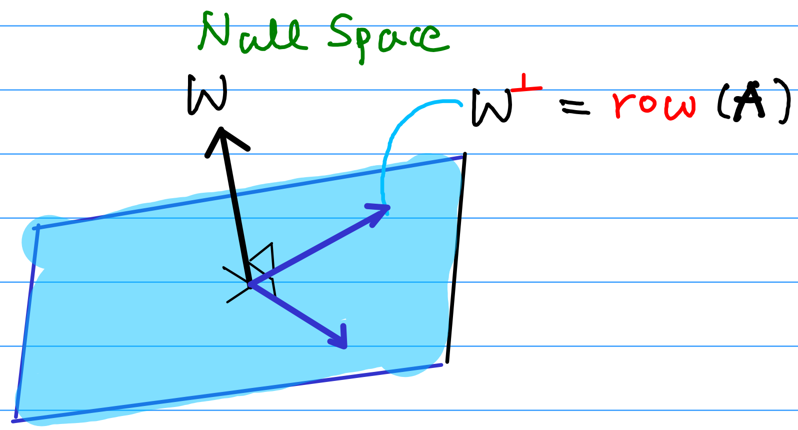
Orthogonal Complements

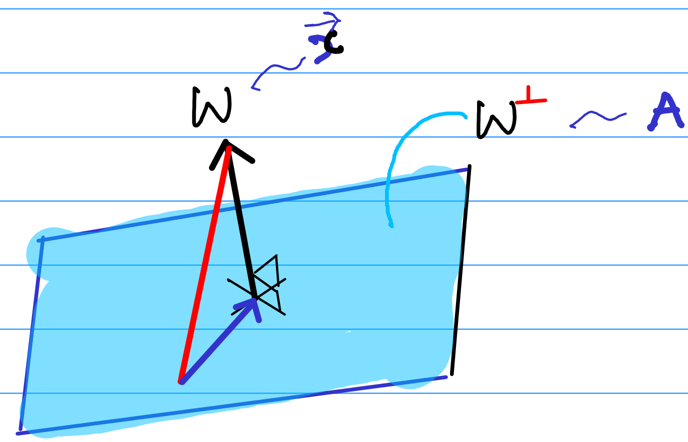
W : solution space $\Rightarrow$ Null Space

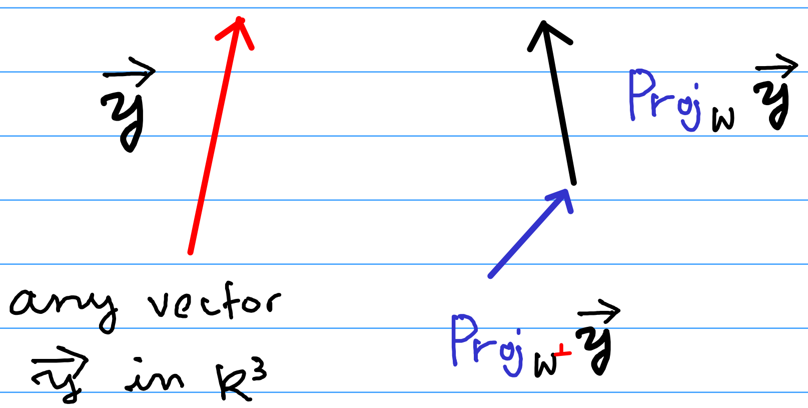
$$W = \{ x \mid Ax = 0 \}$$

Orthogonal complements of row (A)

ex) nullity = 1







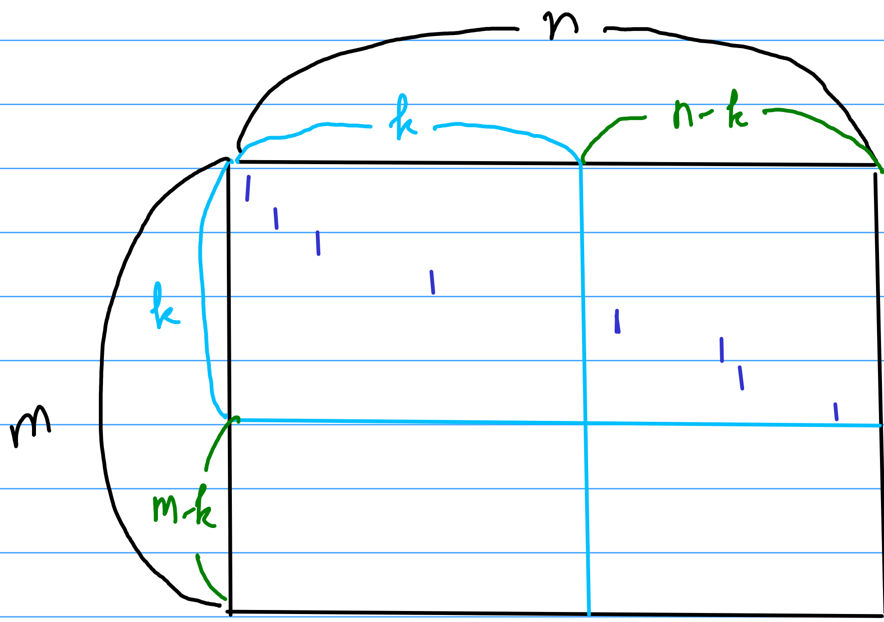
$$Ax = 0$$

$$\begin{array}{ccc} \mathrm{row}(A) & \perp & \mathrm{null}(A) \\ \Big\Downarrow \text{basis} & & \Big\Downarrow \text{basis} \\ \{v_1, v_2, \dots, v_k\} & & \{w_{k+1}, w_{k+2}, \dots, w_n\} \end{array}$$

$$\{v_1, v_2, \dots, v_k, w_{k+1}, w_{k+2}, \dots, w_n\} \quad \text{basis} \quad \mathbb{R}^n$$

$$Ax = 0$$

$$\text{row}(A) \perp \text{null}(A)$$



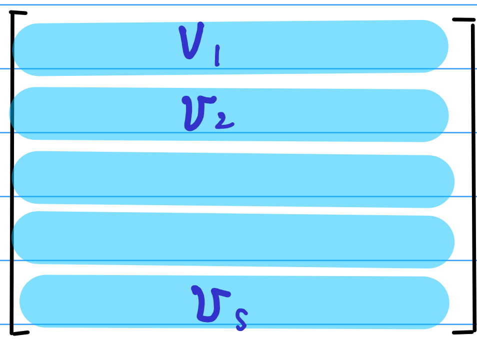
$\text{rank}(A) = k$

$\text{nullity}(A) = n-k$

Finding a basis in W

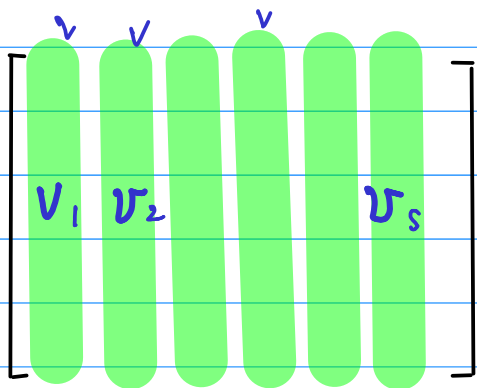
$\text{span}(S) = W \quad , \quad S = \{v_1, v_2, \cdots, v_s\}$

① Finding any basis in W



$\Rightarrow$ REF

② Finding a basis in W that includes vectors S



$\Rightarrow$ REF

pivot column

EX 1)

1

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & -4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

Col (A) basis

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -2 \\ -9 \\ -18 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 10 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ -14 \\ -28 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ -4 \\ 0 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}$$

EX 1) ②

RREF

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccc} & & & s & t \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

row(A) basis

$$\begin{matrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{matrix}$$

$\perp$

null(A) basis

$$\begin{matrix} 0 & -2 \\ -1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}$$

$3+2=5$

EX 1) ③

$$[A|I] = \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ -2 & 1 & -3 & -2 & -4 & 0 & 1 & 0 & 0 \\ 0 & 5 & -14 & -9 & 0 & 0 & 0 & 1 & 0 \\ 2 & 10 & -28 & -18 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

row (A^T)

$\boxed{\text{col }(A)\ \text{basis}}$

Elementary Matrix

$$[U|E] = \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & +\frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & 0 & -14 & 31 & -14 \\ 0 & 0 & 1 & 1 & 0 & 0 & -5 & 11 & -5 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

$\boxed{\text{null }(A^T)\ \text{basis}}$

$\begin{bmatrix} 1 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \perp \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ -14 \\ -28 \end{bmatrix}$

$3 + 1 = 4$

EX 2)

①

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

$\boxed{\begin{array}{l} \text{Col}(A) \\ \text{basis} \end{array}}$

$$RREF(A) = \begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -14 \\ 3 \\ 0 \\ 0 \end{bmatrix} = -14 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -37 \\ 4 \\ 5 \\ 0 \end{bmatrix} = -37 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ -6 \\ -6 \\ 3 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} -2 \\ -1 \\ -1 \\ 2 \end{bmatrix} = -14 \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 2 \\ 7 \\ -4 \end{bmatrix} = -37 \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} + 5 \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

EX 2) ②

RREF

$$\begin{array}{cccccc} & s & & t & & u \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array}$$
$$\begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 14 \\ 6 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} 37 \\ 6 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

$\boxed{\text{row}(A) \text{ basis}}$

$$\begin{matrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{matrix}$$

$\perp$

$\boxed{\text{null}(A) \text{ basis}}$

$$\begin{matrix} 3 & 14 & 37 \\ 1 & 6 & 6 \\ 0 & -3 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & 1 \end{matrix}$$

$3+3=6$

EX 2) ③

$$[A|I] = \left[\begin{array}{cccccc|cccc} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 2 & -6 & 9 & -1 & 8 & 2 & 0 & 1 & 0 & 0 \\ 2 & -6 & 9 & -1 & 9 & 7 & 0 & 0 & 1 & 0 \\ -1 & 3 & -4 & 2 & -5 & -4 & 0 & 0 & 0 & 1 \end{array}\right]$$

row (A^T)

$\boxed{\text{col } (A) \text{ basis}}$

Elementary Matrix

$$[U|E] = \left[\begin{array}{cccccc|cccc} 1 & -3 & 0 & -14 & 0 & -37 & 0 & 9 & -13 & -9 \\ 0 & 0 & 1 & 3 & 0 & 4 & 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\boxed{\text{null } (A^T) \text{ basis}}$

$3 + 1 = 4$

$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$ $\perp$ $\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$

EX 3)

1

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

$\boxed{\text{Col}(A) \text{ basis}}$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -5 \\ -3 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ -8 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \\ -7 \end{bmatrix}$$

EX 3) ②

RREF

$$\begin{array}{ccccc} & & s & & t \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\boxed{\text{row}(A) \text{ basis}}$

$$\begin{matrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{matrix}$$

$\perp$

$\boxed{\text{null}(A) \text{ basis}}$

$$\begin{matrix} -2 & -1 \\ 1 & -1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{matrix}$$

$3+2=5$

EX 3)

3

$$[A|I] = \left[\begin{array}{ccccc|cccc} 1 & 2 & 0 & 2 & 5 & 1 & 0 & 0 & 0 \\ -2 & -5 & 1 & -1 & -8 & 0 & 1 & 0 & 0 \\ 0 & -3 & 3 & 4 & 1 & 0 & 0 & 1 & 0 \\ 3 & 6 & 0 & -7 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

row (A^T)

$\boxed{\text{col } (A) \text{ basis}}$

Elementary Matrix

$$[U|E] = \left[\begin{array}{ccccc|cccc} 1 & 0 & 2 & 0 & 1 & 0.00000 & 0.04762 & 0.65079 & 0.36508 \\ 0 & 1 & -1 & 0 & 1 & -0.00000 & -0.19048 & -0.26984 & -0.12698 \\ 0 & 0 & 0 & 1 & 1 & -0.00000 & -0.14286 & 0.04762 & -0.09524 \\ 0 & 0 & 0 & 0 & 0 & 1.00000 & 0.61905 & -0.20635 & 0.07937 \end{array}\right]$$

$\boxed{\text{null } (A^T) \text{ basis}}$

$3+1=4$

1.00000 0.61905 -0.20635 0.07937 $\perp$ $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -5 \\ -3 \\ 6 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -1 \\ 4 \\ -7 \end{bmatrix}$

```
A = [1 0 0 0 2; -2 1 -3 -2 -4; 0 5 -14 -9 0; 2 10 -28 -18 4]
rref(A)
I = eye(4)
C = [ A I]
rref(C)
rref(C)
D = rref(C)
E = [ D(:,6) D(:,7) D(:,8) D(:,9) ]
E * A
A = [1 -3 4 -2 5 4; 2 -6 9 -1 8 2 ; 2 -6 9 -1 9 7; -1 3 -4 2 -5 -4]
C = [ A I]
D = rref(C)
E = [ D(:,7) D(:,8) D(:,9) D(:,10)]
E * A
A
A = [1 2 0 2 5; -2 -5 1 -1 -8; 0 -3 3 4 1; 3 6 0 -7 2 ]
C = [ A I]
D = rref(C)
A
B
C = [ A I]
E = [ D(:,6) D(:,7) D(:,8) D(:,9) ]
E * A
```