The Growth of Functions (2A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Functions and Ranges

$$\left\{ \begin{array}{l} x^2 + 2x + 1 \\ x^2 \\ 2x \\ 1 \end{array} \right.$$

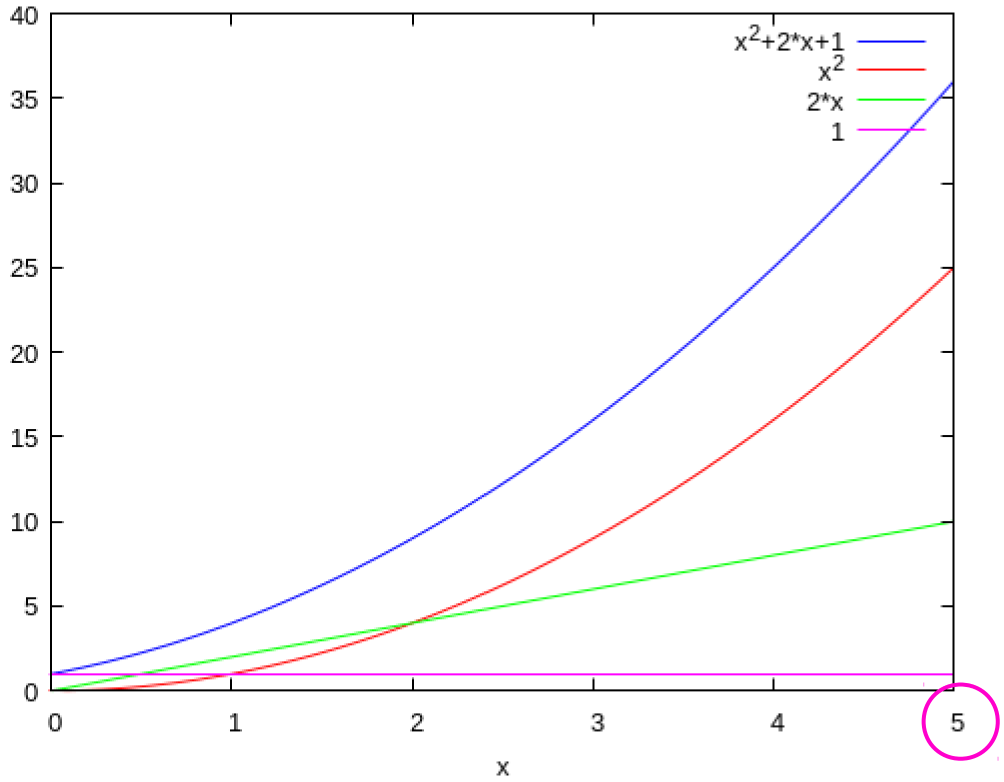
$$A_1 = [0, 5]$$

$$A_2 = [0, 100]$$

$$A_3 = [0, 500]$$

Small Range

All are distinguishable



x^2+2x+1

x^2

$2x$

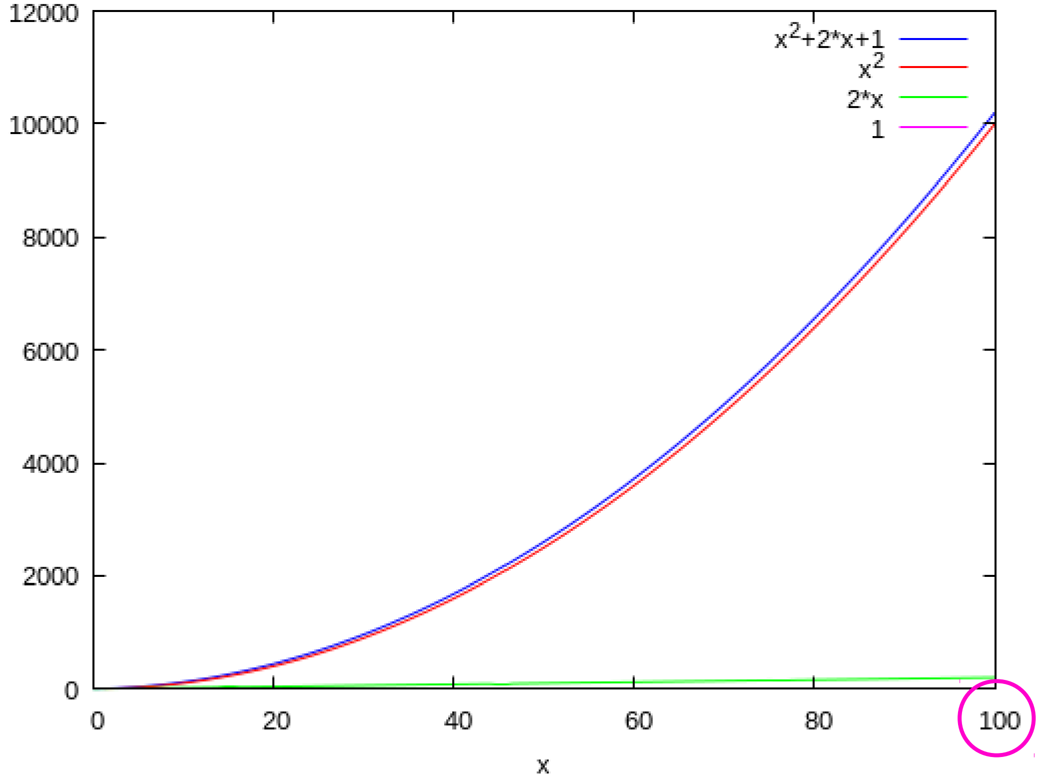
1

Zoom Out

for $x > -0.5$ $\quad x^2 < x^2+2x+1$

Medium Range

similar



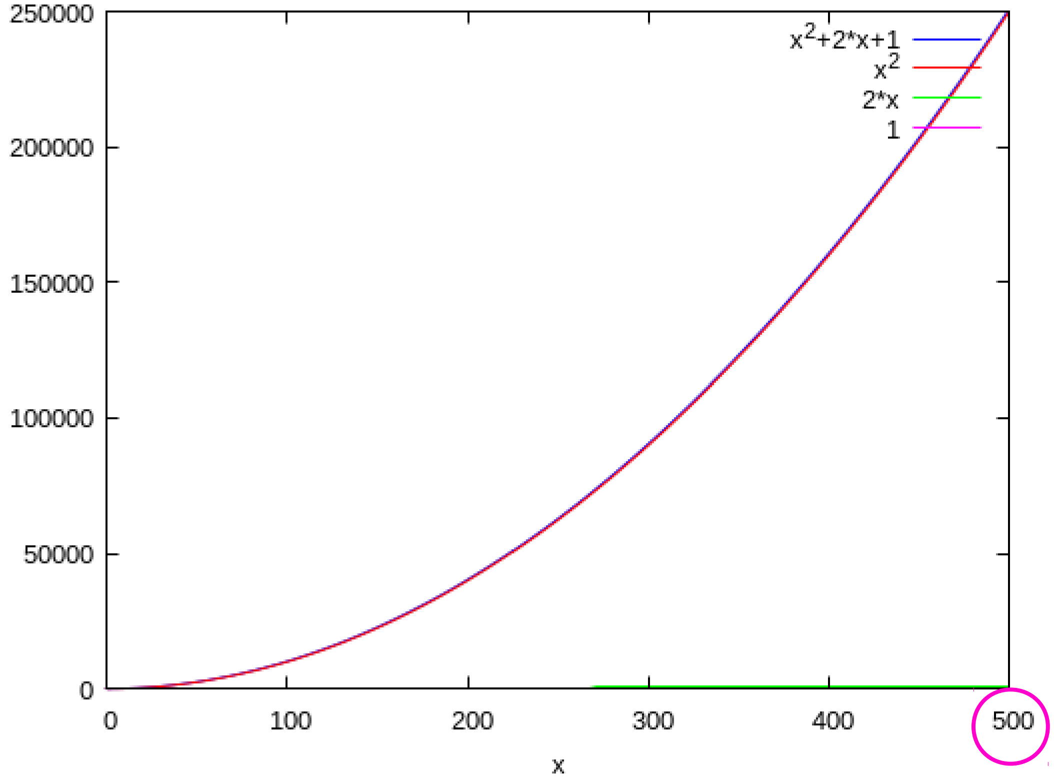
$x^2 + 2x + 1$ $\quad$ $10000 + 201$

x^2 $\quad$ 10000

$2x$

Zoom Out More

for $x > -0.5$ $\qquad$ $x^2 < x^2 + 2x + 1$



Indistinguishable

$$\begin{cases} x^2+2x+1 & 250000+1001 \\ x^2 & 250000 \end{cases}$$

for $x > -0.5$ $\qquad x^2 < x^2+2x+1$

Functions and Ranges

$$\begin{cases} 2 \cdot x^2 \\ x^2 + 2x + 1 \\ 2x \\ 1 \end{cases}$$

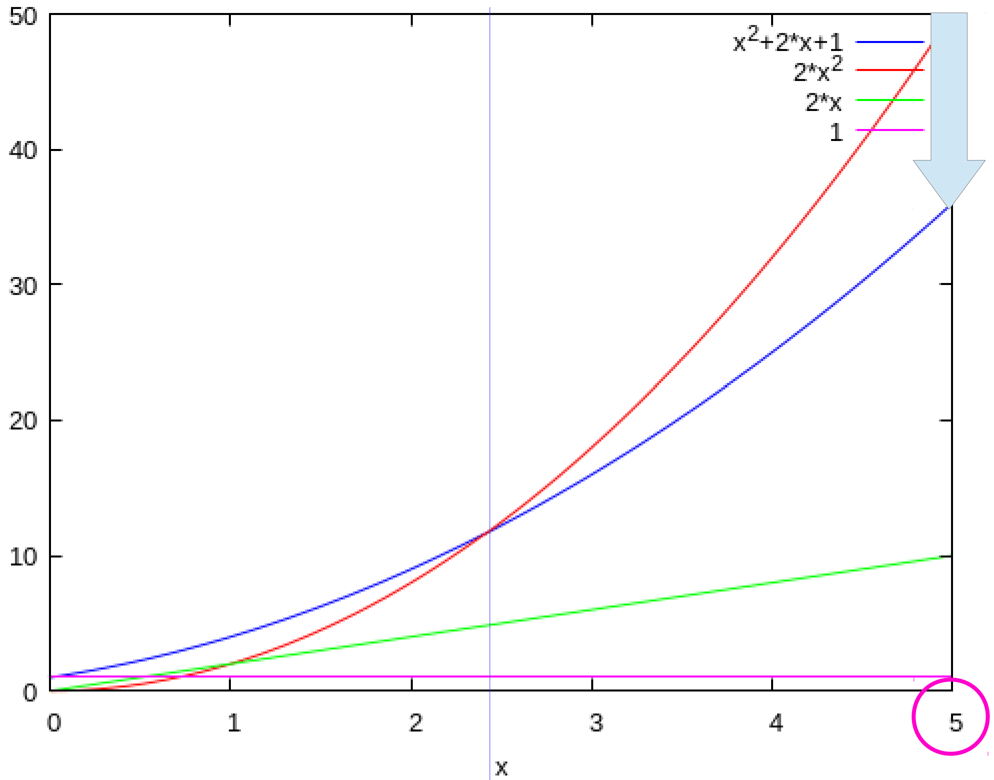
$$B_1 = [0, 5]$$

$$B_2 = [0, 100]$$

$$B_3 = [0, 500]$$

Small Range, $2x^2$

distinguishable



$2x^2$ 50

x^2+2x+1 25+11

$2x$

1

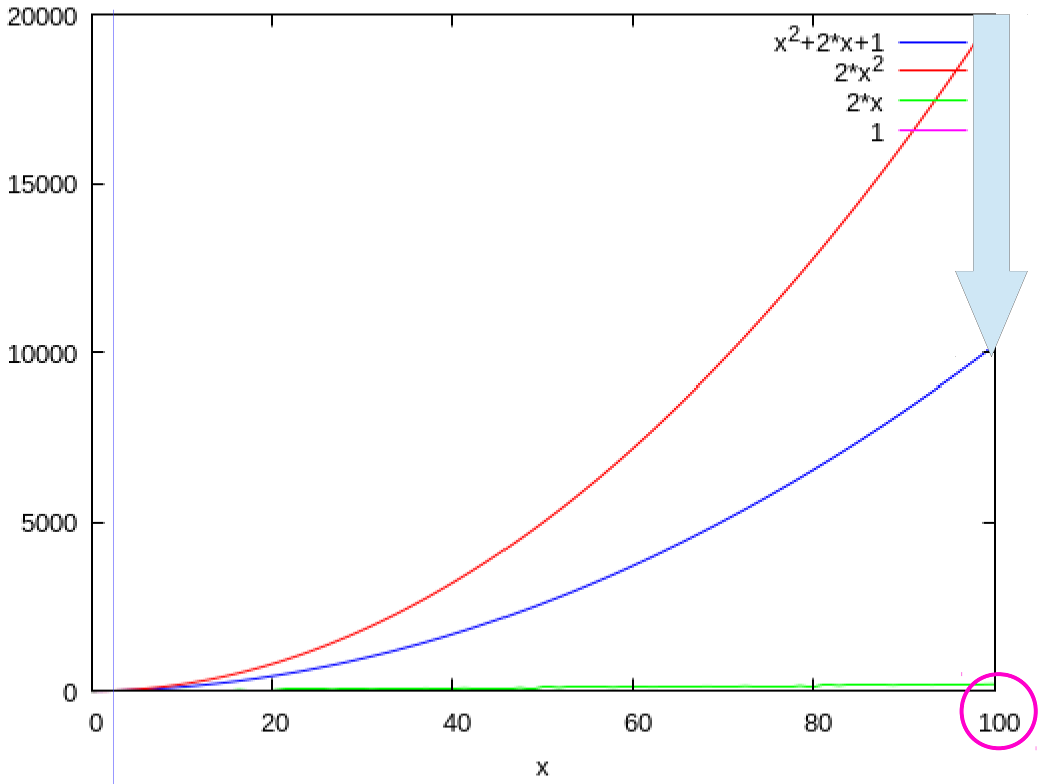
Zoom Out

for $x > 2.414$ $x^2+2x+1 < 2x^2$

for $x > k$ $f(n) < C g(n)$

Medium Range, $2x^2$

distinguishable



$2x^2$ 20000

x^2+2x+1 10000+201

$2x$

1

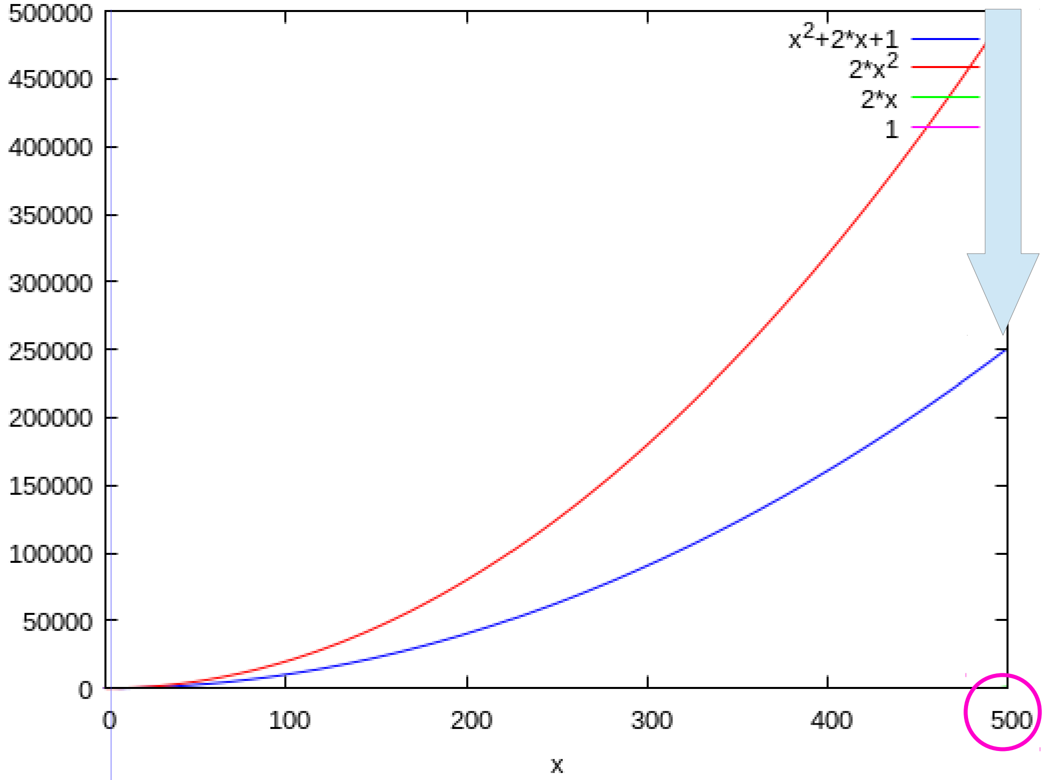
Zoom Out

for $x > 2.414$ $x^2+2x+1 < 2x^2$

for $x > k$ $f(n) < Cg(n)$

Large Range, $2x^2$

distinguishable



$2x^2$ 500000

x^2+2x+1 $250000+1001$

$2x$

1

Zoom Out

$for \quad x > 2.414$ $\quad x^2+2x+1 < 2x^2$

$for \quad x > k$ $\quad f(n) < Cg(n)$

Functions and Ranges

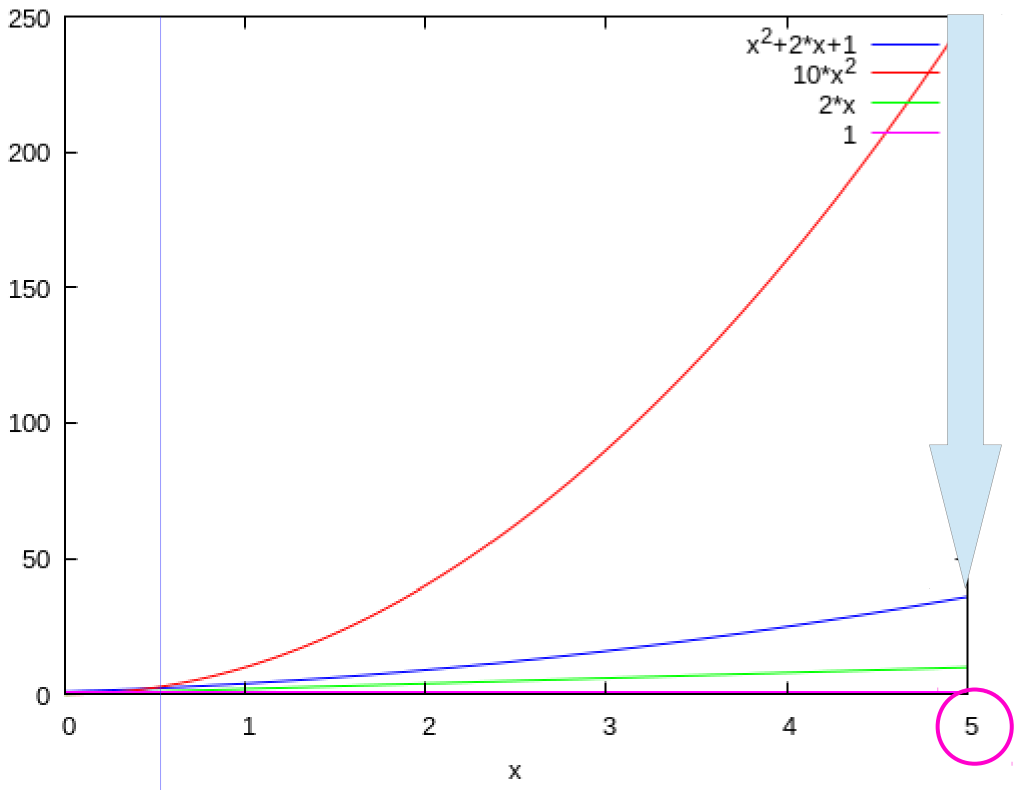
$$\left\{ \begin{array}{l} 10 \cdot x^2 \\ x^2 + 2x + 1 \\ 2x \\ 1 \end{array} \right.$$

$$C_1 = [0, 5]$$

$$C_2 = [0, 100]$$

$$C_3 = [0, 500]$$

Small Range, $10x^2$



distinguishable

$\mathbf{10}\, x^2$ 250

x^2+2x+1 $25+11$

$2x$

1

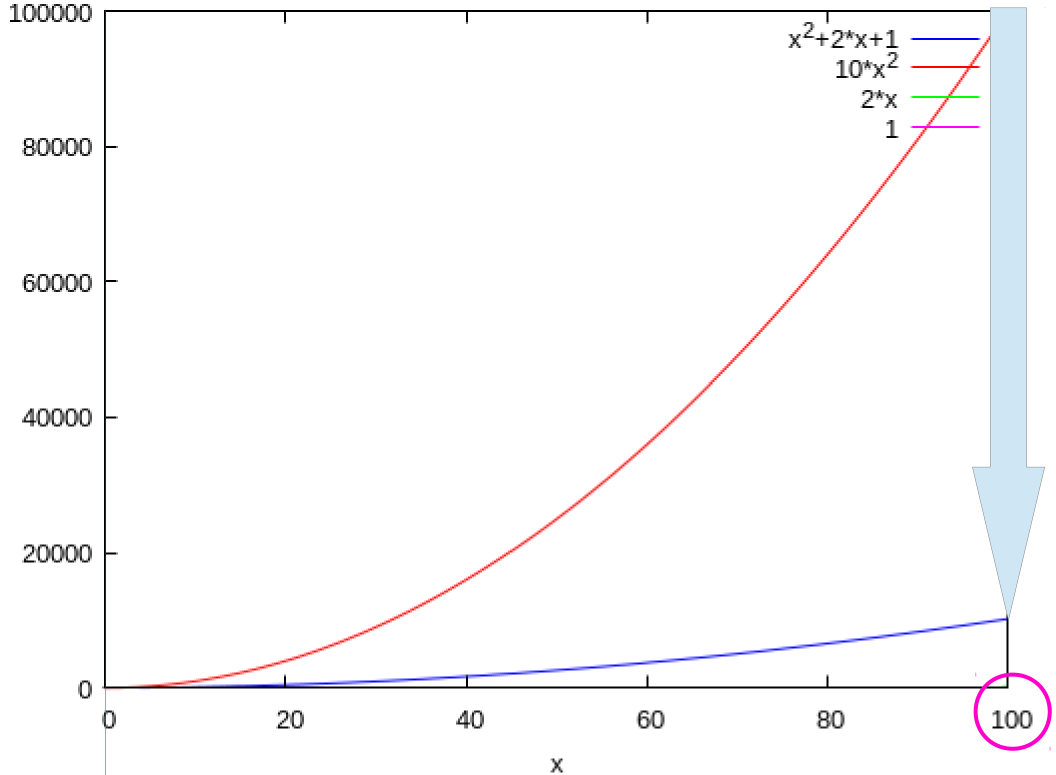
Zoom Out

$for \quad x > 0.462 \qquad x^2+2x+1 < \mathbf{10}\, x^2$

$for \quad x > k \qquad f(n) < \mathbf{C}\, g(n)$

Medium Range, $10x^2$

distinguishable



$10x^2$ 100000

x^2+2x+1 10000+201

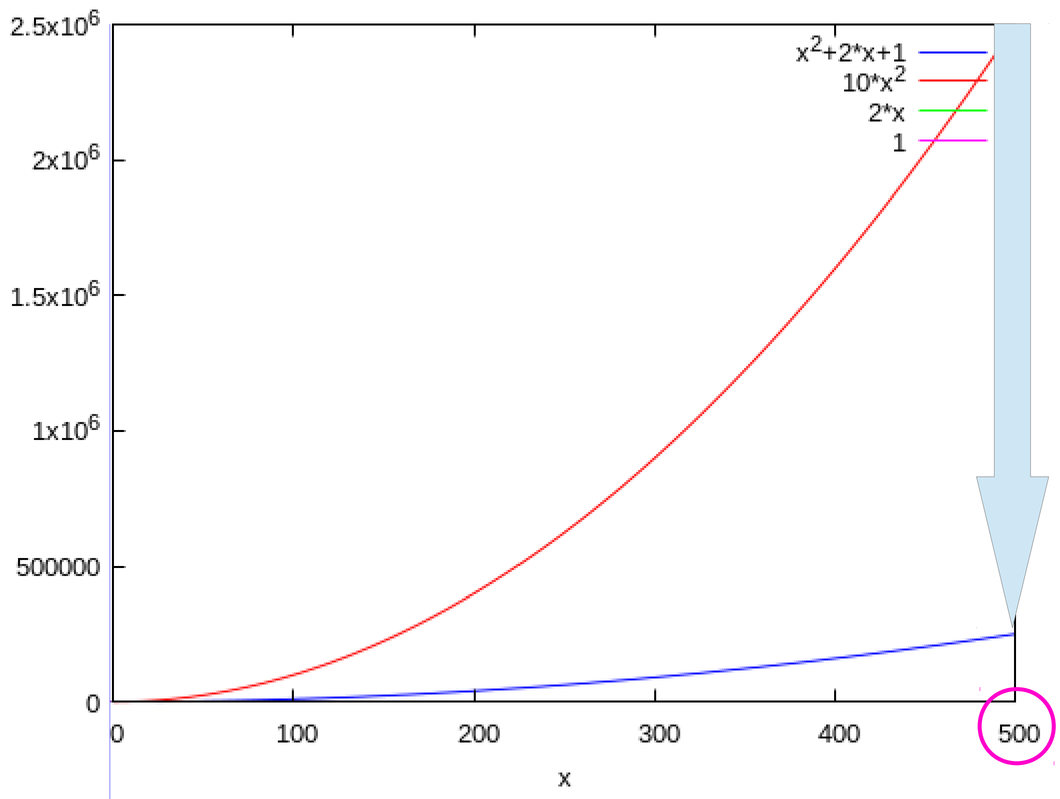
Zoom Out

for $x > 0.462$ $x^2+2x+1 < \mathbf{10}\,x^2$

for $x > k$ $f(n) < C\,g(n)$

Large Range, $10x^2$

distinguishable



$10x^2$ 2500000

x^2+2x+1 $250000+1001$

$for \quad x > 0.462 \qquad x^2+2x+1 < 10x^2$

$for \quad x > k \qquad f(n) < C\,g(n)$

Functions and Ranges

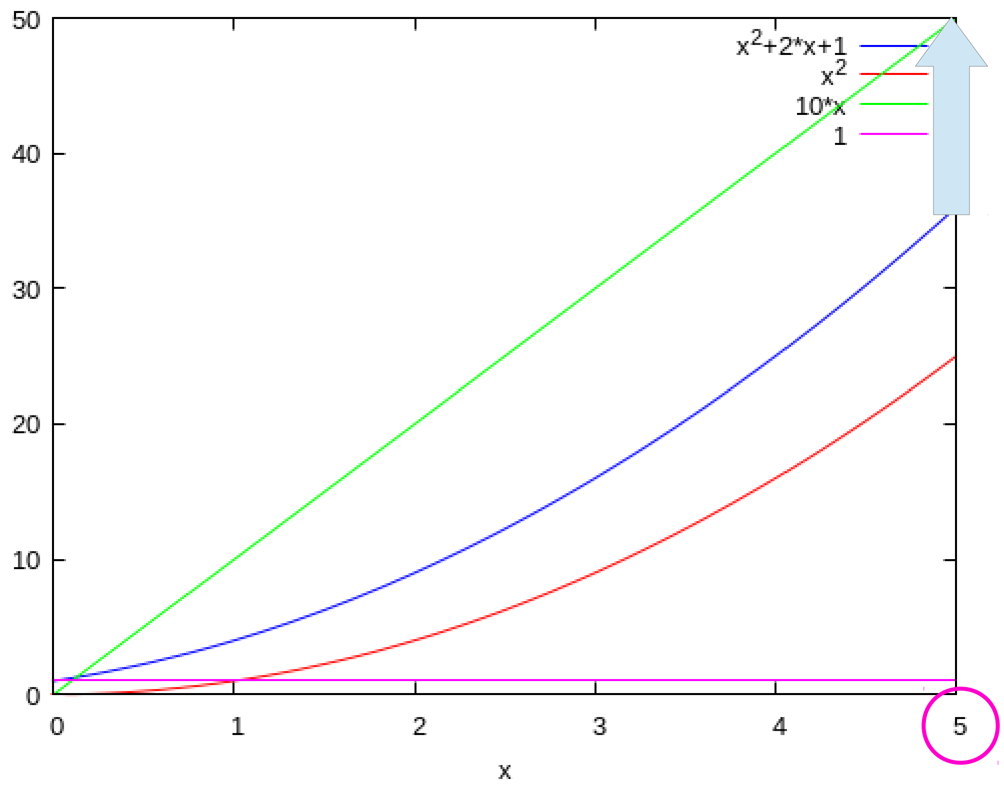
$$\begin{cases} 10 \cdot x \\ x^2 + 2x + 1 \\ x^2 \\ 1 \end{cases}$$

$$D_1 = [0, 5]$$

$$D_2 = [0, 100]$$

$$D_3 = [0, 500]$$

Small Range, *10x*



$10\,x$

$x^2+2\,x+1$

distinguishable

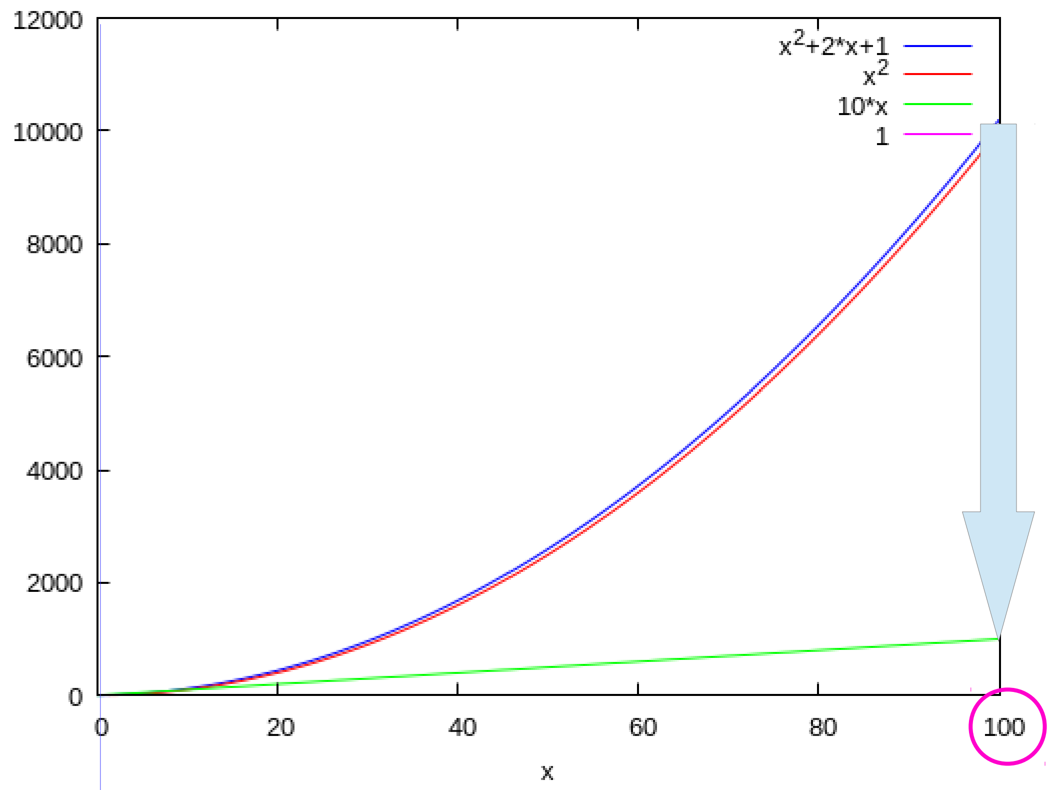
x^2

1

Zoom Out

for $0.127 < x < 7.873 \qquad x^2+2\,x+1 < 10\,x$

Medium Range, *10x*



x^2+2x+1

x^2

indistinguishable

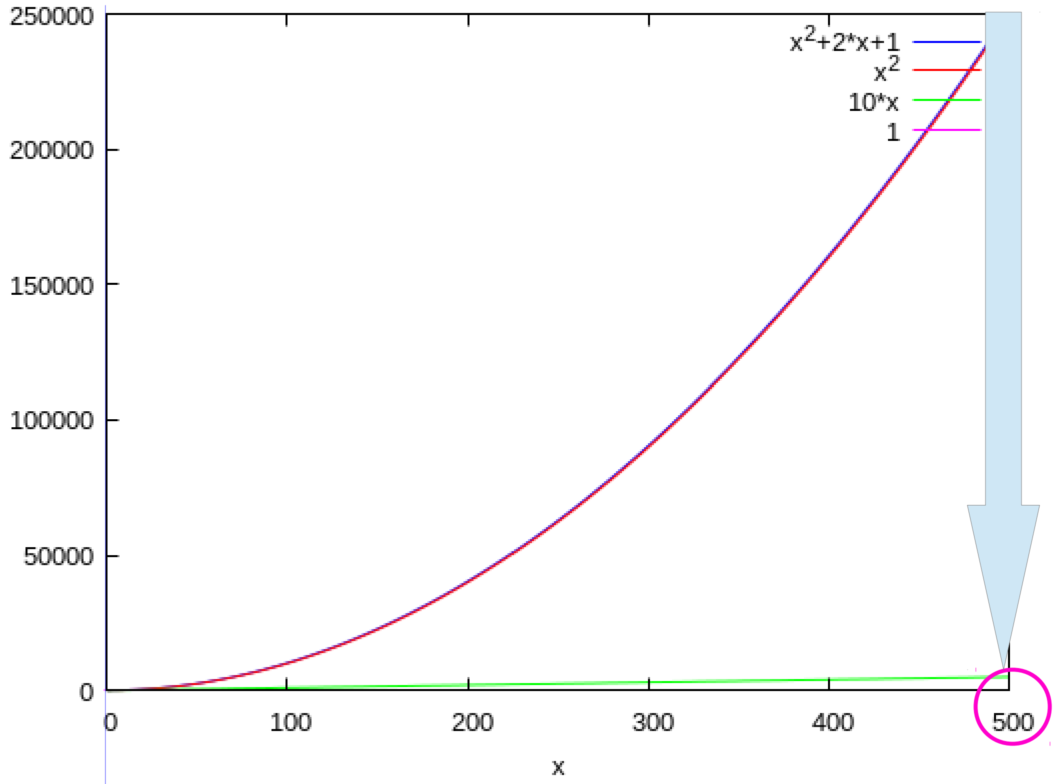
$\mathbf{10}x$

Zoom Out

for $x > 7.873$ $\mathbf{10}x < x^2+2x+1$

for $x > k$ $\mathbf{C}\,g(n) < f(n)$

Large Range, *10x*



x^2+2x+1

x^2

indistinguishable

$\mathbf{10}\,x$

$for \quad x > 7.873 \qquad\qquad \mathbf{10}\,x < x^2+2x+1$

$for \quad x > k \qquad\qquad \mathbf{C}\,g(n) < f(n)$

Big-**O** Definition

Let f and g be functions $(Z \to R \text{ or } R \to R)$
from the set of integers or
the set of real numbers
to the set of real numbers.

We say f(x) is **O(**g(x)**)** "f(x) is **big-oh** of g(x)"

If there are constants C and k such that

$$|f(x)| \leq C|g(x)| \quad \text{whenever } x > k.$$

↑

g(x) : **upper bound** of f(x)

Big-Ω Definition

Let f and g be functions (Z → R or R → R)
from the set of integers or
the set of real numbers
to the set of real numbers.

We say $f(x)$ is **Ω(**$g(x)$**)** "$f(x)$ is **big-omega** of $g(x)$"

If there are constants C and k such that

$$C|g(x)| \le |f(x)| \quad \text{whenever } x > k.$$

↑

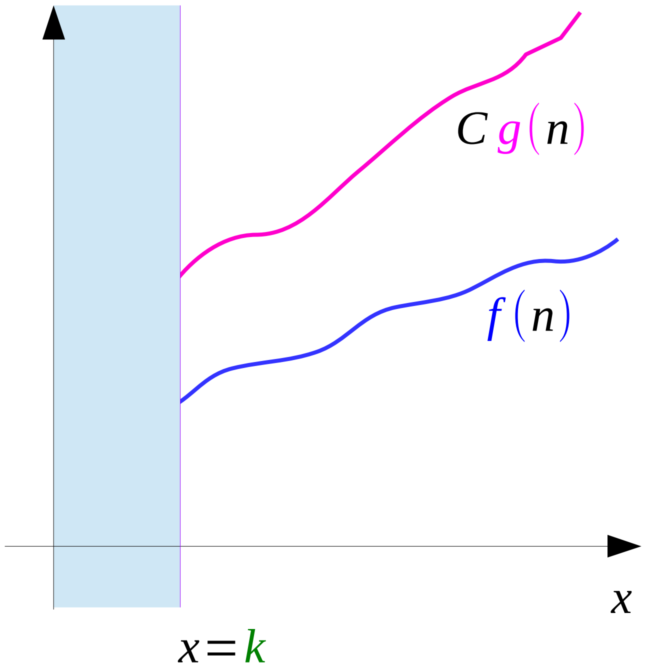
$g(x)$: **lower bound** of $f(x)$

Big-**O** Definition

$$\text{for} \quad k < x$$

$$f(x) \leq C|g(x)|$$

$$f(x) \ \text{is} \ O(g(x))$$



g(x) : **upper bound** of f(x)

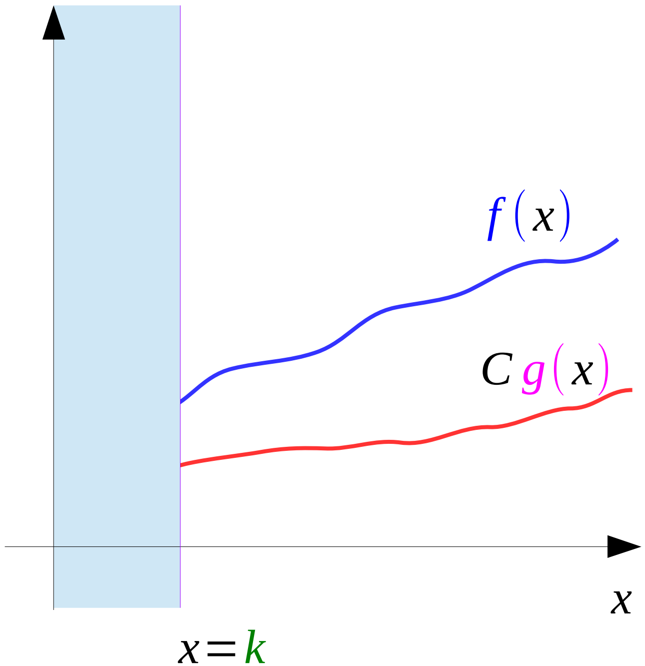
g(x) has a simpler form than f(x)
is usually a single term

Big-Ω Definition

$$for \quad k < x$$

$$f(x) \geq C|g(x)|$$

$$f(x) \quad is \quad \Omega(g(x))$$



g(x) : **lower bound** of f(x)

g(x) has a simpler form than f(x)

is usually a single term

Big-Θ definition

$$for \quad k < x$$

$$f(x) \le C|g(x)| \iff f(x) \ is \ O(g(x))$$

$$C|g(x)| \le f(x) \iff f(x) \ is \ \Omega(g(x))$$

$$C_1|g(x)| \le f(x) \le C_2|g(x)| \iff f(x) \ is \ \Theta(g(x))$$

Big-Θ = Big-Ω ∩ Big-O

$for \quad k < x$

$$\overbrace{\underbrace{C_1|g(x)| \le f(x)}_{\Omega(g(x))} \le C_2|g(x)|}^{O(g(x))} \quad \Longleftrightarrow \quad f(x) \ is \ \Theta(g(x))$$

$$\Omega(g(x)) \wedge O(g(x)) \quad \Longleftrightarrow \quad \Theta(g(x))$$

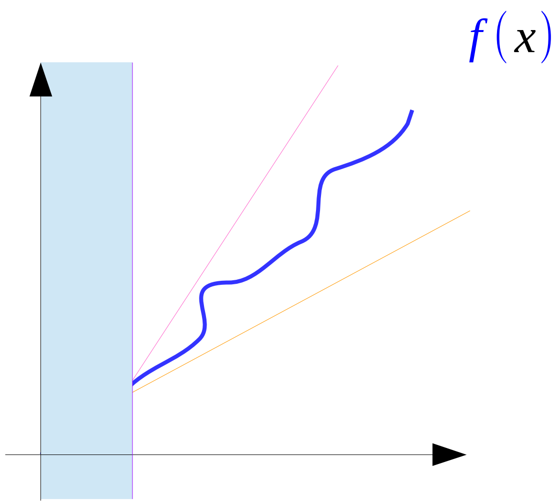
Θ(x) and Θ(1)

$for \quad 0 < k < x$

$f(x) \leq C\,x \iff f(x)$ is $O(x)$

$C\,x \leq f(x) \iff f(x)$ is $\Omega(x)$

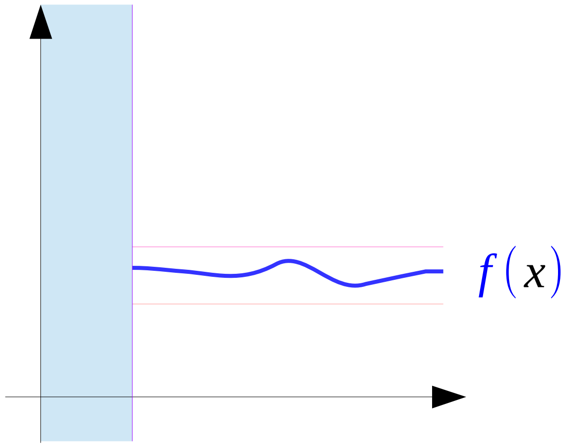
$C_1\,x \leq f(x) \leq C_2\,x \iff f(x)$ is $\Theta(x)$



$f(x) \leq C \cdot 1 \iff f(x)$ is $O(1)$

$C \cdot 1 \leq f(x) \iff f(x)$ is $\Omega(1)$

$C_1 \cdot 1 \leq f(x) \leq C_2 \cdot 1 \iff f(x)$ is $\Theta(1)$



Big-**O**, Big-**Ω**, Big-**Θ** Examples

for $x > -0.5$ $\quad \mathbf{1}\,x^2 < x^2+2x+1 \qquad x^2+2x+1$ *is* $\boldsymbol{\Omega}(x^2)$

lower bound

upper bound

for $x > 2.414$ $\quad x^2+2x+1 < \mathbf{2}\,x^2 \qquad x^2+2x+1$ *is* $\boldsymbol{O}(x^2)$

x^2+2x+1 *is* $\boldsymbol{\Theta}(x^2)$

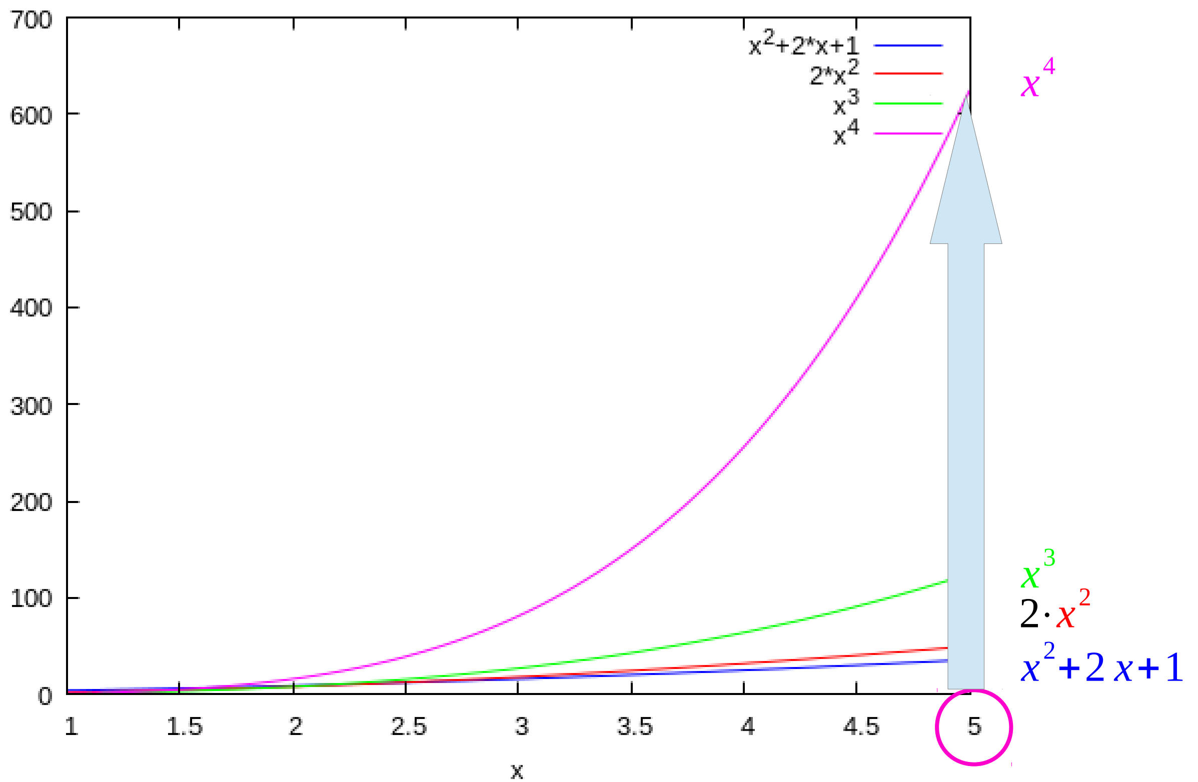
upper bound

for $x > 0.462$ $\quad x^2+2x+1 < \mathbf{10}\,x^2 \qquad x^2+2x+1$ *is* $\boldsymbol{O}(x^2)$

for $x > 7.873$ $\quad \mathbf{10}\,x < x^2+2x+1 \qquad x^2+2x+1$ *is* $\boldsymbol{\Omega}(x)$

lower bound

Many Larger Upper Bounds



x^2+2x+1 *is* $O(x^2)$

x^2+2x+1 *is* $O(x^3)$

x^2+2x+1 *is* $O(x^4)$

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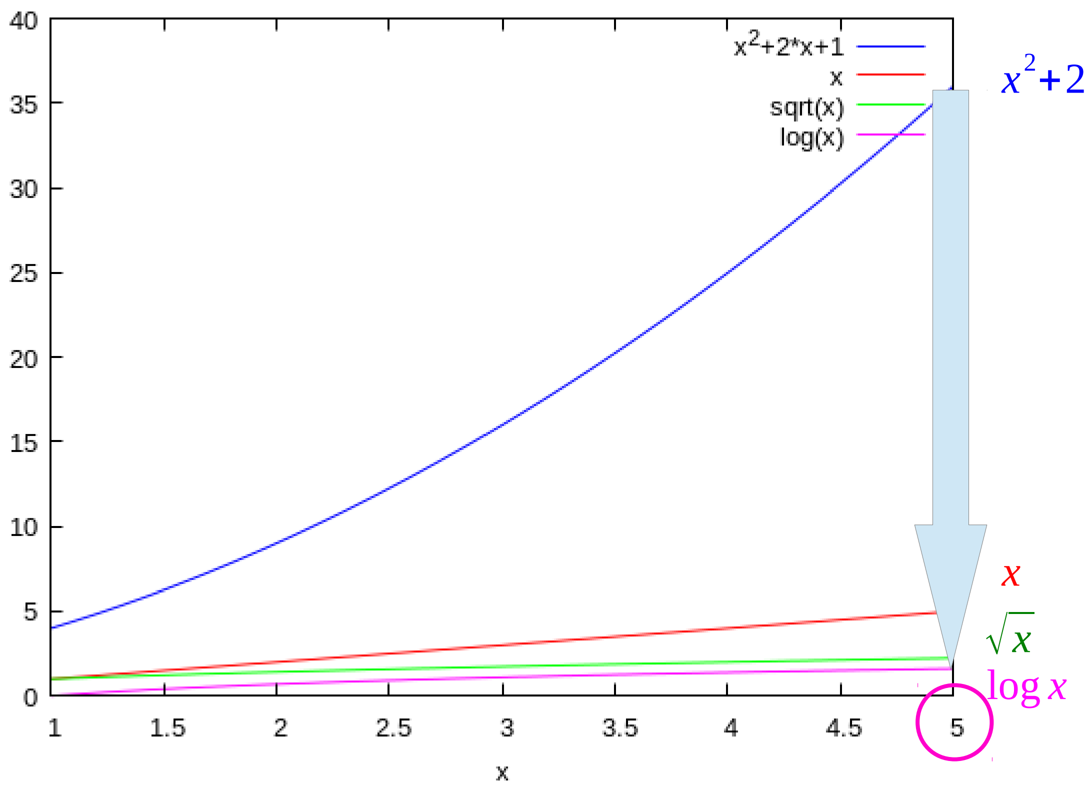
•
•

the least upper bound?

Many Smaller Lower Bound



x^2+2x+1 *is* $\Omega(x^2)$

x^2+2x+1 *is* $\Omega(x)$

x^2+2x+1 *is* $\Omega(\sqrt{x})$

x^2+2x+1 *is* $\Omega(\log x)$

•
•
•

the <u>greatest</u> lower bound?

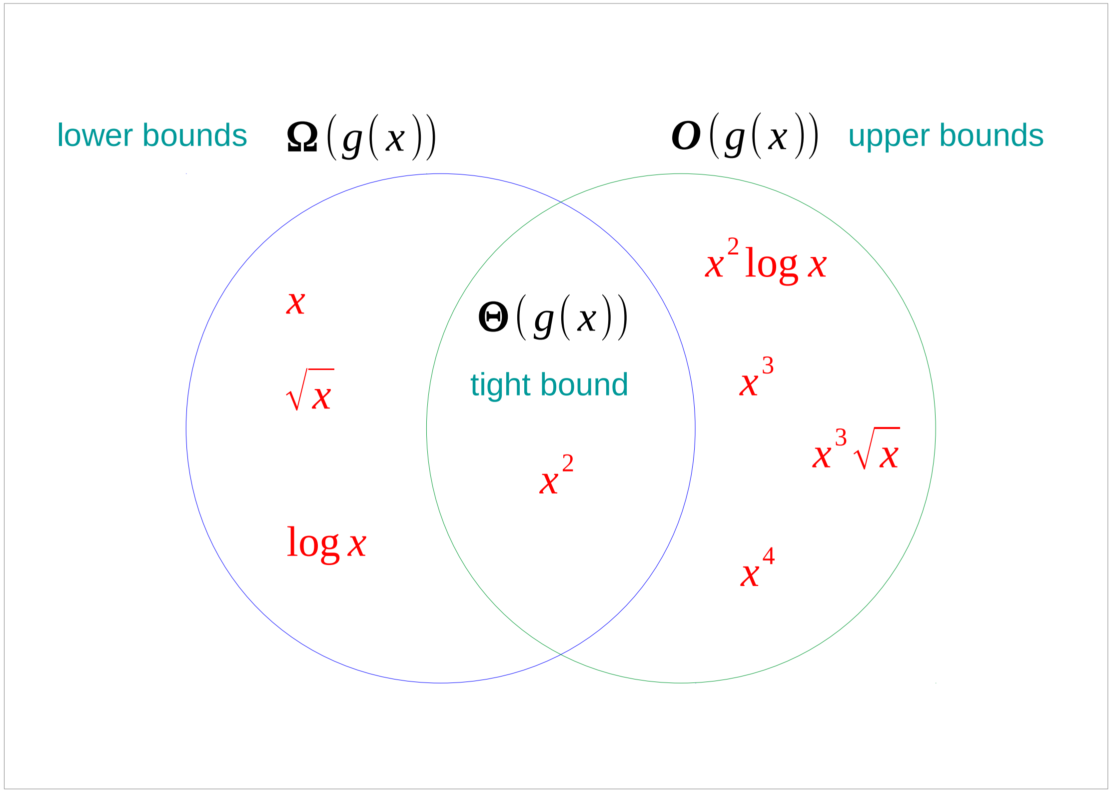
Many Upper and Lower Bounds

x^2+2x+1 *is* $\boldsymbol{O}(x^2)$	$\Leftrightarrow$	$x^2+2x+1 \le C\,x^2$	upper bound	the least
x^2+2x+1 *is* $\boldsymbol{O}(x^3)$	$\Leftrightarrow$	$x^2+2x+1 \le C\,x^3$	upper bound	
x^2+2x+1 *is* $\boldsymbol{O}(x^4)$	$\Leftrightarrow$	$x^2+2x+1 \le C\,x^4$	upper bound	
⋮		⋮		

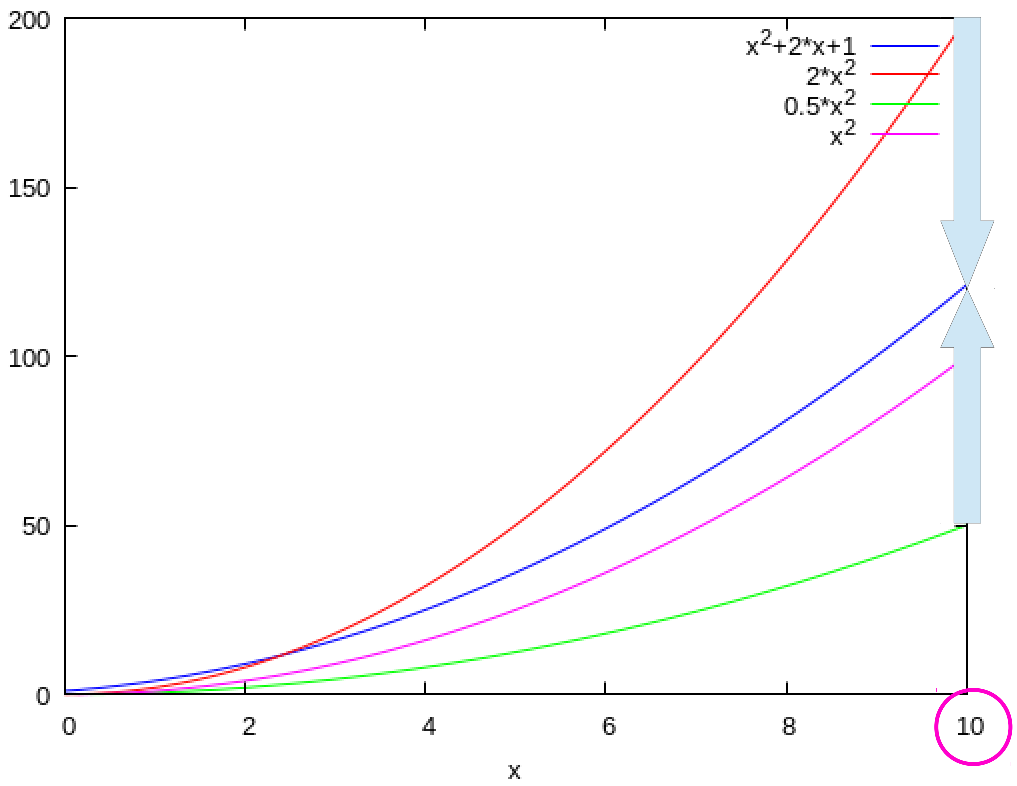
x^2+2x+1 *is* $\boldsymbol{O}(x^2)$	$\Leftrightarrow$	$x^2+2x+1 \ge C\,x^2$	lower bound	the greatest
x^2+2x+1 *is* $\boldsymbol{\Omega}(x)$	$\Leftrightarrow$	$x^2+2x+1 \ge C\,x$	lower bound	
x^2+2x+1 *is* $\boldsymbol{\Omega}(\sqrt{x})$	$\Leftrightarrow$	$x^2+2x+1 \ge C\sqrt{x}$	lower bound	
x^2+2x+1 *is* $\boldsymbol{\Omega}(\log x)$	$\Leftrightarrow$	$x^2+2x+1 \ge C\log x$	lower bound	
⋮		⋮		

Simultaneously being lower and upper bound

$$f(x) = x^2 + 2x + 1$$



Big-Θ Examples (1)



$2 \cdot x^2$

$x^2 + 2 * x + 1$

$1 \cdot x^2$

$0.5 \cdot x^2$

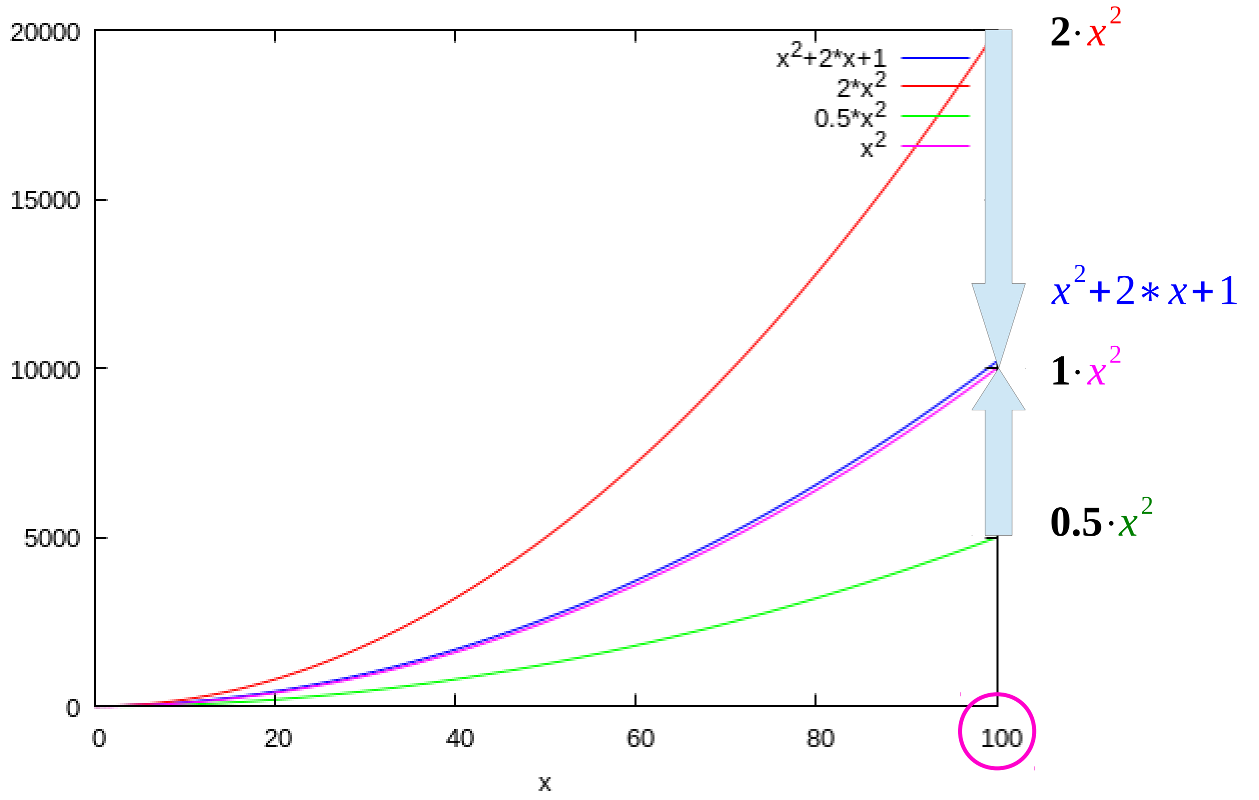
$x^2 + 2x + 1$ *is* $\boldsymbol{O}(x^2)$

$x^2 + 2x + 1$ *is* $\boldsymbol{\Theta}(x^2)$

$x^2 + 2x + 1$ *is* $\boldsymbol{\Omega}(x^2)$

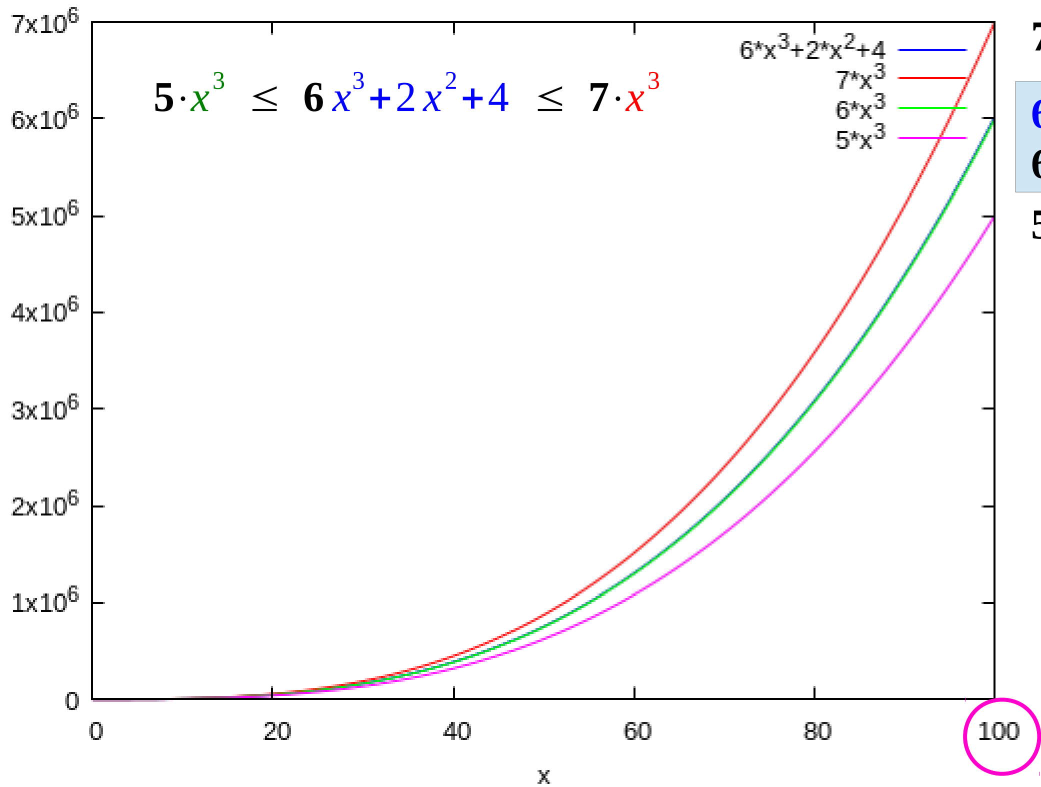
Zoom Out

Big-Θ Examples (2)



x^2+2x+1 *is* $\Theta(x^2)$

Big-Θ Examples (3)



$7 \cdot x^3$

$6x^3 + 2x^2 + 4$

$6 \cdot x^3$

$5 \cdot x^3$

indistinguishable

$6x^3 + 2x^2 + 4$ *is* $\Theta(x^3)$

$6x^3 + 2x^2 + 4 \approx 6 \cdot x^3$

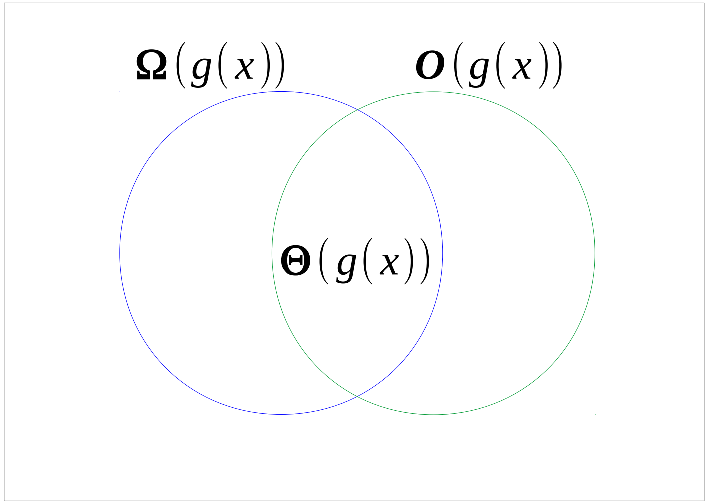
Tight bound Implications

$f(x)$ is $\Theta(g(x)) \longrightarrow f(x)$ is $O(g(x))$

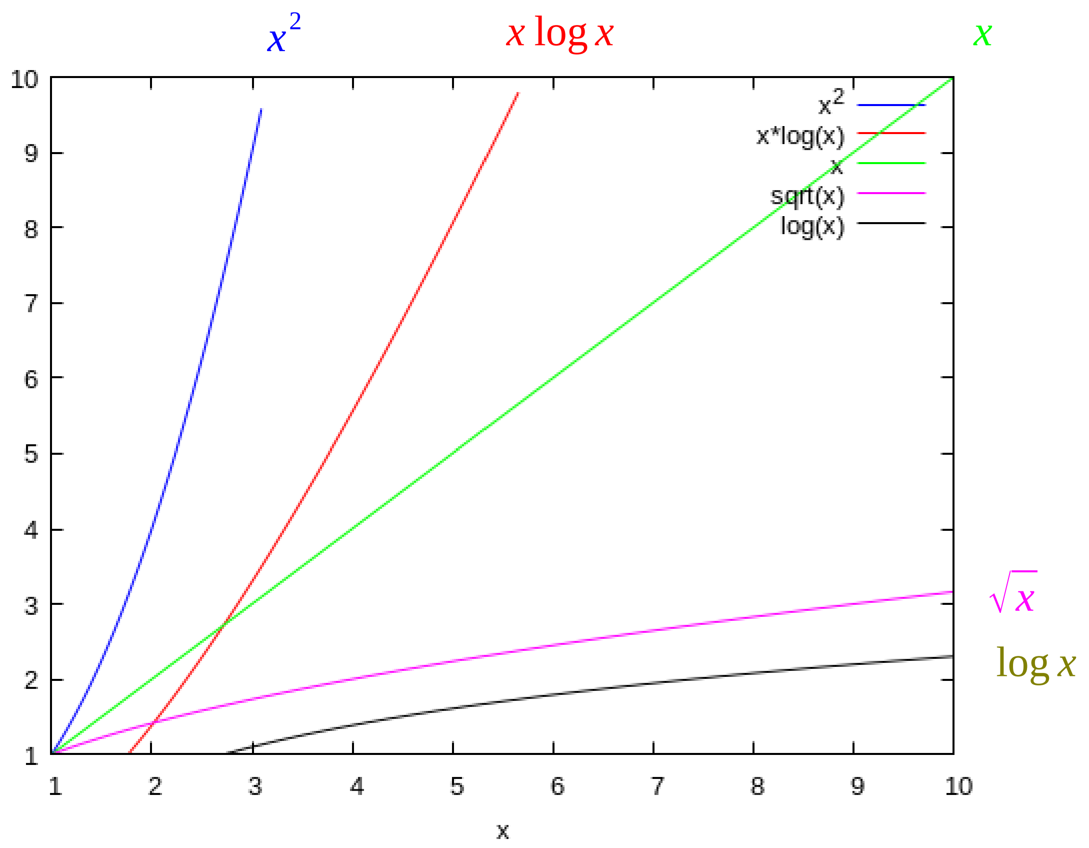
$f(x)$ is $\Theta(g(x)) \longrightarrow f(x)$ is $\Omega(g(x))$

$f(x)$ is $\Theta(g(x)) \not\longleftarrow f(x)$ is $O(g(x))$

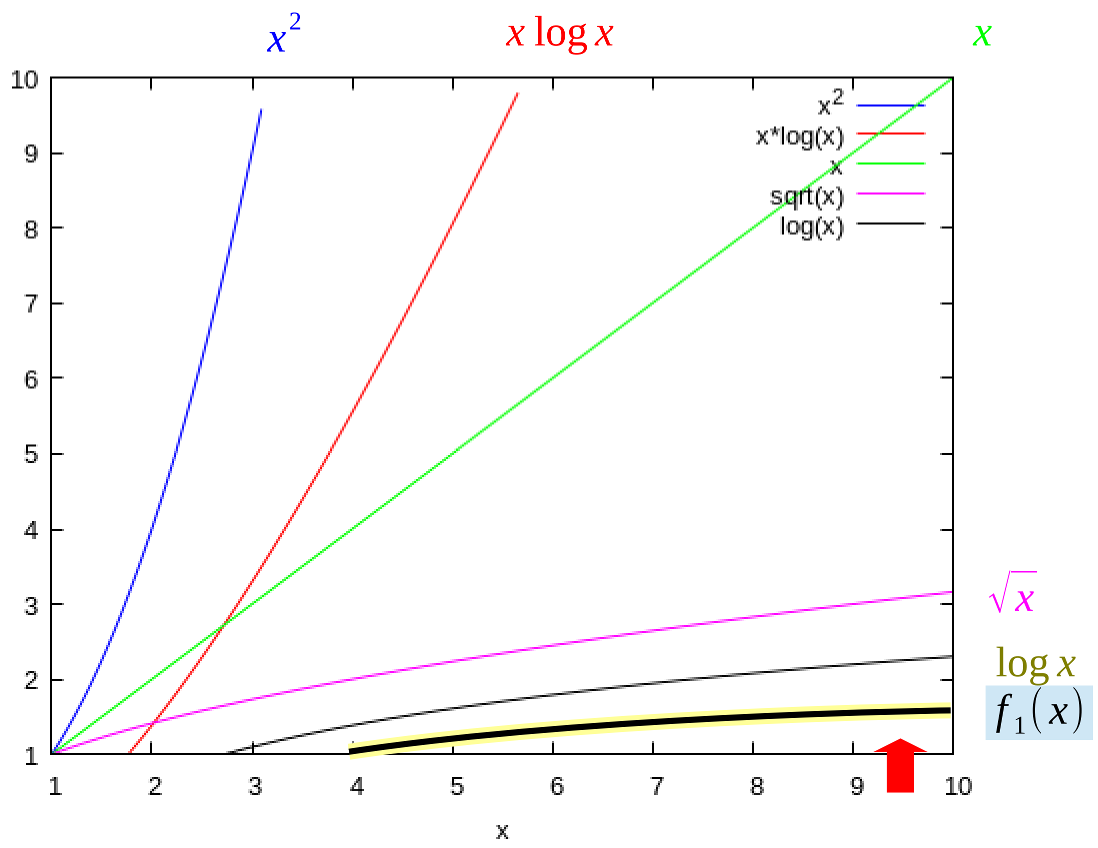
$f(x)$ is $\Theta(g(x)) \not\longleftarrow f(x)$ is $\Omega(g(x))$



Common Growth Functions

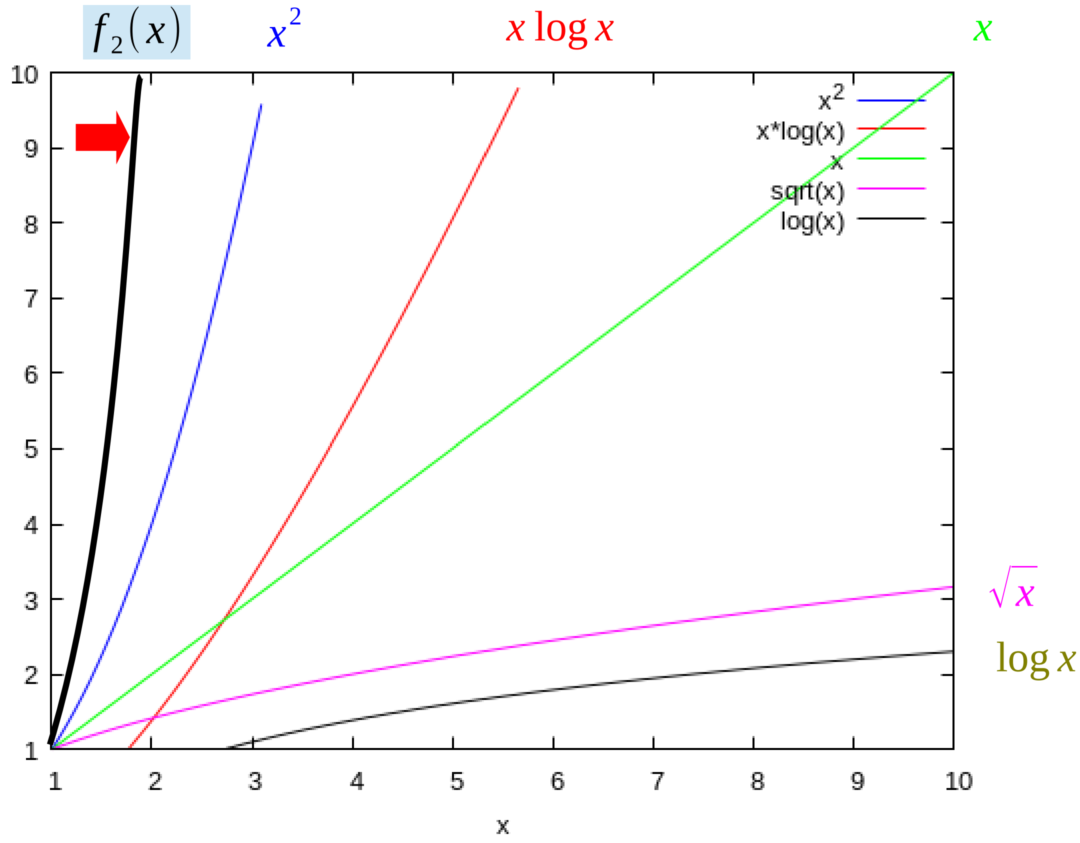


Upper bounds



$f_1(x)$ *is* $\boldsymbol{O}(\log x)$ → $\boldsymbol{O}(\sqrt{x})$ → $\boldsymbol{O}(x)$ → $\boldsymbol{O}(x \log x)$ → $\boldsymbol{O}(x^2)$

Lower bounds



$f_2(x)$ *is* $\mathbf{\Omega}(x^2)$ ➡ $\mathbf{\Omega}(x \log x)$ ➡ $\mathbf{\Omega}(x)$ ➡ $\mathbf{\Omega}(\sqrt{x})$ ➡ $\mathbf{\Omega}(\log x)$

Example 1

$f(n)=n^6+3n$	$f(n)=O(n^6)$	$f(n)=\Omega(n)$
$f(n)=2^n+12$	$f(n)=O(2^n)$	$f(n)=\Omega(1)$
$f(n)=2^n+3^n$	$f(n)=O(3^n)$	$f(n)=\Omega(2^n)$
$f(n)=n^n+n$	$f(n)=O(n^n)$	$f(n)=\Omega(n)$

https://discrete.gr/complexity/

Example 2

$f(n)=n^6+3n$	$f(n)=O(n^6)$	$f(n)=\Omega(n^6)$	$f(n)=\Theta(n^6)$
$f(n)=2^n+12$	$f(n)=O(2^n)$	$f(n)=\Omega(2^n)$	$f(n)=\Theta(2^n)$
$f(n)=2^n+3^n$	$f(n)=O(3^n)$	$f(n)=\Omega(3^n)$	$f(n)=\Theta(3^n)$
$f(n)=n^n+n$	$f(n)=O(n^n)$	$f(n)=\Omega(n^n)$	$f(n)=\Theta(n^n)$

https://discrete.gr/complexity/

Example 3

$\Theta(n) \longrightarrow O(n)$

$\Theta(n) \longrightarrow O(n^2)$

$\Theta(n^2) \longrightarrow O(n^3)$

$\Theta(n) \not\longrightarrow O(1)$

$O(1) \longrightarrow \Theta(1)$

$O(n) \not\longrightarrow \Theta(1)$

generally not true

$\Theta(n) \rightarrow O(n) \rightarrow O(n^2)$

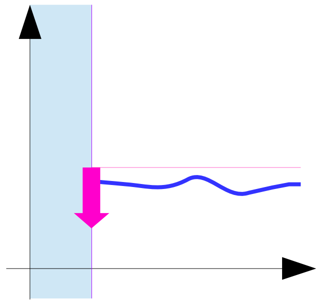
$\Theta(n^2) \rightarrow O(n^2) \rightarrow O(n^3)$

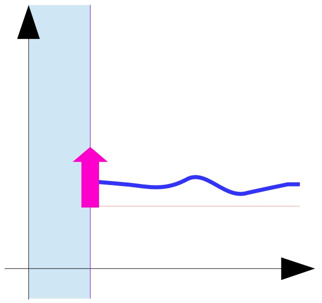
$\Theta(n) \rightarrow \Omega(n) \not\rightarrow O(1)$

$\Theta(1) \equiv \Omega(1) \equiv O(1)$

$\log(n) \; : \; O(\log n) \rightarrow O(n)$

$\log(n) \; : \; \Omega(\log n) \not\rightarrow O(1)$





https://discrete.gr/complexity/

Example 4

$\Theta(n)$	$\longrightarrow$	$O(n)$	upper bound	tight	
$\Theta(n^2)$	$\longrightarrow$	$O(n^3)$	upper bound		
$\Theta(1)$	$\longrightarrow$	$O(n)$	upper bound		
$\Theta(n)$	$\not\longrightarrow$	$O(1)$	upper bound	wrong	$\Theta(n) \rightarrow \Omega(n) \not\rightarrow O(1)$
$\Theta(n)$	$\longrightarrow$	$O(2n)$	upper bound	tight	$O(2n) = O(n)$

https://discrete.gr/complexity/

Example 5

$\Theta(1)$	$\longrightarrow$	$O(1)$	tight upper bound	$\longrightarrow$ $O(n)$	upper bound
$\Theta(\sqrt{n})$	$\longrightarrow$	$O(\sqrt{n})$	tight upper bound	$\longrightarrow$ $O(n)$	upper bound
$\Theta(n)$	$\longrightarrow$	$O(n)$	tight upper bound	$\longrightarrow$ $O(n^2)$	upper bound
$\Theta(n^2)$	$\longrightarrow$	$O(n^2)$	tight upper bound	$\longrightarrow$ $O(n^3)$	upper bound
$\Theta(n^3)$	$\longrightarrow$	$O(n^3)$	tight upper bound	$\longrightarrow$ $O(n^4)$	upper bound

$\Theta(1)$	$\longrightarrow$	$\Omega(1)$	tight lower bound		
$\Theta(\sqrt{n})$	$\longrightarrow$	$\Omega(\sqrt{n})$	tight lower bound	$\longrightarrow$ $\Omega(1)$	lower bound
$\Theta(n)$	$\longrightarrow$	$\Omega(n)$	tight lower bound	$\longrightarrow$ $\Omega(\sqrt{n})$	lower bound
$\Theta(n^2)$	$\longrightarrow$	$\Omega(n^2)$	tight lower bound	$\longrightarrow$ $\Omega(n)$	lower bound
$\Theta(n^3)$	$\longrightarrow$	$\Omega(n^3)$	tight lower bound	$\longrightarrow$ $\Omega(n^2)$	lower bound

https://discrete.gr/complexity/

Example 6

$f(n) = 6n^2 + 3n$

$f(n) = O(n^2) \longrightarrow f(n) = O(n^3)$ upper bound

$f(n) = \Theta(n^2) \longrightarrow f(n) \neq \Theta(n^3)$

$f(n) = \Omega(n^2) \longrightarrow f(n) \neq \Omega(n^3)$

$c_1 n^2 - c_2 n^3 = n^2(c_1 - c_2 n)$

$f(n) = O(n^2) \longrightarrow f(n) \neq O(n)$

$f(n) = \Theta(n^2) \longrightarrow f(n) \neq \Theta(n)$

$f(n) = \Omega(n^2) \longrightarrow f(n) = \Omega(n)$ lower bound

$c_1 n^2 - c_2 n = n(c_1 n - c_2)$

Example 7

$O(n^2)$	$\longrightarrow O(n^3)$	always true
$\Theta(n^2)$	$\not\longrightarrow \Theta(n^3)$	always false
$\Omega(n^2)$	$\not\longrightarrow \Omega(n^3)$	generally not true

$O(n^2)$	$\not\longrightarrow O(n)$	generally not true
$\Theta(n^2)$	$\not\longrightarrow \Theta(n)$	always false
$\Omega(n^2)$	$\longrightarrow \Omega(n)$	always true

References

[1] http://en.wikipedia.org/
[2]