

Fundamental Vector Spaces (H.1)

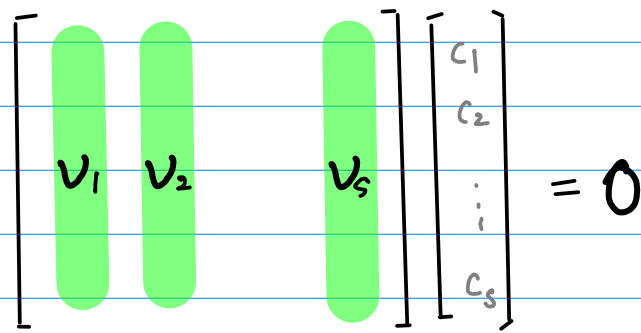
20151211

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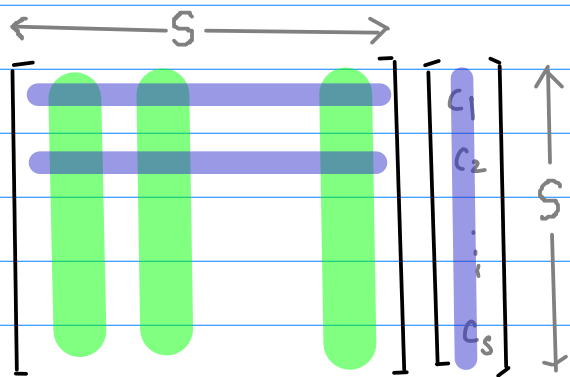
$\{v_1, v_2, \dots, v_s\}$ linear independent

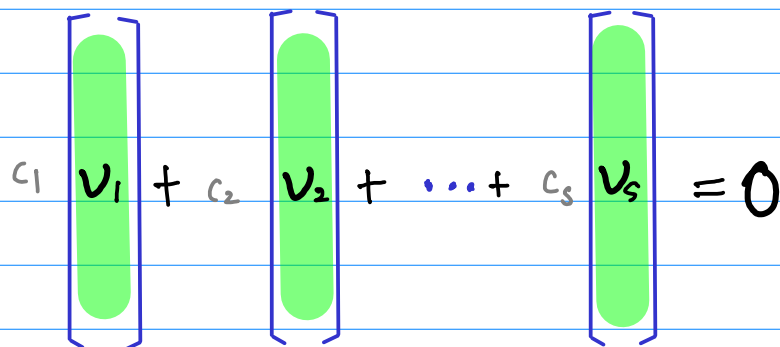
$$c_1 v_1 + c_2 v_2 + \dots + c_s v_s = 0 \quad c_1 = c_2 = \dots = c_s = 0$$



A diagram representing the matrix equation $[v_1 \ v_2 \ \dots \ v_s] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{bmatrix} = 0$. The vectors v_1, v_2, \dots, v_s are shown as green vertical bars within a large square bracket. To their right is a column vector of coefficients c_1, c_2, \dots, c_s also enclosed in a square bracket. An equals sign followed by a zero is to the right of the coefficient vector.

only
trivial solution
 $c_1 = c_2 = \dots = c_s = 0$



$$c_1 v_1 + c_2 v_2 + \dots + c_s v_s = 0$$


A diagram showing the linear combination $c_1 v_1 + c_2 v_2 + \dots + c_s v_s = 0$. Each vector v_1, v_2, \dots, v_s is enclosed in a blue square bracket. The coefficients c_1, c_2, \dots, c_s are placed to the left of their respective vectors. The entire expression is followed by an equals sign and a zero.

Consistent Linear System

- Consistent homogeneous linear system

$Ax = 0$ W x_h homogeneous solution
solution space

translation ↓

$Ax = b$ $W + x_0$ x_p particular solution
solution space

- Consistent non-homogeneous linear system

$$Ax_h = 0$$

$$+) \quad Ax_p = b$$

$$A(x_h + x_p) = b \quad \Rightarrow \quad (x_h + x_p) \text{ is also solution to}$$

$$Ax = b$$

"general"
✓

⑥ consistent homogeneous linear system

homogeneous sol.

$$\boxed{Ax = 0}$$

W : Null space

$$\mathbf{x}_h$$

always consistent

$$\left\{ \begin{array}{l} \text{rank}(A) = n \Rightarrow \text{unique solution } \mathbf{x} = \mathbf{0} \quad (\because A^{-1}\mathbf{0}) \\ \text{rank}(A) < n \Rightarrow \text{infinitely many solution} \\ \quad (n-r) \text{ parameters} = \text{nullity} \end{array} \right.$$

⑥ consistent non-homogeneous linear system

$$\boxed{Ax = b}$$

$\mathbf{x}_p + W$: solution space

$$\mathbf{x}_p$$

$$\text{rank}(A) = \text{rank}(A|b) \Rightarrow \text{consistent}$$

b is in column space of A

$$\left\{ \begin{array}{l} \text{rank}(A) = n \Rightarrow \text{unique solution } \mathbf{x} = A^{-1}b \\ \text{rank}(A) < n \Rightarrow \text{infinitely many solution} \\ \quad (n-r) \text{ parameters} = \text{nullity} \end{array} \right.$$

$$\text{rank}(A) < \text{rank}(A|b) \Rightarrow \text{inconsistent} \quad \text{no solution}$$

Translation

Line

$$\mathcal{X} = t \mathbf{v}$$

↓ translate

$$\mathcal{X} = t \mathbf{v} + \mathbf{x}_0$$

$$\mathcal{X} = t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2$$

↓ translate

$$\mathcal{X} = t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \mathbf{x}_0$$

plane

$$\mathcal{X} = t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \dots + t_s \mathbf{v}_s$$

↓ translate

$$\mathcal{X} = t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \dots + t_s \mathbf{v}_s + \mathbf{x}_0$$

hyperplane

$$A \vec{x} = \vec{0} \quad \mathcal{W}: \text{solution space}$$

↓ translation

$$A \vec{x} = \vec{b} \quad \mathcal{W} + \mathbf{x}_0: \text{solution space}$$

Line, Plane, Hyperplane

Line

$$a_1 x_1 + a_2 x_2 = b$$

$$\vec{a} \cdot \vec{x} = b$$

$$(a_1 \neq 0) \& (a_2 \neq 0)$$

Plane

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = b$$

$$\vec{a} \cdot \vec{x} = b$$

$$(a_1 \neq 0) \& (a_2 \neq 0) \& (a_3 \neq 0)$$

Hyperplane
in \mathbb{R}^n

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

$$\vec{a} \cdot \vec{x} = b$$

$$(a_1 \neq 0) \& (a_2 \neq 0) \& \dots \& (a_n \neq 0)$$

Hyperplane

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$$

$$\vec{a} \cdot \vec{x} = 0$$

in \mathbb{R}^n that
passes through
the origin

$$(a_1 \neq 0) \& (a_2 \neq 0) \& \dots \& (a_n \neq 0)$$

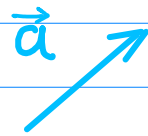
Hyperplane : the orthogonal complement of a line

$$\boxed{a \cdot x = b}$$

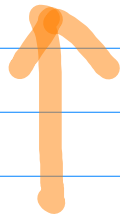
$$\Leftrightarrow A \cdot x = b$$

$$(a_1, a_2, \dots, a_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

compare

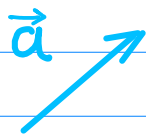


normal vector



translate

$$\boxed{a \cdot x = 0}$$



normal vector

hyperplane
through the origin
with normal a

Orthogonal
complement
of a

line

$$\{x \mid x \in \mathbb{R}^n, x \perp a\}$$

for a given $a \neq 0$

Hyperplane a^\perp (a perp)

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

$$a \cdot x = b \quad (a \neq 0)$$

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$$

$$a \cdot x = 0 \quad (a \neq 0)$$

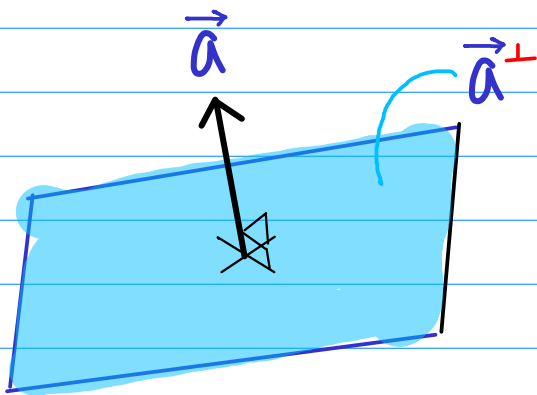
hyperplane that
pass through the origin
with a normal vector a

the normal vector

Orthogonal complement of a

a^\perp (a perp)

a normal vector



(a perp)

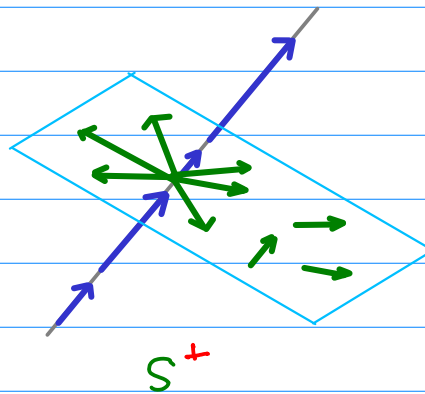
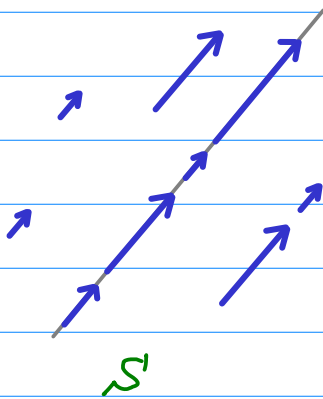
hyperplane
through the origin

Orthogonal Complement

S : a non-empty set in \mathbb{R}^n

S^\perp : the orthogonal complement of S

= { all vectors in \mathbb{R}^n
that are orthogonal
to every vector in S }



in this case,
also a hyperplane
since S is a set
of vectors in a line

$$\dim(S) = 1$$

$$\dim(S^\perp) = 2$$

$$\dim(S) + \dim(S^\perp) = n \quad (\mathbb{R}^n)$$

Null Space

$$A \cdot x = 0$$

$$\begin{matrix} & \xrightarrow{n} \\ \begin{matrix} \uparrow m \\ \left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_m \end{array} \right] \end{matrix} \cdot \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{matrix}$$

$$\begin{array}{l} a_1 \cdot x = 0 \\ a_2 \cdot x = 0 \\ \vdots \\ a_m \cdot x = 0 \end{array}$$

solution space \Rightarrow $\text{null}(A)$

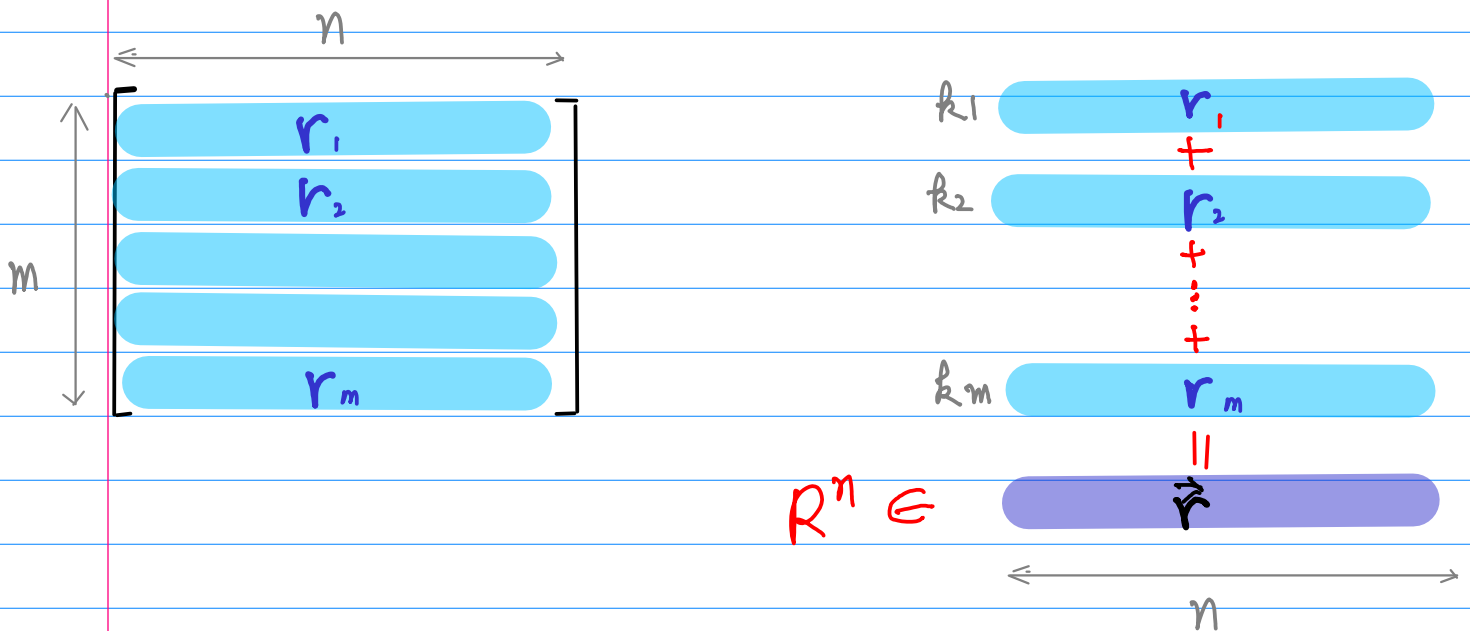
$$\begin{aligned} &= \{ x \mid a_1 \cdot x = 0 \ \& \ a_2 \cdot x = 0 \ \& \ \dots \ \& \ a_m \cdot x = 0 \} \\ &= \{ x \mid a_1 \perp x \ \& \ a_2 \perp x \ \& \ \dots \ \& \ a_m \perp x \} \end{aligned}$$

$$\text{null}(A) = \left\{ \begin{array}{c} \uparrow n \\ x \end{array} \right\}$$

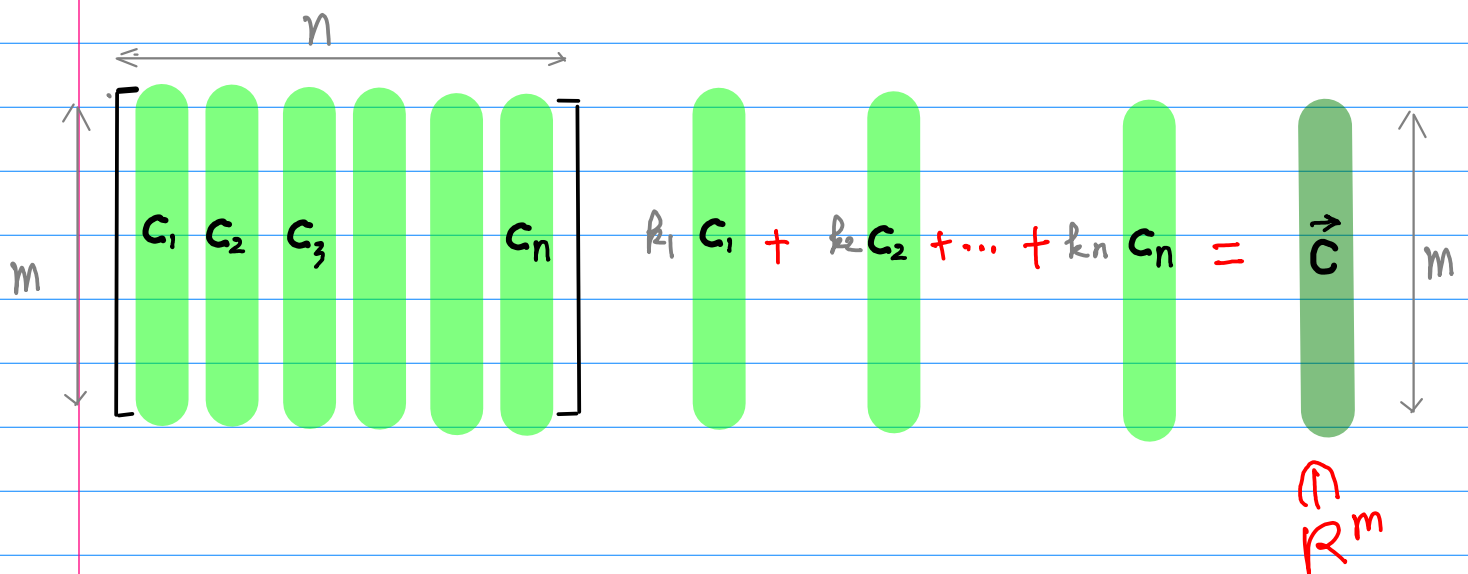
all vectors in \mathbb{R}^n
that are orthogonal to
every row vector of A

Row Space & Column Space

the row space of A $\text{row}(A)$
 the subspace of \mathbb{R}^n
 that is spanned by the row vectors of A



the column space of A $\text{col}(A)$
 the subspace of \mathbb{R}^m
 that is spanned by the column vectors of A



Finding W basis and W^\perp basis

Anton's Examples

$$\begin{array}{l}
 \textcircled{1} \quad v_1 = (1, 0, 0, 0, 2) \\
 \textcircled{2} \quad v_2 = (-2, 1, -3, -2, +4) \\
 \textcircled{3} \quad v_3 = (0, 5, -14, -9, 0) \\
 \textcircled{4} \quad v_4 = (2, 10, -28, -18, 4) \\
 \textcircled{5}
 \end{array}
 \left. \vphantom{\begin{array}{l} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}} \right\} \Rightarrow \mathbb{R}^5 \text{ subspace } W \text{ basis}$$

$$\left. \vphantom{\begin{array}{l} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}} \right\} \Rightarrow \mathbb{R}^5 \text{ subspace } W^\perp \text{ basis}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

$$W = \text{row}(A)$$

row space

\Downarrow REF

$$\begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & 2 \\ 0 & \textcircled{1} & -3 & -2 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim(W) = 3$$

$$\left. \begin{array}{l} w_1 = (1, 0, 0, 0, 2) \\ w_2 = (0, 1, -3, -2, 0) \\ w_3 = (0, 0, 1, 1, 0) \end{array} \right\} W = \text{row}(A)$$

W basis

row(A)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & 4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

$W = \text{row}(A)$

row space

\Downarrow RREF

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim(W) = 3$$

RREF \Rightarrow

$$\left. \begin{aligned} w_1' &= (1, 0, 0, 0, 2) \\ w_2' &= (0, 1, 0, 1, 0) \\ w_3' &= (0, 0, 1, 1, 0) \end{aligned} \right\}$$

$W = \text{row}(A)$

REF \Rightarrow

$$\left. \begin{aligned} w_1 &= (1, 0, 0, 0, 2) \\ w_2 &= (0, 1, -3, -2, 0) \\ w_3 &= (0, 0, 1, 1, 0) \end{aligned} \right\}$$

$W = \text{row}(A)$

W^\perp basis

$\text{null}(A)$

$W^\perp \Rightarrow AX=0$ solution space

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

leading variables

$$x_1 + 2x_5 = 0$$

$$x_2 + x_4 = 0$$

$$x_3 + x_4 = 0$$

$$x_1 = -2x_5$$

$$x_2 = -x_4$$

$$x_3 = -x_4$$

free variables

$$x_5 = s$$

$$x_4 = t$$

$$x_1 = -2s$$

$$x_2 = -t$$

$$x_3 = -t$$

$$x_4 = t$$

$$x_5 = s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

W^\perp basis

$$\begin{cases} u_1 = (-2, 0, 0, 0, 1)^T \\ u_2 = (0, -1, -1, 1, 0)^T \end{cases}$$

basis for $\text{col}(A)$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

$W = \text{row}(A)$

row space

\Downarrow RREF (Reduced Row Echelon Form)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot column

$$A = \begin{bmatrix} \downarrow 1 & \downarrow 0 & \downarrow 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ \downarrow 2 & \downarrow 10 & \downarrow -28 & -18 & 4 \end{bmatrix}$$

Column space basis

$$A = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 & C_5 \\ 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

$$C_4 = C_2 + C_3$$

$$C_5 = 2C_1$$

↓ RREF

$$C'_1 \quad C'_2 \quad C'_3 \quad C'_4 \quad C'_5$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C'_4 = C'_2 + C'_3$$

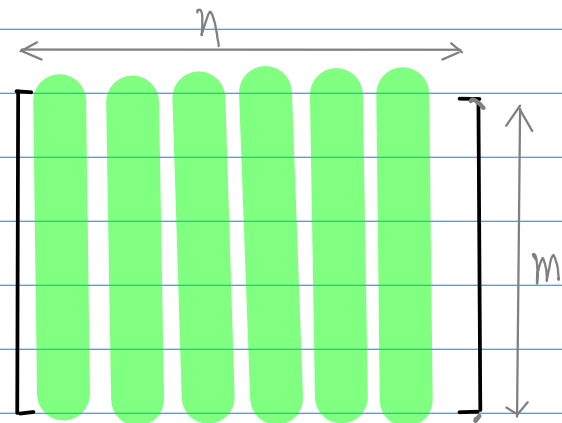
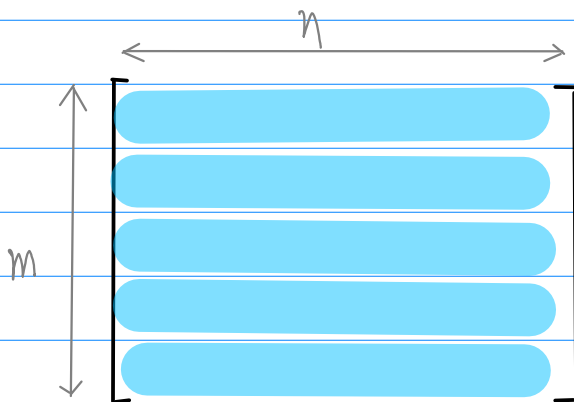
$$C'_5 = 2C'_1$$

pivot column

$$A = \begin{bmatrix} \downarrow & \downarrow & \downarrow & & \\ 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & +4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

↓ ↓ ↓
Column space basis

row(A) & col(A)



$$\begin{aligned} \text{row}(A) = & c_1 \cdot \text{---} \\ & + \\ & c_2 \cdot \text{---} \\ & + \\ & \vdots \\ & + \\ & c_m \cdot \text{---} \end{aligned}$$

The diagram shows the row space of A as a linear combination of rows. Each row is represented by a horizontal blue bar. The width of the bars is labeled n at the top. The rows are labeled c_1 , c_2 , and c_m on the left.

$$\text{col}(A) = c_1 \cdot \text{---} + c_2 \cdot \text{---} + \dots + c_n \cdot \text{---}$$

The diagram shows the column space of A as a linear combination of columns. Each column is represented by a vertical green bar. The height of the bars is labeled m on the right. The columns are labeled c_1 , c_2 , and c_n at the top.

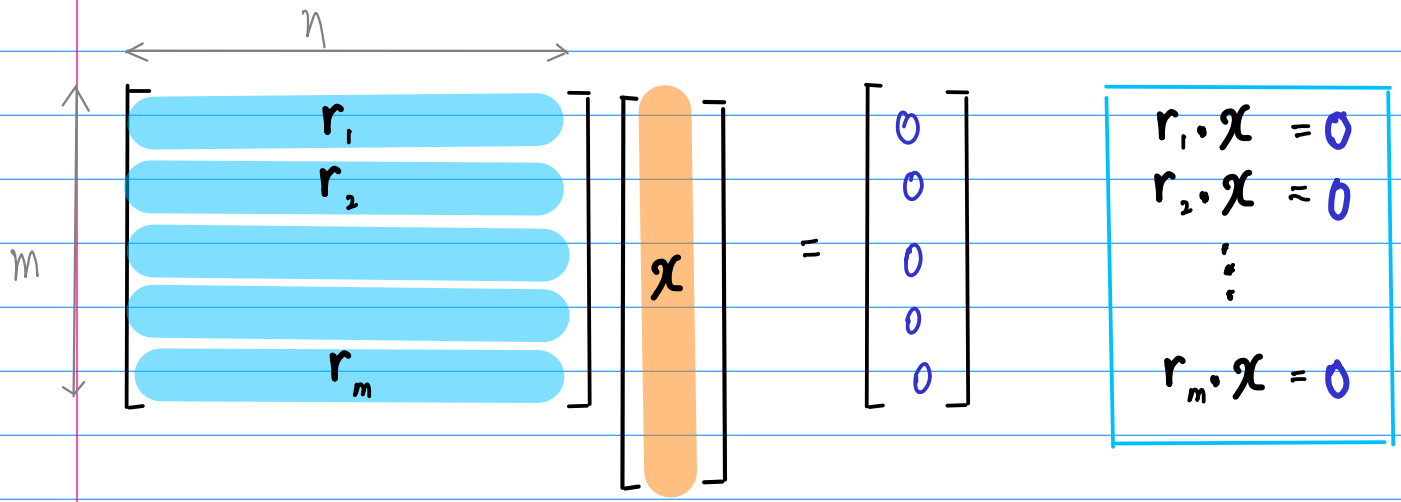
$\text{row}(A) \Rightarrow$ subspace of \mathbb{R}^n

$\text{col}(A) \Rightarrow$ subspace of \mathbb{R}^m

$$\mathbb{R}^3$$
$$(x, y, 0)$$

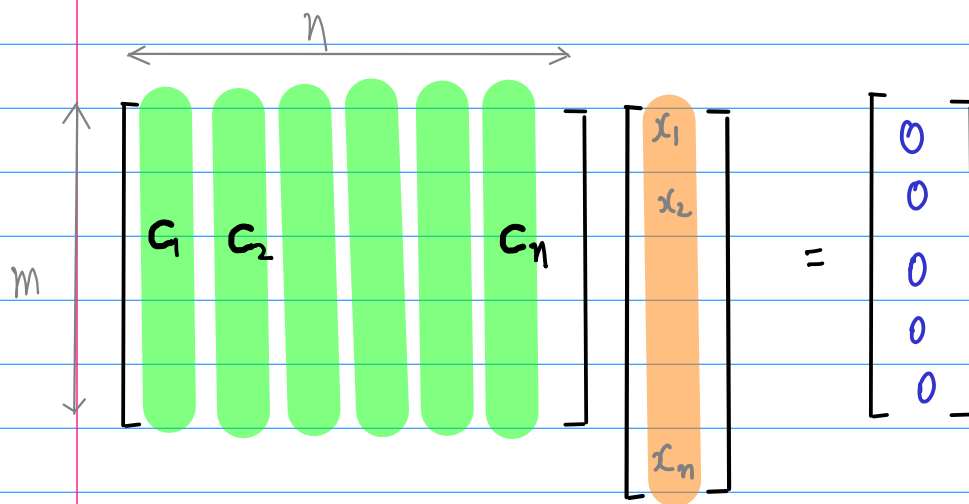
$$\mathbb{R}^3 \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\mathbf{A} \mathbf{x} = \mathbf{0}$$



A diagram illustrating the matrix equation $\mathbf{A} \mathbf{x} = \mathbf{0}$. The matrix \mathbf{A} is represented as a collection of m horizontal blue rounded rectangles representing row vectors r_1, r_2, \dots, r_m . The width of the matrix is labeled n . To the right of the matrix is a vertical orange rounded rectangle representing the vector \mathbf{x} . An equals sign follows, leading to a vertical column of five zeros. To the right of this is a blue-bordered box containing the following equations:

$$\begin{aligned} r_1 \cdot \mathbf{x} &= 0 \\ r_2 \cdot \mathbf{x} &= 0 \\ &\vdots \\ r_m \cdot \mathbf{x} &= 0 \end{aligned}$$



A diagram illustrating the matrix equation $\mathbf{A} \mathbf{x} = \mathbf{0}$ using column vectors. The matrix \mathbf{A} is represented as a collection of n vertical green rounded rectangles representing column vectors c_1, c_2, \dots, c_n . The height of the matrix is labeled m . To the right of the matrix is a vertical orange rounded rectangle representing the vector \mathbf{x} , with its components x_1, x_2, \dots, x_n labeled. An equals sign follows, leading to a vertical column of five zeros.

$$x_1 c_1 + x_2 c_2 + \dots + x_n c_n = \mathbf{0}$$

$$\mathbf{A} \mathbf{x} = \mathbf{0}$$

&

$$\mathbf{A} \mathbf{x} = \mathbf{b}$$

$$\mathbf{A} \mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$r_1 \cdot x = 0$
 $r_2 \cdot x = 0$
 \vdots
 $r_m \cdot x = 0$

$$(k_1 r_1 + k_2 r_2 + \dots + k_m r_m) \cdot x = 0$$

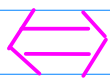
$$\text{row}(A) \perp \text{null}(A)$$

$$\mathbf{A} \mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 c_1 + x_2 c_2 + \dots + x_n c_n = b$$

$$\mathbf{A} \mathbf{x} = \mathbf{b} \text{ consistent}$$



$$b \in \text{col}(A)$$

$$\mathbf{A} \mathbf{x} = \mathbf{0}$$

$$\text{row}(A) \perp \text{null}(A)$$

$$\mathbf{A} \mathbf{x} = \mathbf{b}$$

$$\begin{array}{l} \mathbf{A} \mathbf{x} = \mathbf{b} \\ \text{consistent} \end{array} \iff \mathbf{b} \in \text{col}(A)$$

$$A x = 0$$

$$A^T y = 0$$

$$\text{row}(A) \perp \text{null}(A)$$

$$\text{row}(A^T) \perp \text{null}(A^T)$$

$$\text{col}(A) \perp \text{null}(A^T)$$

$$\text{row}(A) \perp \text{null}(A)$$

$$\text{col}(A) \perp \text{null}(A^T)$$

$$\text{row}(A) \cup \text{null}(A) = \mathbb{R}^n$$

$$\text{col}(A) \cup \text{null}(A^T) = \mathbb{R}^n$$

Orthogonal Complements

$$\text{row}(A)^\perp = \text{null}(A)$$

$$\text{null}(A)^\perp = \text{row}(A)$$

$$\text{col}(A)^\perp = \text{null}(A^T)$$

$$\text{null}(A^T)^\perp = \text{col}(A)$$

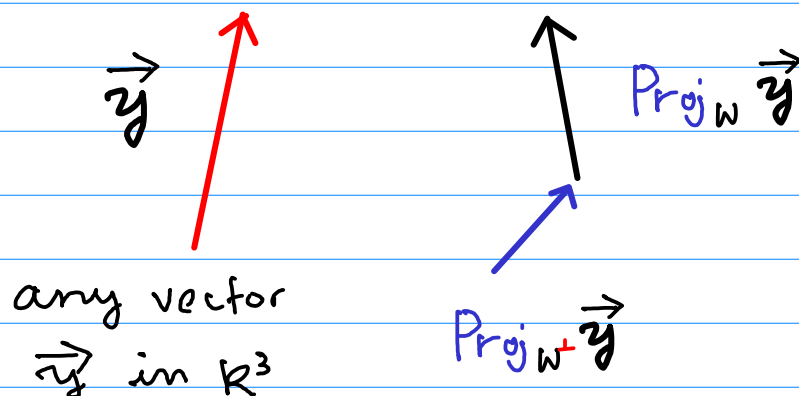
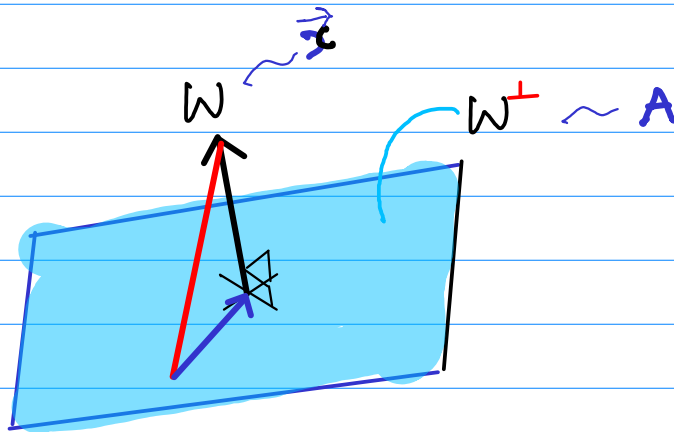
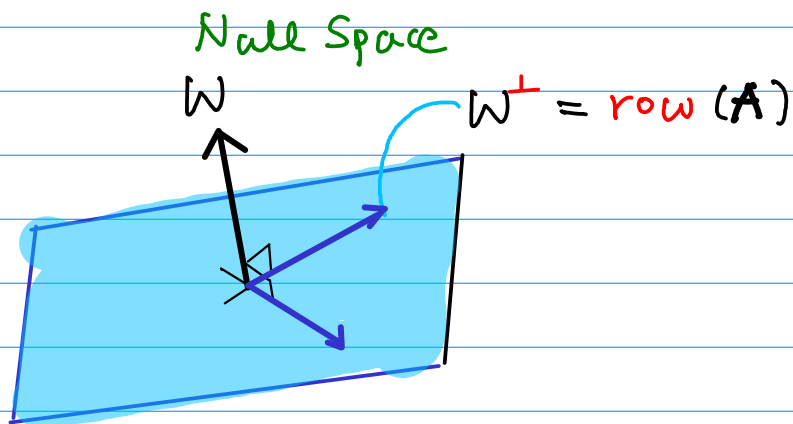
Orthogonal Complements

W : solution space \Rightarrow Null Space

$$W = \{x \mid Ax = 0\}$$

Orthogonal Complements of $\text{row}(A)$

ex) nullity = 1



$$A x = 0$$

$$\text{row}(A) \quad \perp \quad \text{null}(A)$$

\Downarrow basis

$\{v_1, v_2, \dots, v_k\}$

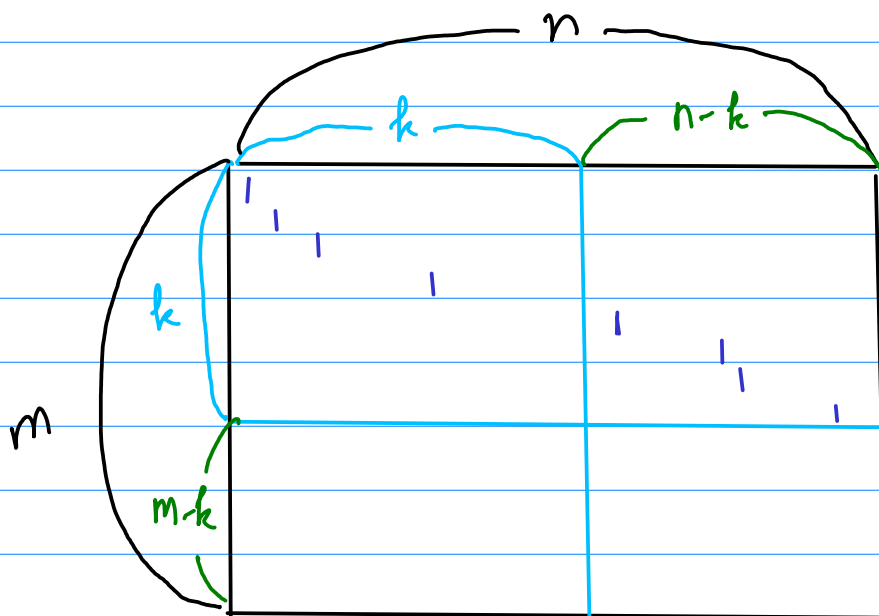
\Downarrow basis

$\{w_{k+1}, w_{k+2}, \dots, w_n\}$

$\{v_1, v_2, \dots, v_k, w_{k+1}, w_{k+2}, \dots, w_n\}$ basis \mathbb{R}^n

$$A x = 0$$

$\text{row}(A) \perp \text{null}(A)$



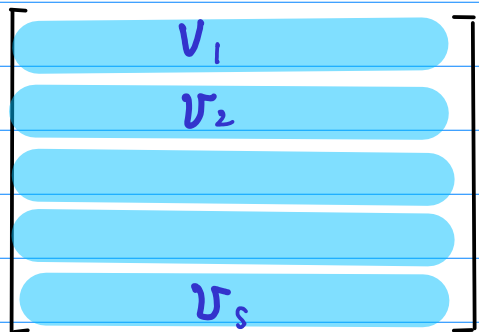
$$\text{rank}(A) = k$$

$$\text{nullity}(A) = n - k$$

Finding a basis in W

$$\text{span}(S) = W, \quad S = \{v_1, v_2, \dots, v_s\}$$

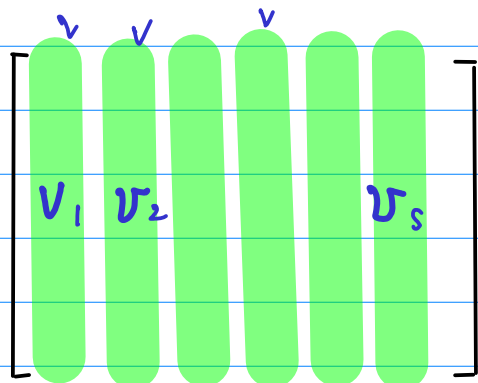
① Finding any basis in W



A matrix with s rows, each highlighted in light blue. The first row is labeled v_1 , the second row is labeled v_2 , and the last row is labeled v_s . The matrix is enclosed in large square brackets. To the right of the matrix is a red arrow pointing to the text "REF".

$$\Rightarrow \text{REF}$$

② Finding a basis in W that includes vectors v_i



A matrix with s columns, each highlighted in light green. The first column is labeled v_1 , the second column is labeled v_2 , and the last column is labeled v_s . Each column has a small blue arrow pointing to its top. The matrix is enclosed in large square brackets. To the right of the matrix is a red arrow pointing to the text "REF".

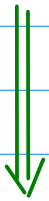
$$\Rightarrow \text{REF}$$

pivot column

EX 1)
①

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & -4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

col(A)
basis



$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -2 \\ -9 \\ -18 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 10 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ -14 \\ -28 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -4 \\ 0 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}$$

EX 1)
②

RREF

$$\begin{array}{ccccc} & & & s & t \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

row(A) basis

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

null(A) basis

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

⊥

$$3+2=5$$

EX 1)

③

$$[A|I] = \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ -2 & 1 & -3 & -2 & -4 & 0 & 1 & 0 & 0 \\ 0 & 5 & -14 & -9 & 0 & 0 & 0 & 1 & 0 \\ 2 & 10 & -28 & -18 & 4 & 0 & 0 & 0 & 1 \end{array} \right]$$

↑ ↑ ↑

row(A^T)

col(A)
basis

Elementary Matrix

$$[U|E] = \left[\begin{array}{ccccc|cccc} \textcircled{1} & 0 & 0 & 0 & 2 & 0 & 0 & -1 & +\frac{1}{2} \\ 0 & \textcircled{1} & 0 & 1 & 0 & 0 & -14 & 31 & -14 \\ 0 & 0 & \textcircled{1} & 1 & 0 & 0 & -5 & 11 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -\frac{1}{2} \end{array} \right]$$

null(A^T)
basis

$$\left[1 \quad 0 \quad 1 \quad -\frac{1}{2} \right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 5 & -14 \\ 2 & 10 & -28 \end{bmatrix}$$

⊥

3 + 1 = 4

EX 2)
①

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

col(A)
basis

$$RREF(A) = \begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -14 \\ 3 \\ 0 \\ 0 \end{bmatrix} = -14 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -37 \\ 4 \\ 5 \\ 0 \end{bmatrix} = -37 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ -6 \\ -6 \\ 3 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ -1 \\ -1 \\ 2 \end{bmatrix} = -14 \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 2 \\ 7 \\ -4 \end{bmatrix} = -37 \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} + 5 \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

EX 2)
②

RREF

$$\begin{array}{c}
 \begin{matrix} s & t & u \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix} \\
 \left[\begin{array}{cccccc}
 1 & -3 & 0 & -14 & 0 & -37 \\
 0 & 0 & 1 & 3 & 0 & 4 \\
 0 & 0 & 0 & 0 & 1 & 5 \\
 0 & 0 & 0 & 0 & 0 & 0
 \end{array} \right]
 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 14 \\ 6 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 37 \\ 6 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

row(A) basis

$$\begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\
 0 & 0 & 1 & 3 & 0 & 4 \\
 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

null(A) basis

$$\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 14 \\ 6 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 37 \\ 6 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

⊥

3 + 3 = 6

EX 2) ③

$$[A|I] = \left[\begin{array}{cccccc|cccc} 1 & -3 & 4 & -2 & 5 & 4 & 1 & 0 & 0 & 0 \\ 2 & -6 & 9 & -1 & 8 & 2 & 0 & 1 & 0 & 0 \\ 2 & -6 & 9 & -1 & 9 & 7 & 0 & 0 & 1 & 0 \\ -1 & 3 & -4 & 2 & -5 & -4 & 0 & 0 & 0 & 1 \end{array} \right]$$

row (A^T)
col (A)
basis

$$[U|E] = \left[\begin{array}{cccccc|cccc} 1 & -3 & 0 & -14 & 0 & -37 & 0 & 9 & -13 & -9 \\ 0 & 0 & 1 & 3 & 0 & 4 & 0 & 7 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Elementary Matrix
null (A^T)
basis

$3 + 1 = 4$

$$\begin{array}{c} \underline{1 \ 0 \ 0 \ 1} \\ \perp \\ \begin{array}{ccc} 1 & 4 & 5 \\ 2 & 9 & 8 \\ 2 & 9 & 9 \\ -1 & -4 & -5 \end{array} \end{array}$$

EX 3) ①

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -1 & 2 \end{bmatrix}$$

col(A)
basis

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -5 \\ -3 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 5 \\ -8 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 4 \\ -1 \end{bmatrix}$$

EX 3)
②

RREF

$$\begin{array}{c} \begin{array}{cc} s & t \end{array} \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

row(A) basis

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

null(A) basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

⊥

$$3+2=5$$

EX 3)

3)

$$[A|I] = \left[\begin{array}{ccccc|cccc} 1 & 2 & 0 & 2 & 5 & 1 & 0 & 0 & 0 \\ -2 & -5 & 1 & -1 & -8 & 0 & 1 & 0 & 0 \\ 0 & -3 & 3 & 4 & 1 & 0 & 0 & 1 & 0 \\ 3 & 6 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \text{row}(A^T) \\ \text{col}(A) \\ \text{basis} \end{array}$$

↑ ↑ ↑

Elementary Matrix

$$[U|E] = \left[\begin{array}{ccccc|cccc} 1 & 0 & 2 & 0 & 1 & 0.00000 & 0.04762 & 0.65079 & 0.36508 \\ 0 & 1 & -1 & 0 & 0 & -0.00000 & -0.19048 & -0.26984 & -0.12698 \\ 0 & 0 & 0 & 1 & 0 & -0.00000 & -0.14286 & 0.04762 & -0.09524 \\ 0 & 0 & 0 & 0 & 0 & 1.00000 & 0.61905 & -0.20635 & 0.07937 \end{array} \right]$$

$\text{null}(A^T)$
 basis

$3+1 = 4$

1.00000 0.61905 -0.20635 0.07937

⊥

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -3 \\ 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \\ -1 \end{bmatrix}$$

```

151 A = [1 0 0 0 2; -2 1 -3 -2 -4; 0 5 -14 -9 0; 2 10 -28 -18 4]
152 rref(A)
153 I = eye(4)
154 C = [ A I]
155 rref(C)
156 rref(C)
157 D = rref(C)
158 E = [ D(:,6) D(:,7) D(:,8) D(:,9) ]
159 E * A
160 A = [1 -3 4 -2 5 4; 2 -6 9 -1 8 2 ; 2 -6 9 -1 9 7; -1 3 -4 2 -5 -4]
161 C = [ A I]
162 D = rref(C)
163 E = [ D(:,7) D(:,8) D(:,9) D(:,10)]
164 E * A
165 A
166 A = [1 2 0 2 5; -2 -5 1 -1 -8; 0 -3 3 4 1; 3 6 0 -7 2 ]
167 C = [ A I]
168 D = rref(C)
169 A
170 B
171 C = [ A I]
172 E = [ D(:,6) D(:,7) D(:,8) D(:,9) ]
173 E * A

```

