Background – Vector Space (3A)

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Vector Space

V: non-empty set of objects

defined operations:

$\mathbf{u} + \mathbf{v}$ addition

$k\,\mathbf{u}$ scalar multiplication

if the following axioms are satisfied

for all object **u**, **v**, **w** and all scalar *k*, *m*

⇒ V: vector space

objects in V: vectors

Vector Space Axioms

if the following axioms are satisfied

for all object **u**, **v**, **w** and all scalar *k*, *m*

⇨

V: vector space

objects in V: vectors

1. if **u** and **v** are objects in V, then **u** + **v** is in V
2. **u** + **v** = **v** + **u**
3. **u** + (**v** + **w**) = (**u** + **v**) + **w**
4. **0** + **u** = **u** + **0** = **u** (zero vector)
5. **u** + (–**u**) = (–**u**) + (**u**) = **0**
6. if *k* is any scalar and **u** is objects in V, then *k***u** is in V
7. *k*(**u** + **v**) = *k***u** + *k***v**
8. (*k* + *m*)**u** = *k***u** + *m***u**
9. *k*(*m***u**) = (*km*)**u**
10. *1*(**u**) = **u**

Test for a Vector Space

1. Identify the set V of objects

2. Identify the addition and scalar multiplication on V

3. Verify **u** + **v** is in V and *k***u** is in V

   closure under addition and scalar multiplication

4. Confirm other axioms.

Vector Space over a Field

A **vector space V** over **F** :

A set **V** of *objects*

with the *operations*

vector addition $\mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$

scalar multiplication $\mathbf{F} \times \mathbf{V} \rightarrow \mathbf{V}$

$\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ $\qquad$ $\boldsymbol{u} + \boldsymbol{v} \in \boldsymbol{V}$

$k \in \boldsymbol{F}$ $\qquad$ $k\boldsymbol{u} \in \boldsymbol{V}$

Complex Vector Space

A **complex vector space** : A **vector space C** over **C**
A **real vector space** : A **vector space R** over **R**

A **complex euclidean space** : An n-space $\mathbf{C}^n$ over **C**
A **real euclidean space** : An n-space $\mathbf{R}^n$ over **R**

complex vector space

$\mathbf{C}^1$ over **C**

~~$\mathbf{R}^2$ over **C**~~

~~$\mathbf{R}^1$ over **C**~~

not closed for scalar multiplication

$\mathbf{C}^1$ over **R**

$\mathbf{R}^2$ over **R**

$\mathbf{R}^1$ over **R**

real vector space

real euclidean space R^2

real euclidean space R^1

Vector Space C^1

C^1 over **R**

$$c_1 \cdot 1 + c_2 \cdot i = 0 \iff c_1 = c_2 = 0$$

$\{1, i\}$ linearly independent

$c_i \in \boldsymbol{R}$

C^1 over **C**

$$c_1 \cdot 1 + c_2 \cdot i = 0 \Leftarrow c_1 = -i, \; c_2 = 1$$

$\{1, i\}$ ~~linearly independent~~

$c_i \in \boldsymbol{C}$

Vector Space R^2

~~R^2 over C~~

$$c_1(u_{1,}\ u_2) + c_2(v_{1,}\ v_2)$$

$$c_1\vec{u} + c_2\vec{v}$$

not closed for scalar multiplication

~~$c_i \in C$~~

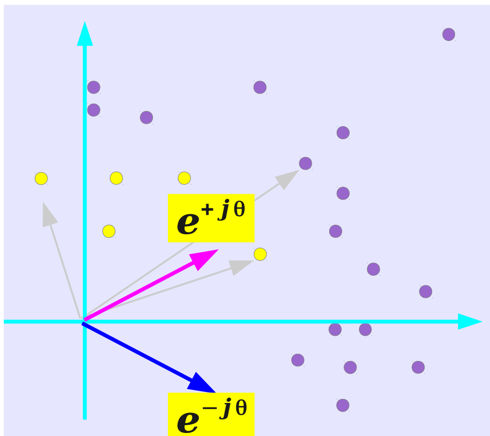
R^2 over R

$$c_1(u_{1,}\ u_2) + c_2(v_{1,}\ v_2)$$

$$c_1\vec{u} + c_2\vec{v}$$

$c_i \in R$

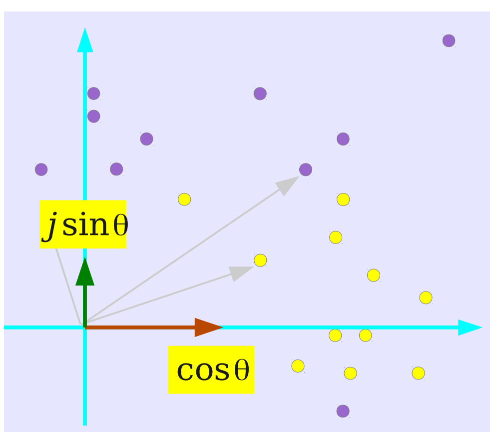
Basis : a set of linear independent spanning vectors



every complex number can be represented by

$$k_1 \, e^{+j\theta} + k_2 \, e^{+j\theta}$$

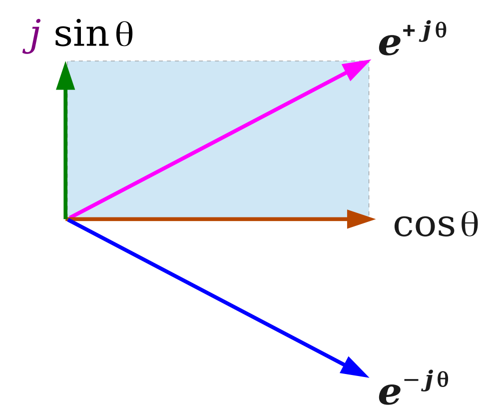
linear combination of $e^{+j\theta}$ and $e^{+j\theta}$
which are one set of linear independent
two vectors



every complex number can also be represented by

$$l_1 \cos\theta + l_2 \, j \sin\theta$$

$$l_1 \cos\theta + l_2 \quad j\sin\theta$$



Basis of the Complex Plane

Basis : a set of linear independent spanning vectors

$e^{+j\theta}$ $\quad$ $e^{+j\theta}$

every complex number can be represented by

$k_1 e^{+j\theta} + k_2 e^{+j\theta}$ $\quad$ C^1 over R

$k_1, k_2 \in \boldsymbol{R}$

every complex number can also be represented by

l_1 $\quad$ $l_2 j$

$l_1 \cos\theta + l_2 j \sin\theta$ $\quad$ C^1 over R

$\cos\theta, \sin\theta \in \boldsymbol{R}$

$\cos\theta$ $\quad$ $j\sin\theta$

$l_1 \cos\theta + l_2 j \sin\theta$ $\quad$ C^1 over R

$l_1, l_2 \in \boldsymbol{R}$

Complex Exponentials

$$c_1 e^{-\sigma} e^{+i\omega} + c_2 e^{-\sigma} e^{-i\omega}$$

$$= e^{-\sigma t}(c_1 e^{+i\omega} + c_2 e^{-i\omega})$$
$$= e^{-\sigma}[c_1(\cos(\omega) + i\sin(\omega)) + c_2(\cos(\omega) - i\sin(\omega))]$$
$$= e^{-\sigma}[(c_1 + c_2)\cos(\omega) + i(c_1 - c_2)\sin(\omega)]$$
$$= c_3 e^{-\sigma}\cos(\omega) + c_4 e^{-\sigma}\sin(\omega)$$

$(c_1 + c_2) = c_3$ *real*

$i(c_1 - c_2) = c_4$ *imag*

$c_1, c_2 \in \boldsymbol{R}$ → $c_3, c_4 \in \boldsymbol{C}$ $c_3(real), c_4(imag)$

C[1] over **R**

$$c_3 e^{-\sigma}\cos(\omega) + c_4 e^{-\sigma}\sin(\omega)$$

$$= c_3 e^{-\sigma}(e^{+i\omega} + e^{-i\omega})/2 + c_4 e^{-\sigma}(e^{+i\omega} - e^{-i\omega})/2i$$
$$= c_3 e^{-\sigma}(e^{+i\omega} + e^{-i\omega})/2 + c_4 e^{-\sigma}(-ie^{+i\omega} + ie^{-i\omega})/2$$
$$= \frac{(c_3 - c_4 i)}{2} e^{-\sigma} e^{+i\omega} + \frac{(c_3 + c_4 i)}{2} e^{-\sigma} e^{-i\omega}$$
$$= c_1 e^{-\sigma} e^{+i\omega} + c_2 e^{-\sigma t} e^{-i\omega}$$

$\frac{(c_3 - c_4 i)}{2} = c_1$ *real*

$\frac{(c_3 + c_4 i)}{2} = c_2$ *real*

$c_3, c_4 \in \boldsymbol{C}$ → $c_1, c_2 \in \boldsymbol{R}$ $c_3(real), c_4(imag)$

C[1] over **R**

Complex Exponentials

$c_1 e^{-\sigma} e^{+i\omega} + c_2 e^{-\sigma} e^{-i\omega}$ → $c_3 e^{-\sigma} \cos(\omega) + c_4 e^{-\sigma} \sin(\omega)$

$\mathbf{C}^1$ over $\mathbf{R}$ | $\mathbf{C}^1$ over $\mathbf{R}$

$c_1, c_2 \in \mathbf{R}$

$(c_1 + c_2) = c_3$

$i(c_1 - c_2) = c_4$

$c_3, c_4 \in \mathbf{C}$

c_3 : *real*

c_4 : *imag*

$c_1 e^{-\sigma} e^{+i\omega} + c_2 e^{-\sigma} e^{-i\omega}$ ← $c_3 e^{-\sigma} \cos(\omega) + c_4 e^{-\sigma} \sin(\omega)$

$\mathbf{C}^1$ over $\mathbf{R}$ | $\mathbf{C}^1$ over $\mathbf{R}$

$c_1, c_2 \in \mathbf{R}$

$c_1 = (c_3 - c_4 i)/2$

$c_2 = (c_3 + c_4 i)/2$

$c_3, c_4 \in \mathbf{C}$

c_3 : *real*

c_4 : *imag*

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Complex Plane Basis $e^{+i\omega}, \quad e^{+i\omega}$

$c_1 e^{+i\omega} \; + \; c_2 e^{-i\omega}$ ⟷ $c_3 \cos(\omega) \; + \; c_4 \sin(\omega)$

real number *real number*

$$c_1 = (c_3 - c_4 i)/2$$
$$c_2 = (c_3 + c_4 i)/2$$

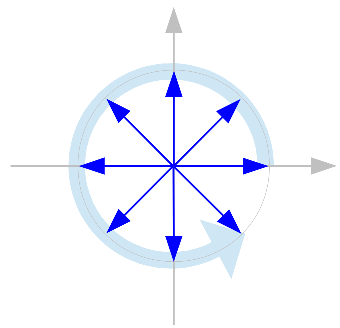
$\mathbf{C}^1$ over $\mathbf{R}$

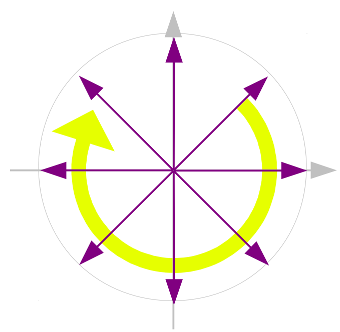
$$c_3 = (c_1 + c_2)$$
$$c_4 = i(c_1 - c_2)$$

real number

imaginary number

$(c_1, \; c_2)$





$(\Re(c_3), \; \Im(c_4))$

$c_1 e^{+i\omega} + c_2 e^{-i\omega}$	$c_3\cos(\omega) + c_4 \sin(\omega)$
$1 \cdot e^{+i\omega} + 0 \cdot e^{-i\omega}$	$1 \cdot \cos(\omega) + 1 i \cdot \sin(\omega)$
$\frac{1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega}$	$\sqrt{2} \cdot \cos(\omega) + 0 i \cdot \sin(\omega)$
$0 \cdot e^{+i\omega} + 1 \cdot e^{-i\omega}$	$1 \cdot \cos(\omega) - 1 i \cdot \sin(\omega)$
$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega}$	$0 \cdot \cos(\omega) - \sqrt{2} i \cdot \sin(\omega)$
$-1 \cdot e^{+i\omega} + 0 \cdot e^{-i\omega}$	$-1 \cdot \cos(\omega) - 1 i \cdot \sin(\omega)$
$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega}$	$-\sqrt{2} \cdot \cos(\omega) - 0 i \cdot \sin(\omega)$
$0 \cdot e^{+i\omega} - 1 \cdot e^{-i\omega}$	$-1 \cdot \cos(\omega) + 1 i \cdot \sin(\omega)$
$\frac{1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega}$	$0 \cdot \cos(\omega) + \sqrt{2} i \cdot \sin(\omega)$

↑ c_1 ↑ c_2 ↑ c_3 ↑ c_4

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Real Coefficients d_1 & d_2

$$c_1 e^{+i\omega} + c_2 e^{-i\omega} \quad \Longleftrightarrow \quad c_3 \cos(\omega) + c_4 \sin(\omega)$$

C^1 over R

real number	$c_1 = (c_3 - c_4 i)/2$
real number	$c_2 = (c_3 + c_4 i)/2$

$c_3 = (c_1 + c_2)$	real number
$c_4 = i(c_1 - c_2)$	imaginary number

$$c_1 e^{+i\omega} + c_2 e^{-i\omega} \quad \Longleftrightarrow \quad d_3 \cos(\omega) + d_4 i \sin(\omega)$$

C^1 over R

$c_1 = (d_3 + d_4)/2$	real number
$c_2 = (d_3 - d_4)/2$	real number

$d_3 = (c_1 + c_2)$	real number
$d_4 = (c_1 - c_2)$	real number

$$d_3 = \Re(c_3)$$
$$d_4 = \Im(c_4)$$

Complex Plane Basis $\cos(\omega), \;\; i\sin(\omega)$

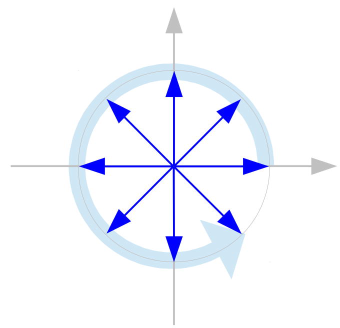
$c_1 e^{+i\omega} + c_2 e^{-i\omega}$ ⟷ $d_3 \cos(\omega) + d_4\, i \sin(\omega)$

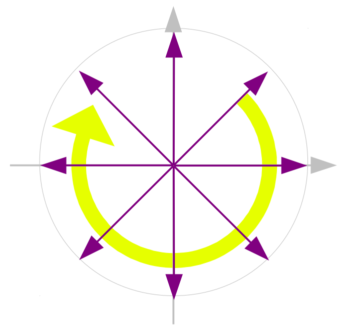
$c_1 = (d_3 + d_4)/2$ *real number*
$c_2 = (d_3 - d_4)/2$ *real number*

C^1 over R

$d_3 = (c_1 + c_2)$ *real number*
$d_4 = (c_1 - c_2)$ *real number*

$(c_1,\; c_2)$





$(d_3,\; d_4)$

c_1, c_2	d_3, d_4
$1 \cdot e^{+i\omega} + 0 \cdot e^{-i\omega}$	$1 \cdot \cos(\omega) + 1\, i \cdot \sin(\omega)$
$\frac{1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega}$	$\sqrt{2} \cdot \cos(\omega) + 0\, i \cdot \sin(\omega)$
$0 \cdot e^{+i\omega} + 1 \cdot e^{-i\omega}$	$1 \cdot \cos(\omega) - 1\, i \cdot \sin(\omega)$
$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega}$	$0 \cdot \cos(\omega) - \sqrt{2}\, i \cdot \sin(\omega)$
$-1 \cdot e^{+i\omega} + 0 \cdot e^{-i\omega}$	$-1 \cdot \cos(\omega) - 1\, i \cdot \sin(\omega)$
$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega}$	$-\sqrt{2} \cdot \cos(\omega) - 0\, i \cdot \sin(\omega)$
$0 \cdot e^{+i\omega} - 1 \cdot e^{-i\omega}$	$-1 \cdot \cos(\omega) + 1\, i \cdot \sin(\omega)$
$\frac{1}{\sqrt{2}} \cdot e^{+i\omega} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega}$	$0 \cdot \cos(\omega) + \sqrt{2}\, i \cdot \sin(\omega)$

↑ c_1 ↑ c_2 ↑ d_3 ↑ d_4

Real Coefficients k_3 & k_4

$$c_1 e^{+i\omega} + c_2 e^{-i\omega} \iff c_3 \cos(\omega) + c_4 \sin(\omega)$$

C^1 over R

real number $c_1 = (c_3 - c_4 i)/2$

real number $c_2 = (c_3 + c_4 i)/2$

$c_3 = (c_1 + c_2)$ *real number*

$c_4 = i(c_1 - c_2)$ *imaginary number*

$c_1 \in R \quad c_1 : real$

$c_2 \in R \quad c_2 : real$

$c_3 \in C \quad c_3 : real$

$c_4 \in C \quad c_4 : imag$

$$m_1 e^{+i\omega} + m_2 e^{-i\omega} \iff k_3 \cos(\omega) + k_4 \sin(\omega)$$

R^1 over R

$m_1 = (k_3 - k_4 i)/2$ *conjugate*

$m_2 = (k_3 + k_4 i)/2$ *complex number*

$+2 * real\ part$ $\quad k_3 = (m_1 + m_2)$ *real number*

$-2 * imag\ part$ $\quad k_4 = i(m_1 - m_2)$ *real number*

$m_1 \in C \quad (m_1 + m_2) : real$

$m_2 \in C \quad i(m_1 - m_2) : real$

$k_3 \in R \quad k_3 : real$

$k_4 \in R \quad k_4 : real$

Subspace : Real Line

$m_1 e^{+i\omega} + m_2 e^{-i\omega}$ ⟷ $k_3 \cos(\omega) + k_4 \sin(\omega)$

$$m_1 = (k_3 - k_4 i)/2$$
$$m_2 = (k_3 + k_4 i)/2$$

conjugate complex number

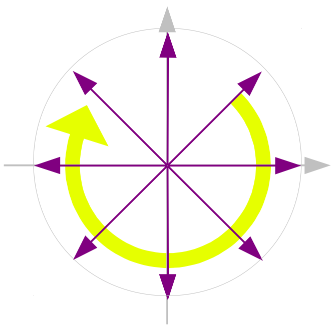
$\mathbf{R}^1$ over $\mathbf{R}$

$+2 * real\ part$ $k_3 = (m_1 + m_2)$ *real number*

$-2 * imag\ part$ $k_4 = i(m_1 - m_2)$ *real number*

$\frac{(+1-0i)}{2} \cdot e^{+i\omega} + \frac{(+1+0i)}{2} \cdot e^{-i\omega}$	$1 \cdot \cos(\omega) + 0 \cdot \sin(\omega)$
$\frac{(+1-i)}{2\sqrt{2}} \cdot e^{+i\omega} + \frac{(+1+i)}{2\sqrt{2}} \cdot e^{-i\omega}$	$\frac{1}{\sqrt{2}} \cdot \cos(\omega) + \frac{1}{\sqrt{2}} \cdot \sin(\omega)$
$\frac{(0-i)}{2} \cdot e^{+i\omega} + \frac{(0+i)}{2} \cdot e^{-i\omega}$	$0 \cdot \cos(\omega) + 1 \cdot \sin(\omega)$
$\frac{(-1-i)}{2\sqrt{2}} \cdot e^{+i\omega} + \frac{(-1+i)}{2\sqrt{2}} \cdot e^{-i\omega}$	$\frac{-1}{\sqrt{2}} \cdot \cos(\omega) + \frac{1}{\sqrt{2}} \cdot \sin(\omega)$
$\frac{(-1-0i)}{2} \cdot e^{+i\omega} + \frac{(-1+0i)}{2} \cdot e^{-i\omega}$	$-1 \cdot \cos(\omega) + 0 \cdot \sin(\omega)$
$\frac{(-1+i)}{2\sqrt{2}} \cdot e^{+i\omega} + \frac{(-1-i)}{2\sqrt{2}} \cdot e^{-i\omega}$	$\frac{-1}{\sqrt{2}} \cdot \cos(\omega) + \frac{-1}{\sqrt{2}} \cdot \sin(\omega)$
$\frac{(0+i)}{2} \cdot e^{+i\omega} + \frac{(0-i)}{2} \cdot e^{-i\omega}$	$0 \cdot \cos(\omega) - 1 \cdot \sin(\omega)$
$\frac{(+1+i)}{2\sqrt{2}} \cdot e^{+i\omega} + \frac{(+1-i)}{2\sqrt{2}} \cdot e^{-i\omega}$	$\frac{1}{\sqrt{2}} \cdot \cos(\omega) - \frac{1}{\sqrt{2}} \cdot \sin(\omega)$

real line



$(k_3,\ k_4)$

↑ m_1 ↑ m_2 ↑ k_3 ↑ k_4

Trigonometric Relationship

$$m_1 e^{+i\omega} + m_2 e^{-i\omega} \quad \Longleftrightarrow \quad k_3 \cos(\omega) + k_4 \sin(\omega)$$

$$m_1 = (k_3 - k_4 i)/2$$
$$m_2 = (k_3 + k_4 i)/2$$

conjugate complex number

R^1 over R

$+2*real\ part$ $\quad k_3 = (m_1 + m_2)$ $\quad$ real number

$-2*imag\ part$ $\quad k_4 = i(m_1 - m_2)$ $\quad$ real number

$$A\cos(\omega t - \varphi) \quad \Longleftrightarrow \quad k_3 \cos(\omega) + k_4 \sin(\omega)$$

$$\sqrt{k_3^2 + k_4^2} = A$$

$$\frac{k_3}{\sqrt{k_3^2 + k_4^2}} = \cos(\varphi)$$

$$\frac{k_4}{\sqrt{k_3^2 + k_4^2}} = \sin(\varphi)$$

Signal Spaces and Phasors

$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 \sin(\omega t)$

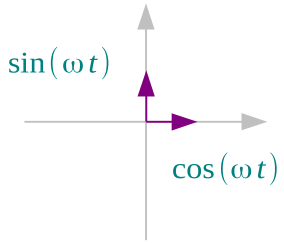
$c_1 = (c_3 - c_4 i)/2$
$c_2 = (c_3 + c_4 i)/2$

conjugate complex number

$+2*real\ part$ $c_3 = (c_1 + c_2)$ *real number*

$-2*imag\ part$ $c_4 = i(c_1 - c_2)$ *real number*

R^2



Functions [edit]

Let K be the set **C** of all complex numbers, and let V be the set $C_C(R)$ of all continuous functions from the real line **R** to the complex plane **C**. Consider the vectors (functions) f and g defined by $f(t) := e^{it}$ and $g(t) := e^{-it}$. (Here, e is the base of the natural logarithm, about 2.71828..., and i is the imaginary unit, a square root of −1.) Some linear combinations of f and g are:

- $\cos t = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$
- $2\sin t = (-i)e^{it} + (i)e^{-it}$.

On the other hand, the constant function 3 is *not* a linear combination of f and g. To see this, suppose that 3 could be written as a linear combination of e^{it} and e^{-it}. This means that there would exist complex scalars a and b such that $ae^{it} + be^{-it} = 3$ for all real numbers t. Setting $t = 0$ and $t = \pi$ gives the equations $a + b = 3$ and $a + b = -3$, and clearly this cannot happen. See Euler's identity.

https://en.wikipedia.org/wiki/Linear_combination

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”