Background – Complex Analysis (1A)

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5/5/15

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Complex Numbers

Complex Numbers

$$i = \sqrt{-1}$$

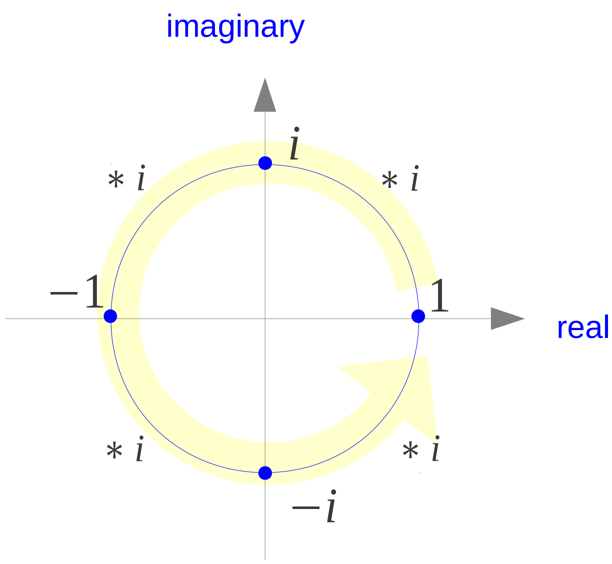
$$i^2 = -1$$

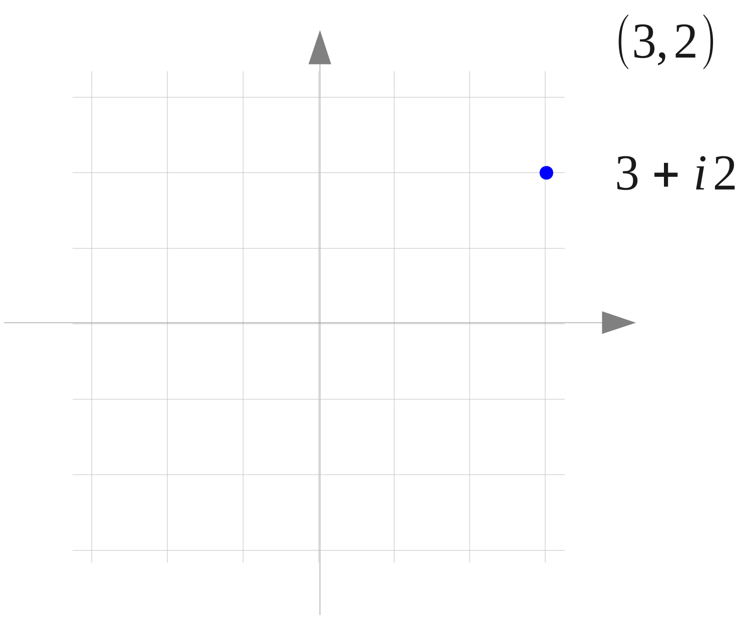
$$i^3 = -i \qquad i^2 \cdot i = -i$$

$$i^4 = +1 \qquad i^2 \cdot i^2 = +1$$

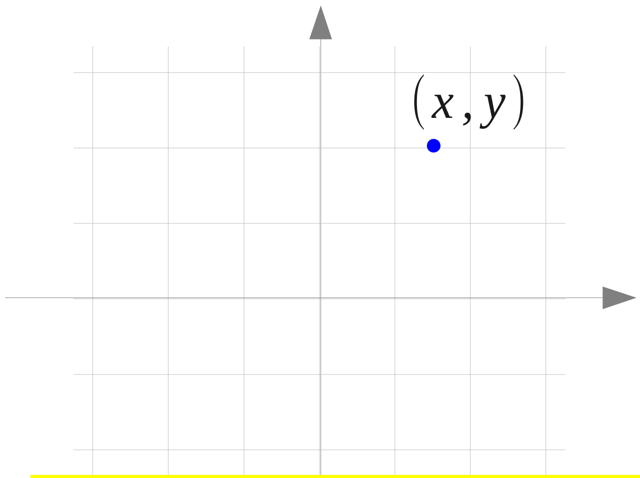
$(3, 2)$ — **two** real numbers
2-d coordinate

$3 + i2$ — **one** complex number
with real part of 3
and imaginary part of 2

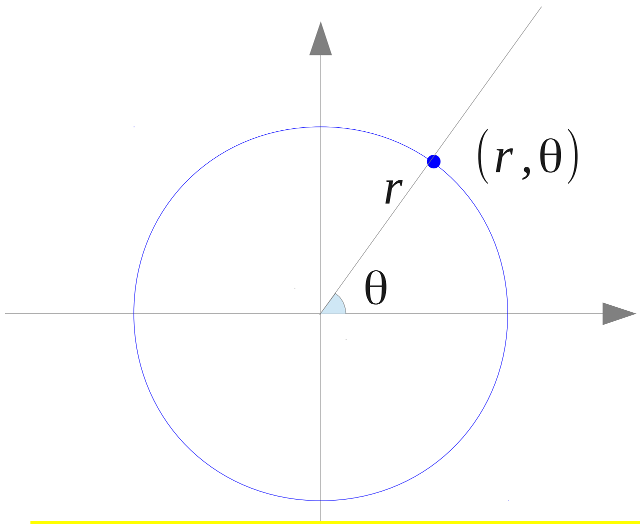




Coordinate Systems



(a) Cartesian Coordinate System

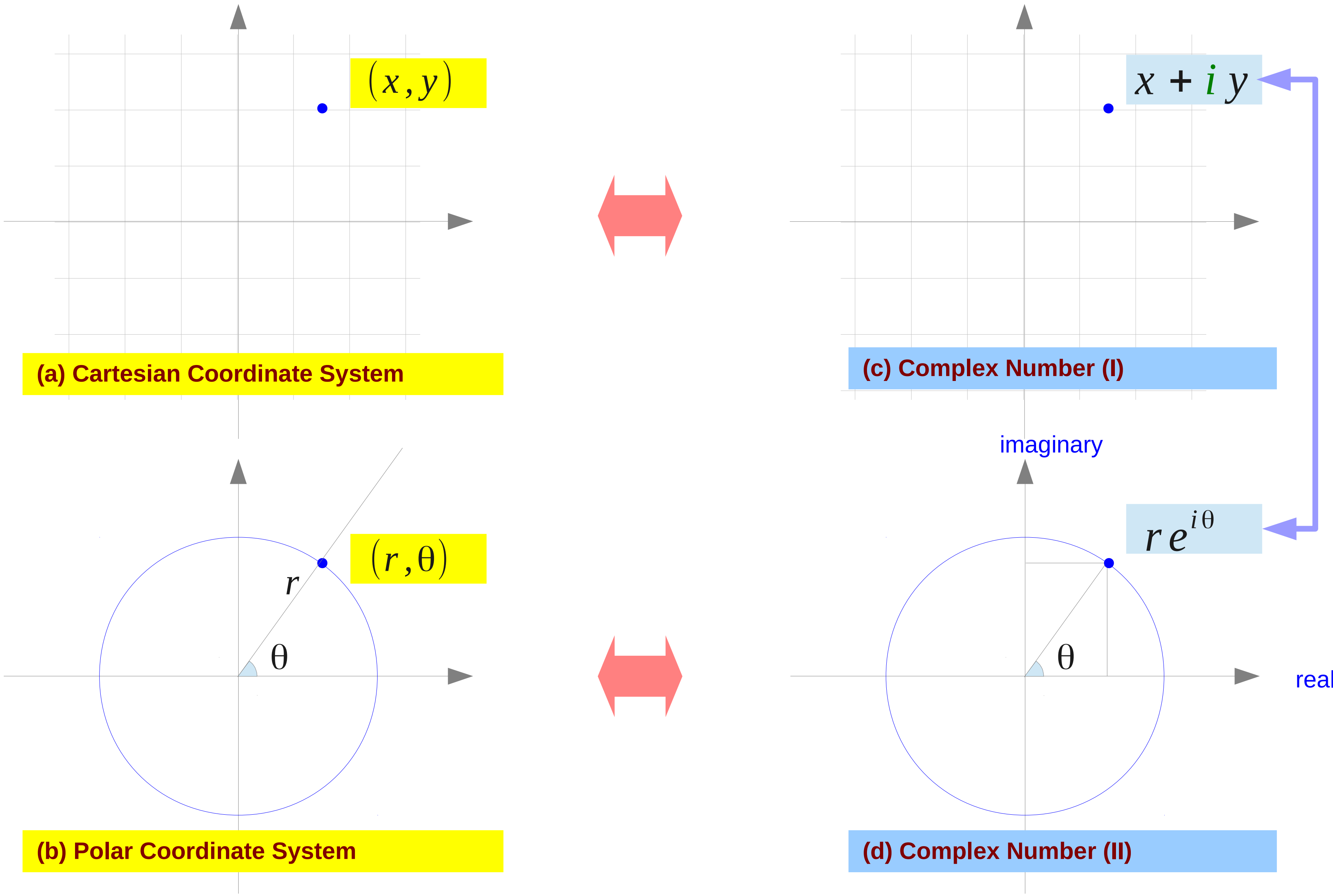


(b) Polar Coordinate System

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

Coordinate Systems and Complex Numbers



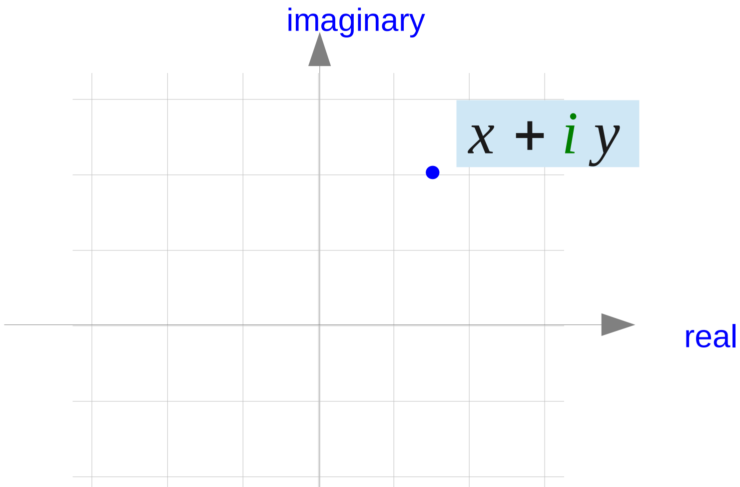
Complex Numbers

$$x = r\cos\theta$$
$$y = r\sin\theta$$

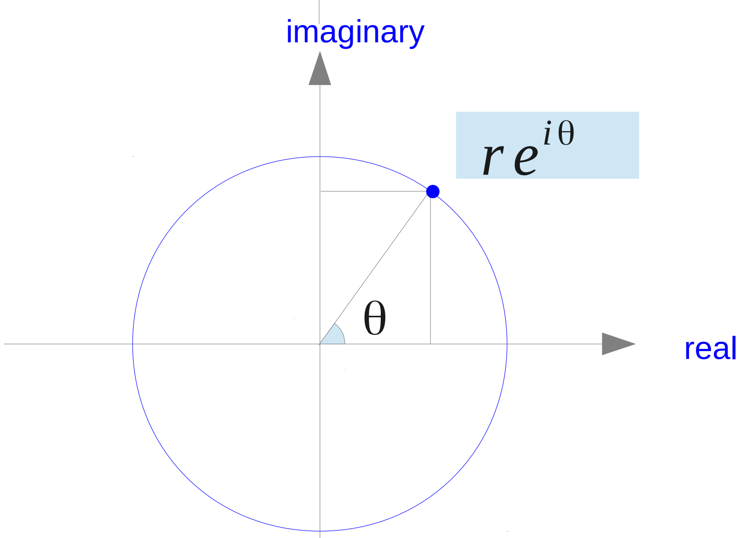
$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

$$\begin{aligned} x + i\,y &= r\cos\theta + i\,r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= r\,e^{i\theta} \end{aligned}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$



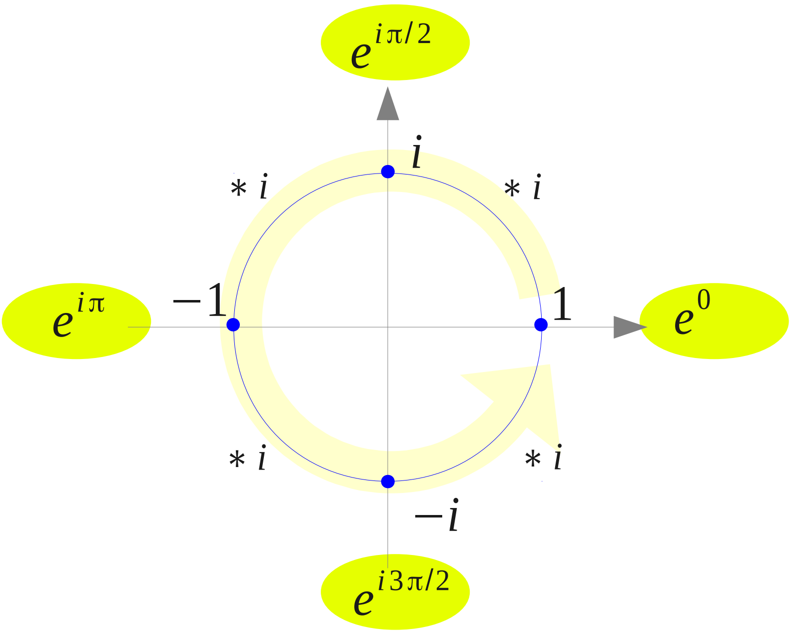
(c) Complex Number (I)



(d) Complex Number (II)

Euler's Formula (1)

$$e^{i\theta} = \cos\theta + i\sin\theta$$



$x \Rightarrow \Re\{e^{i\theta}\} = \cos\theta$

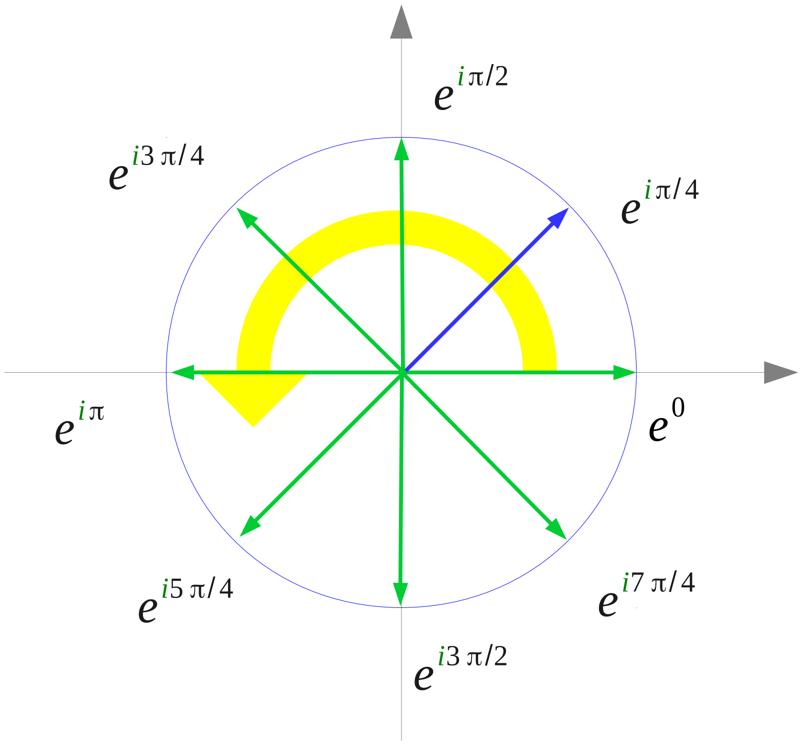
$y \Rightarrow \Im\{e^{i\theta}\} = \sin\theta$

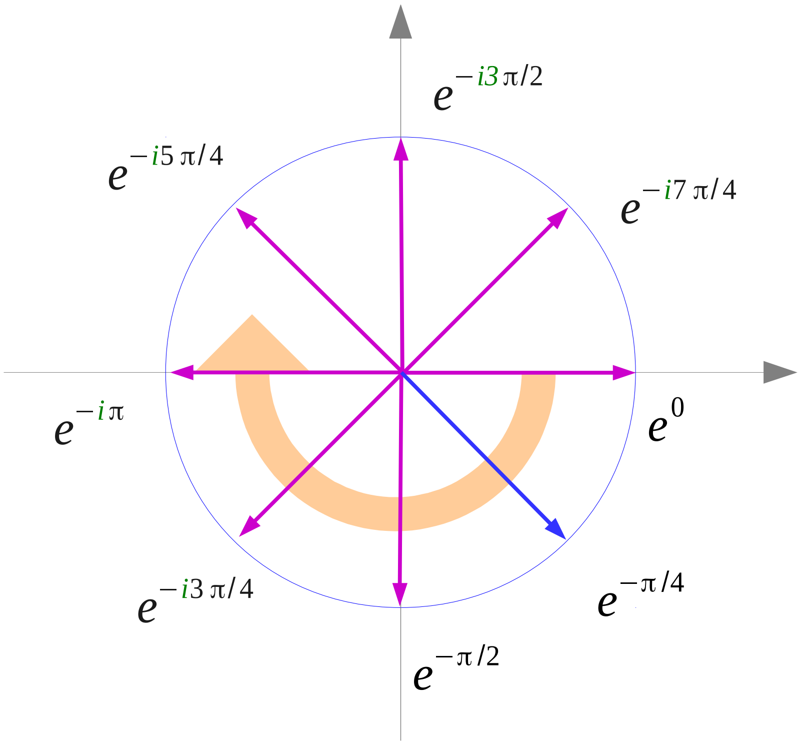
$$\begin{aligned}
e^{0} &= +1 + 0i & &= +1 & &= e^{-i2\pi} \\
e^{i\pi/2} &= 0 + 1i & &= +i & &= e^{-i3\pi/2} \\
e^{i\pi} &= -1 + 0i & &= -1 & &= e^{-i\pi} \\
e^{i3\pi/2} &= 0 - 1i & &= -i & &= e^{-i\pi/2}
\end{aligned}$$

Euler's Formula (2)

$$e^{+i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$





The Euler constant e

$$\frac{d}{dx} a^x = \lim_{h\to 0} \frac{a^{(x+h)} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} \Rightarrow a^x$$

such a,
we call e

$$\lim_{h\to 0} \frac{a^h - 1}{h} = 1 \iff \lim_{h\to 0} \frac{a^h - a^0}{h - 0} = 1 \iff f'(0) = 1$$

$$\frac{d}{dx} e^x = e^x$$

$$e = 2.71828\cdots$$

$$f(x) = e^x \rightarrow f'(x) = e^x \rightarrow f''(x) = e^x$$

The Euler constant e

$$\frac{d}{dx} e^x = e^x$$

$$e = 2.71828\cdots$$

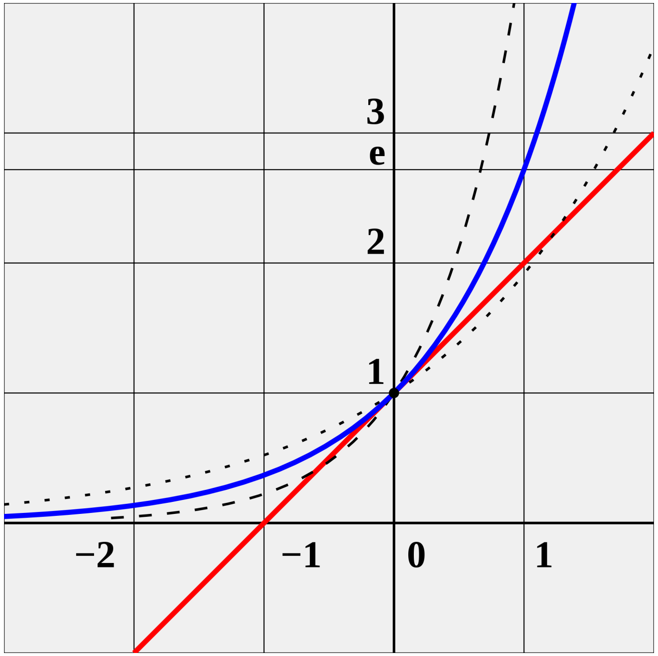
$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1 \quad iif \quad a = e$$

$$f'(0) = 1 \qquad \lim_{h \to 0} \frac{a^h - a^0}{h - 0} = 1$$



Functions f(x) = a^x are shown for several values of a. e is the unique value of a, such that the derivative of f(x) = a^x at the point x = 0 is equal to 1. The blue curve illustrates this case, ex. For comparison, functions 2^x (dotted curve) and 4^x (dashed curve) are shown; they are not tangent to the line of slope 1 and y-intercept 1 (red).

http://en.wikipedia.org/wiki/Derivative

The Derivative of a^x

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

$$\frac{d}{dx}\{a^x\} = \frac{d}{dx}\{e^{x \ln a}\}$$

$$= \{e^{x \ln a}\} \frac{d}{dx}\{x \ln a\}$$

$$\frac{d}{dx}\{a^x\} = \{a^x\} \ln a$$

$$\frac{d}{dx}\{e^x\} = \{e^x\} \ln e = \{e^x\}$$

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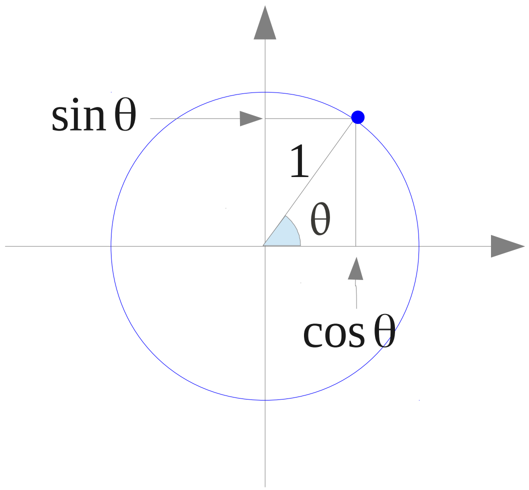
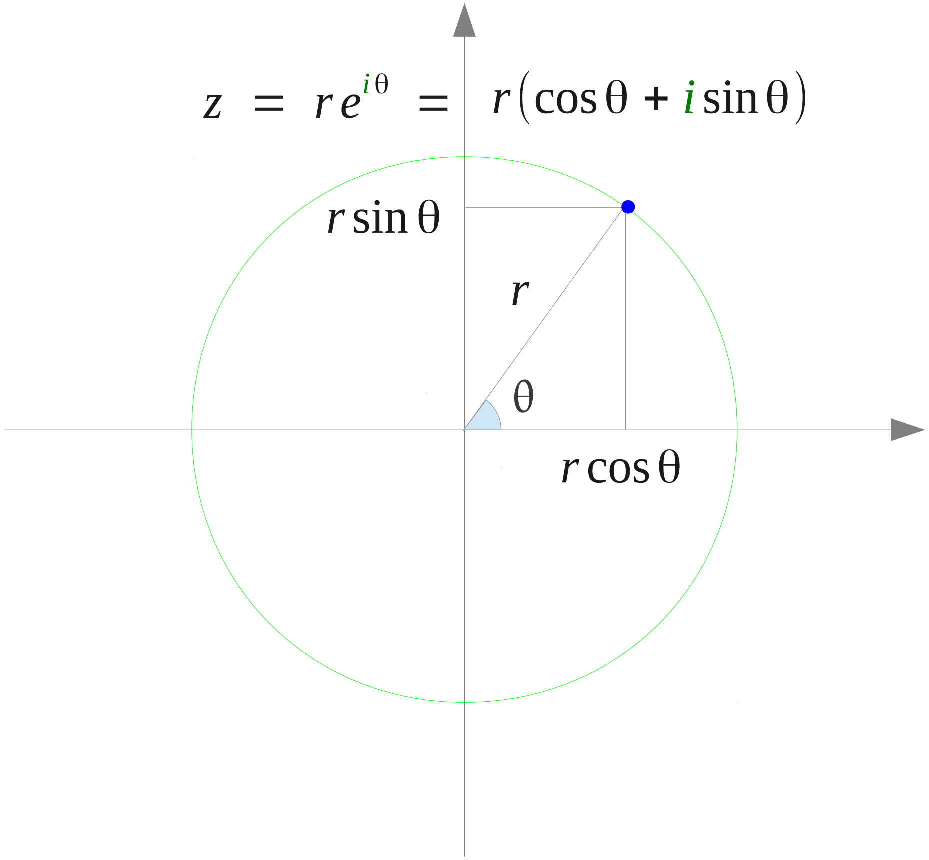
Absolute Values and Arguments

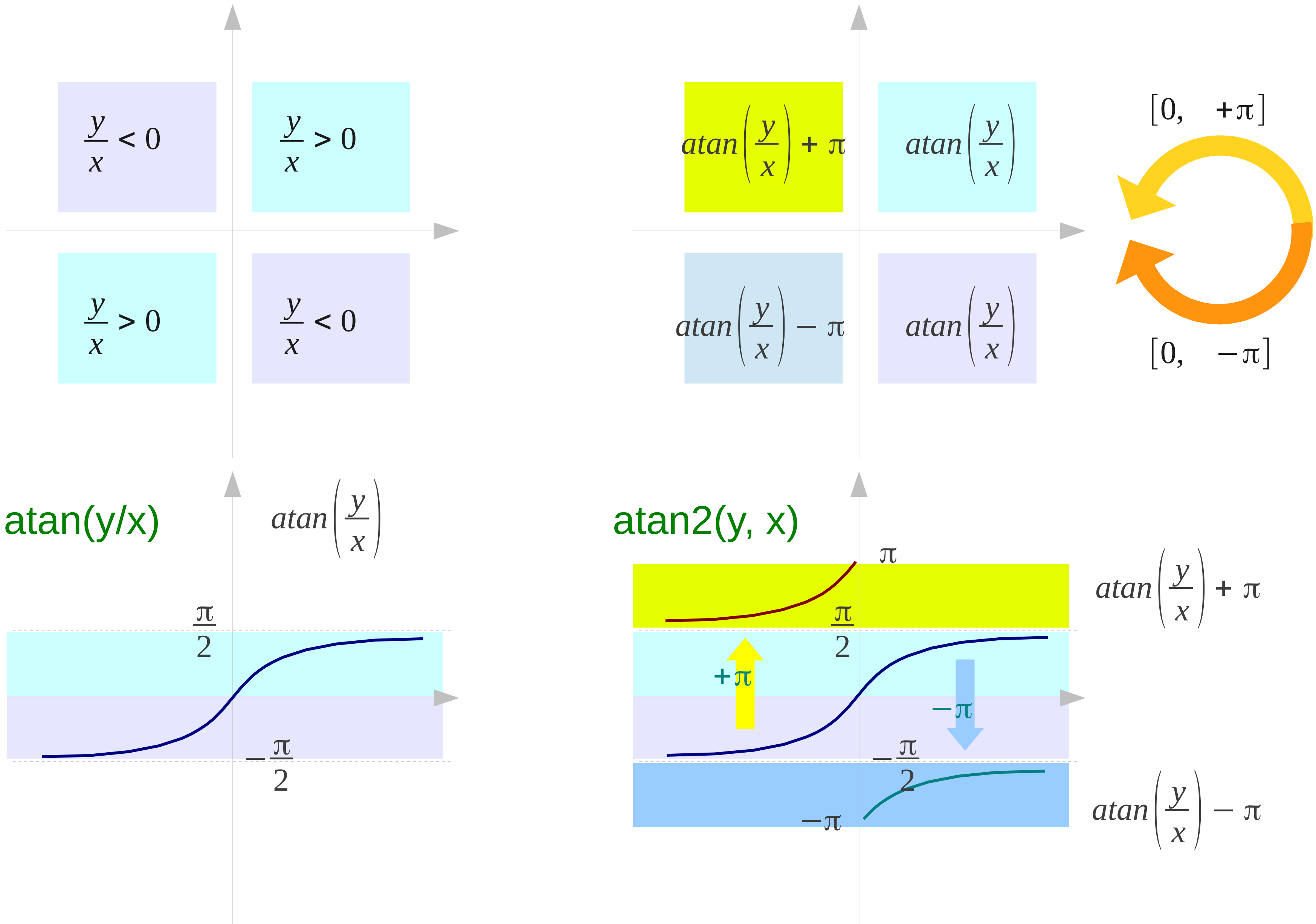


$$z = r e^{i\theta} = r(\cos\theta + i\sin\theta)$$



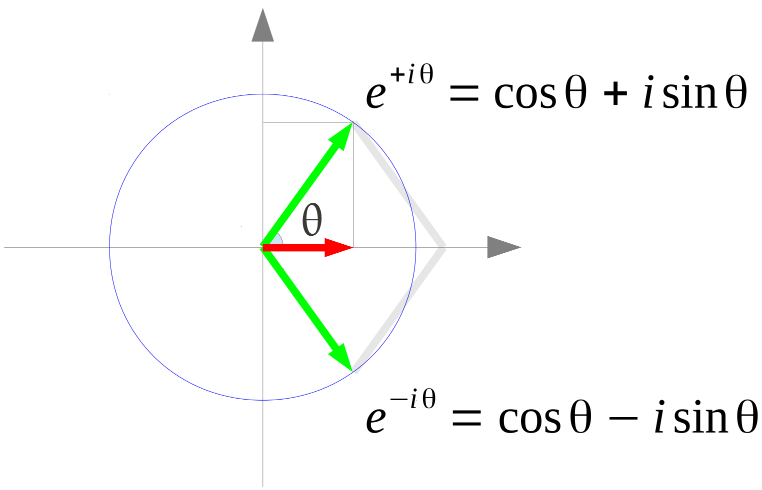
absolute value	$\lvert z\rvert = r$	➡	$\lvert r e^{i\theta}\rvert = \lvert r\rvert\,\lvert e^{i\theta}\rvert = r\sqrt{\cos^2\theta + \sin^2\theta}$
argument, phase	$arg(z) = \theta$	➡	$arg(r e^{i\theta})$

Computing Complex Number Argument



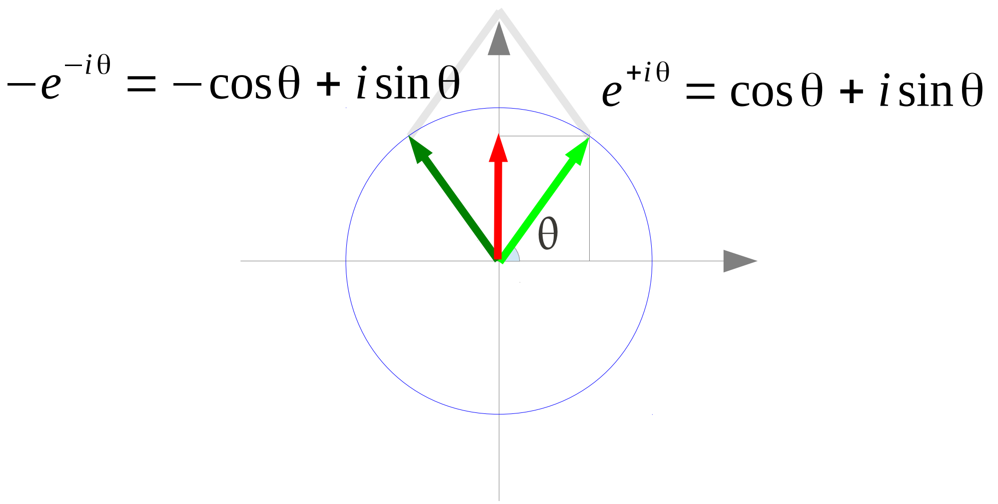
$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$



$$\Re\{e^{i\theta}\} = \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

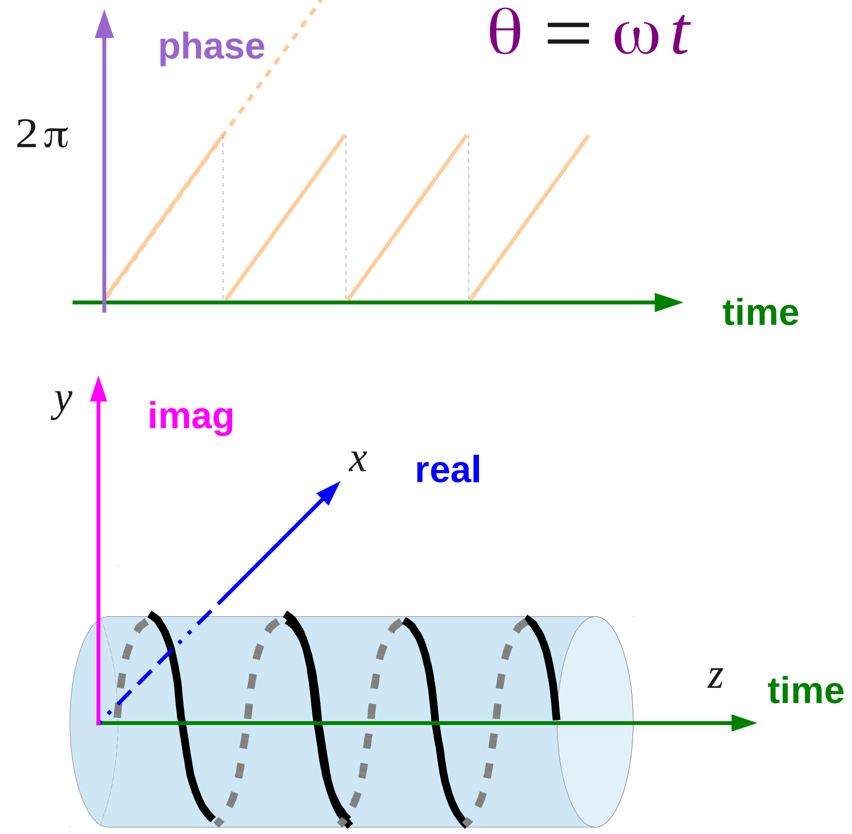
$$\Im\{e^{i\theta}\} = \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

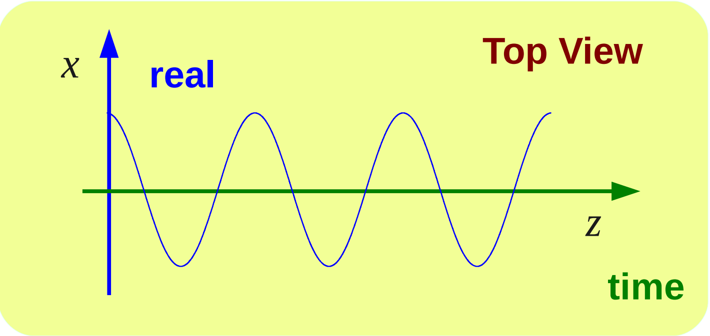


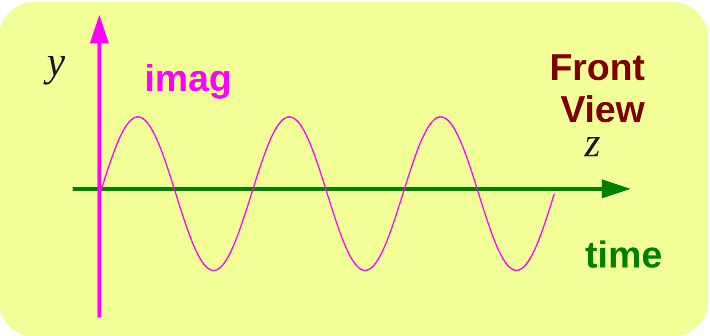
Complex Exponential

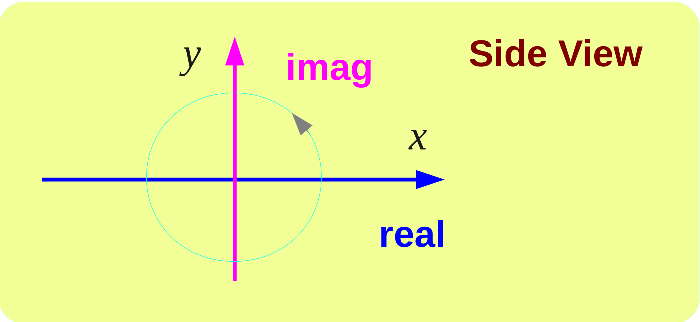
$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$

$$\theta = \omega t$$

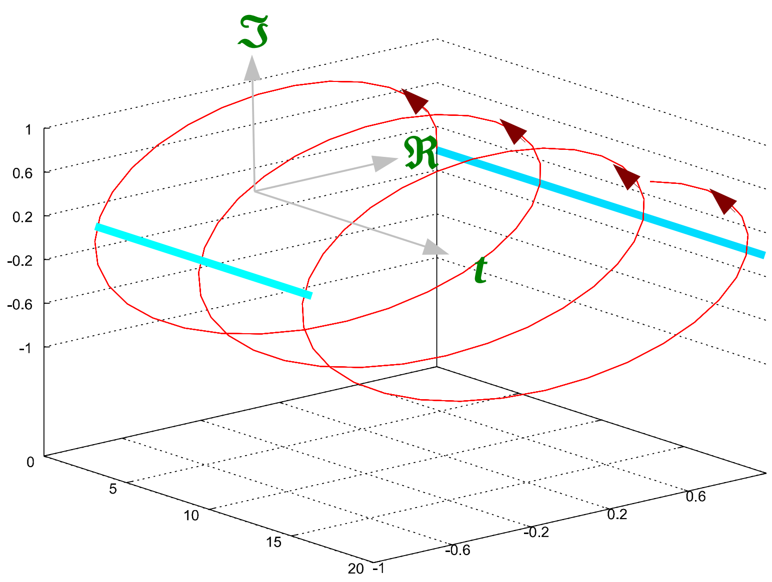


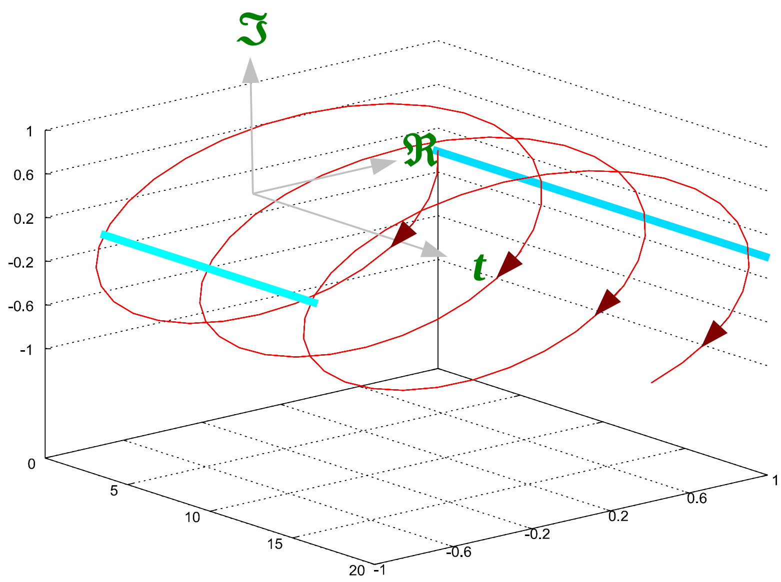






Conjugate Complex Exponential





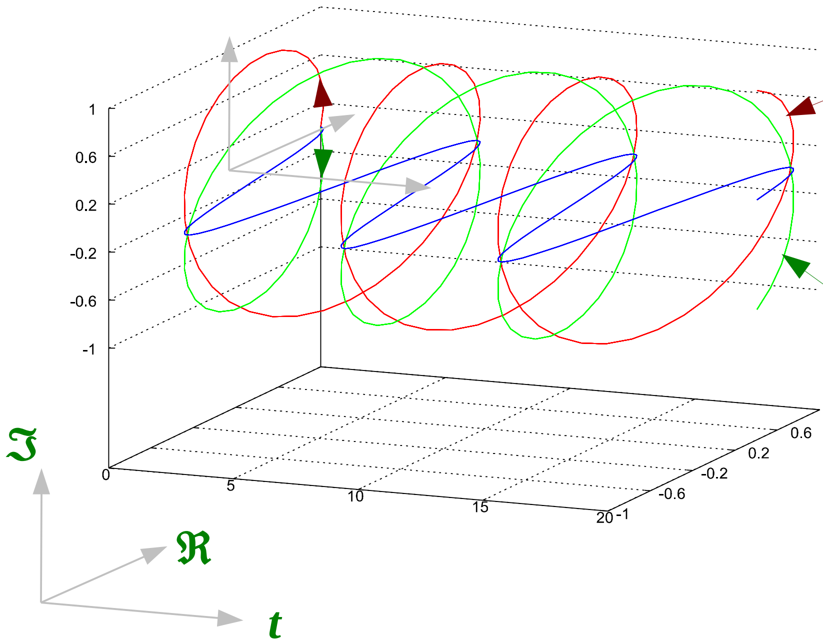
$$e^{+j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

$$e^{-j\omega t} = \cos(\omega_0 t) - j\sin(\omega_0 t)$$

$e^{+jt} = \cos(t) + j\sin(t) \qquad (\omega_0 = 1)$

$e^{-jt} = \cos(t) - j\sin(t) \qquad (\omega_0 = 1)$

$\cos(\omega_0 t)$



$$e^{+jt} = \cos(t) + j\sin(t)$$

$$\left(e^{+jt} + e^{-jt}\right) = 2\cos(t)$$

$$e^{-jt} = \cos(t) - j\sin(t)$$

$$x(t) = A\cos(\omega_0 t)$$

$$= \frac{A}{2} e^{+j\omega_0 t} + \frac{A}{2} e^{-j\omega_0 t}$$

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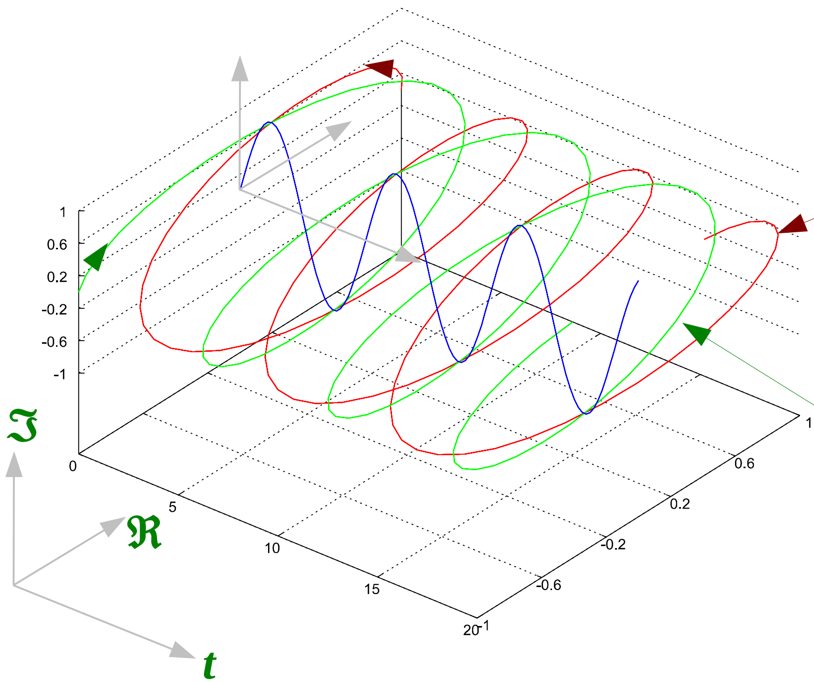
Sin(ω_0t)



$$e^{+jt} = \cos(t) + j\sin(t)$$

$$(e^{+jt} - e^{-jt}) = 2j\sin(t)$$

$$-e^{-jt} = -\cos(t) + j\sin(t)$$

$$x(t) = A\sin(\omega_0 t)$$

$$= \frac{A}{2j} e^{+j\omega_0 t} - \frac{A}{2j} e^{-j\omega_0 t}$$

Complex Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

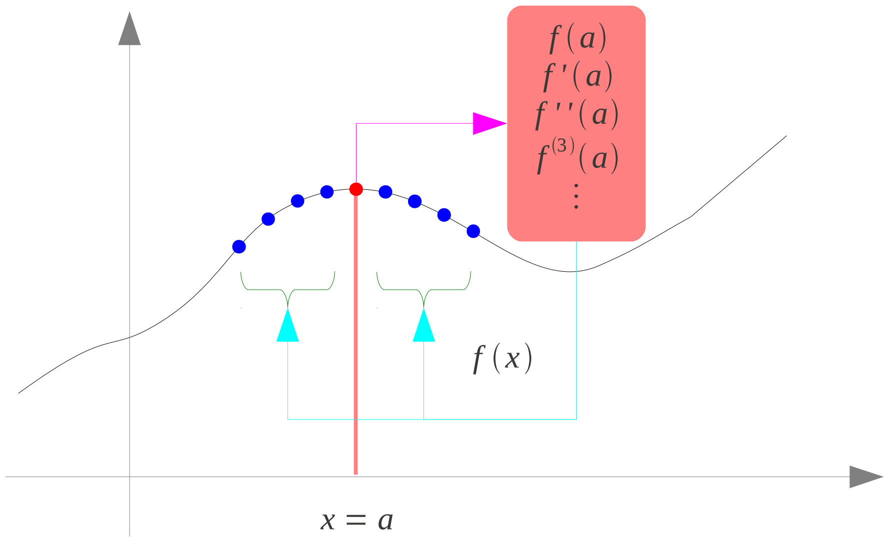
$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$



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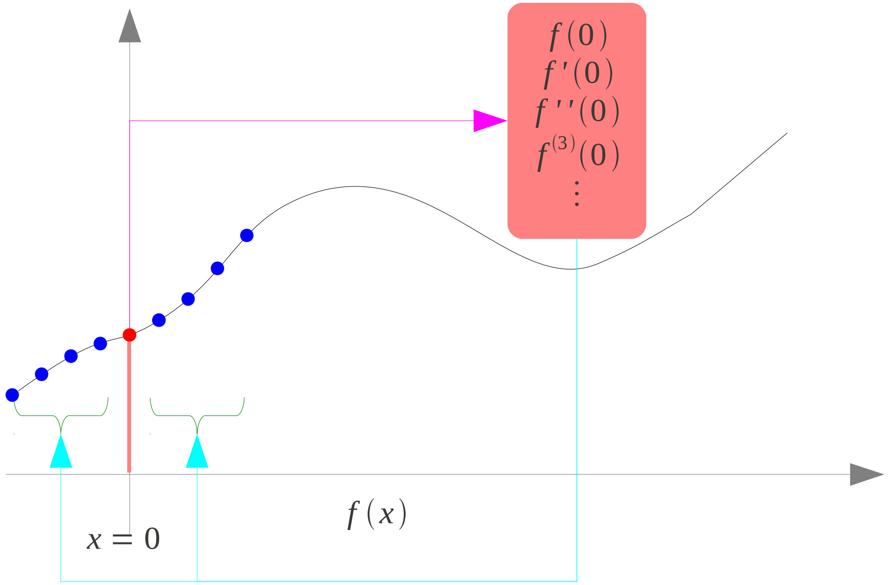
Maclaurin Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$



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Power Series Expansion

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Complex Arithmetic

$$z = a + ib$$
$$w = c + id$$
$$z + w = (a + c) + i(b + d)$$

$$z = a + ib$$
$$w = c + id$$
$$z - w = (a - c) + i(b - d)$$

$$z = a + ib$$
$$w = c + id$$
$$zw = (ac - bd) + i(ad + bc)$$

$$z = a + ib$$
$$w = c + id$$
$$\frac{z}{w} = \left(\frac{a + ib}{c + id}\right)$$
$$= \left(\frac{a + ib}{c + id}\right)\left(\frac{c - id}{c - id}\right)$$
$$= \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{-ad + bc}{c^2 + d^2}\right)$$

Complex Conjugate

$$z = x + iy = \Re\{z\} + i\Im\{z\}$$

$$\bar{z} = x - iy = \Re\{z\} - i\Im\{z\}$$

$$\Re\{z\} = \frac{1}{2}(z + \bar{z})$$

$$\Im\{z\} = \frac{1}{2i}(z - \bar{z})$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{z - w} = \bar{z} - \bar{w}$$

$$\overline{zw} = \bar{z}\,\bar{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

$$\frac{1}{z} = \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{x^2 + y^2} = \frac{\bar{z}}{|z|^2}$$

Complex Power (1)

$$a = e^{\log_e a} = e^{\ln a}$$

$$a^{b} = \left(e^{\log_e a}\right)^{b} = \left(e^{\ln a}\right)^{b} = e^{b \ln a}$$

$$a^{ib} = \left(e^{\log_e a}\right)^{ib} = \left(e^{\ln a}\right)^{ib} = e^{ib \ln a}$$

$$= \cos(b \ln a) + i \sin(b \ln a)$$

$$a^{c+ib} = a^{c}\left(e^{\log_e a}\right)^{ib} = a^{c}\left(e^{\ln a}\right)^{ib} = a^{c} e^{ib \ln a}$$

Complex Power (2)

$$a = e^{\ln a}$$

$$a^{b} = e^{b \ln a}$$

$$a^{ib} = e^{ib \ln a} = [\cos(b \ln a) + i \sin(b \ln a)]$$

$$a^{c+ib} = a^{c} e^{ib \ln a} = a^{c} [\cos(b \ln a) + i \sin(b \ln a)]$$

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[5] www.chem.arizona.edu/~salzmanr/480a