Graph Overview (1A)

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This document was produced by using LibreOffice and Octave.

Some class of graphs (1)

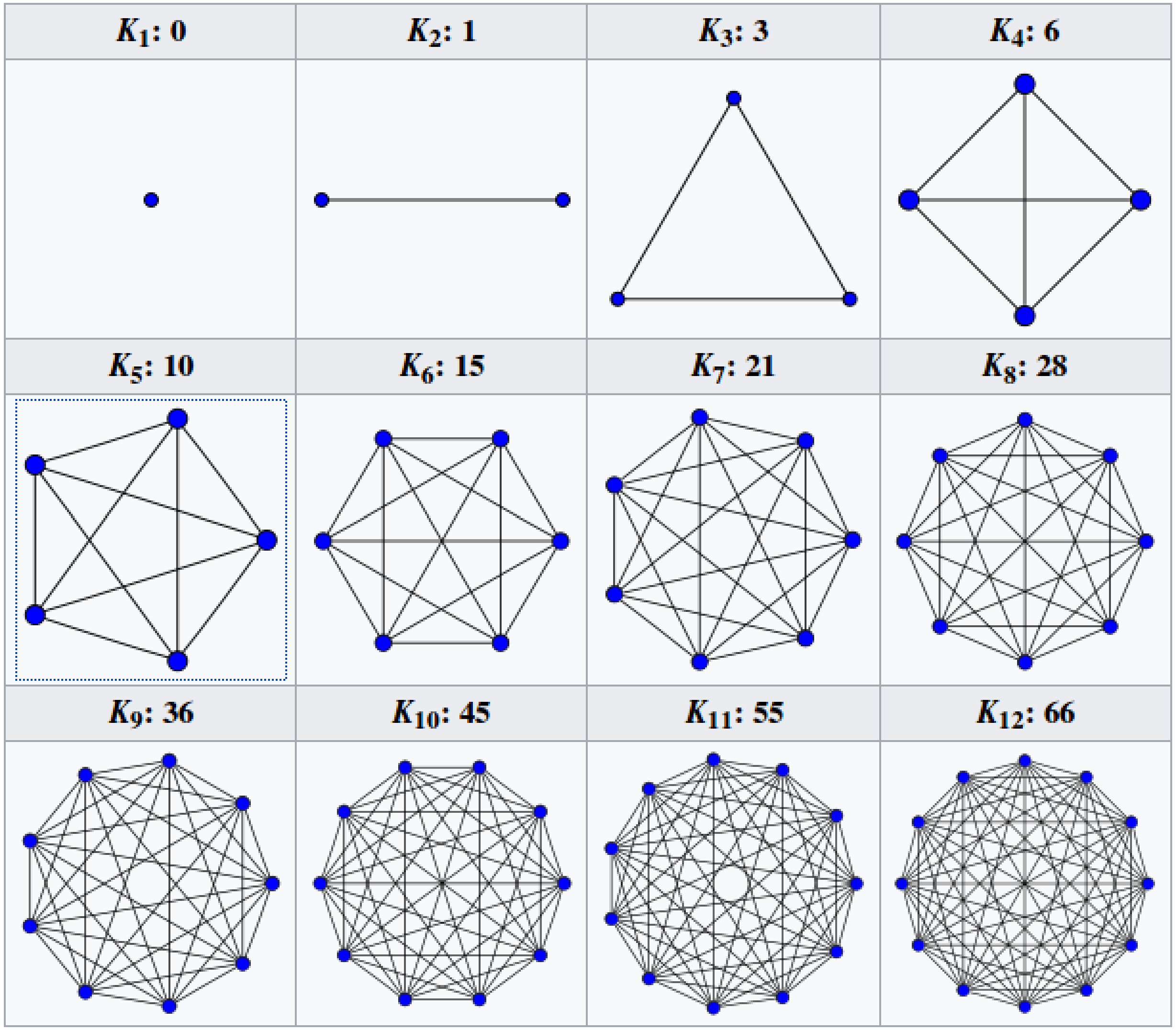
Complete graph
A complete graph is a graph in which each pair of vertices is joined by an edge.
A complete graph contains all possible edges.

Connected graph
In an undirected graph, an unordered pair of vertices {x, y} is called connected if a path leads from x to y. Otherwise, the unordered pair is called disconnected.

Bipartite graph
A bipartite graph is a graph in which the vertex set can be partitioned into two sets, W and X, so that no two vertices in W share a common edge and no two vertices in X share a common edge.
Alternatively, it is a graph with a chromatic number of 2.

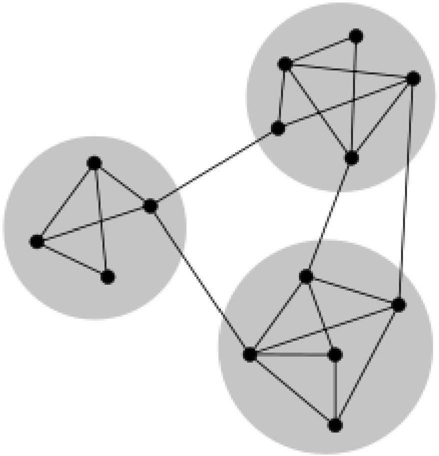
https://en.wikipedia.org/wiki/Graph_(discrete_mathematics)

Complete Graphs

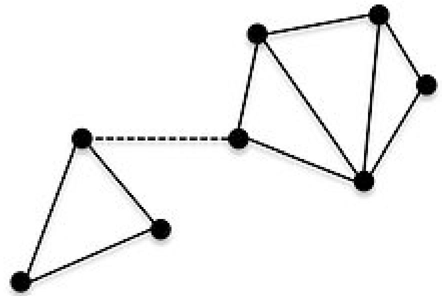


https://en.wikipedia.org/wiki/Complete_graph

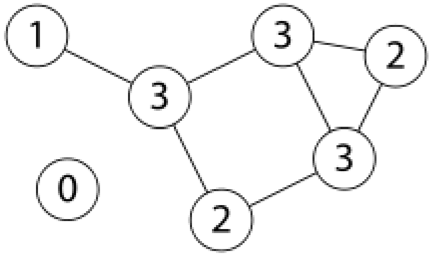
Connected Graphs



This graph becomes disconnected when the right-most node in the gray area on the left is removed



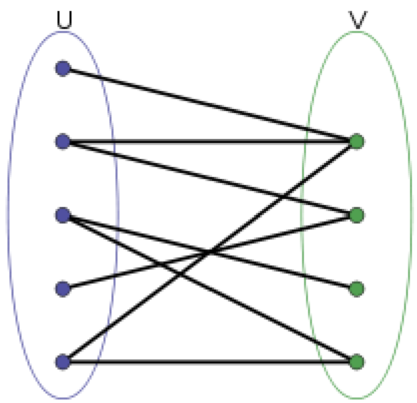
This graph becomes disconnected when the dashed edge is removed.



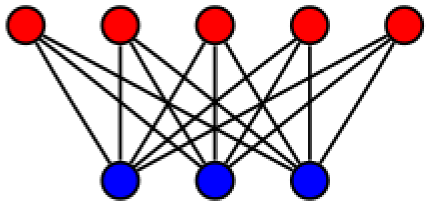
With vertex 0 this graph is disconnected, the rest of the graph is connected.

https://en.wikipedia.org/wiki/Connectivity_(graph_theory)

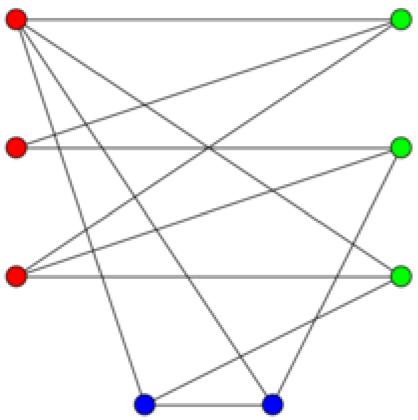
Bipartite Graphs



Example of a bipartite graph without cycles



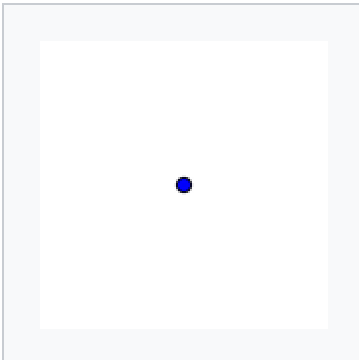
A complete bipartite graph with m = 5 and n = 3



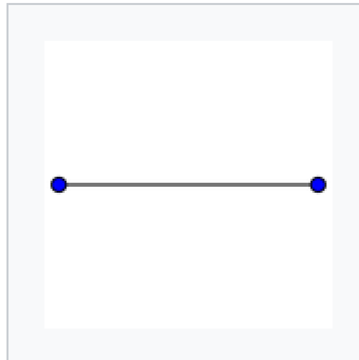
A graph with an odd cycle transversal of size 2: removing the two blue bottom vertices leaves a bipartite graph.

https://en.wikipedia.org/wiki/Bipartite_graph

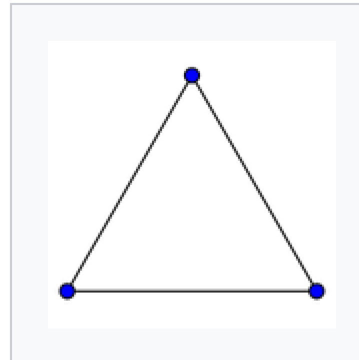
Complete Graphs



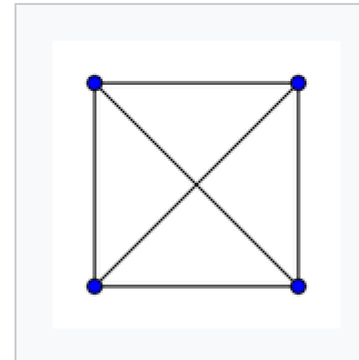
K_1



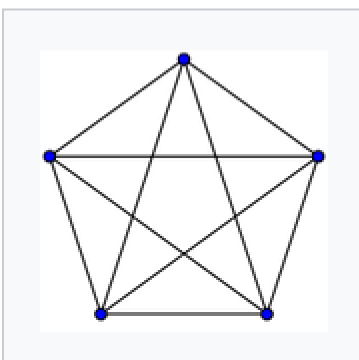
K_2



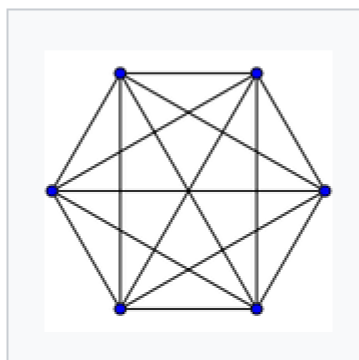
K_3



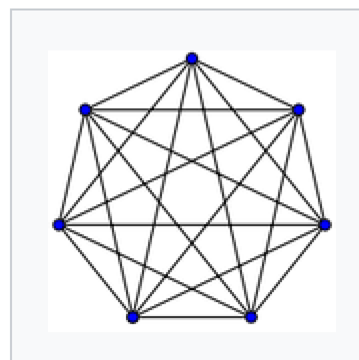
K_4



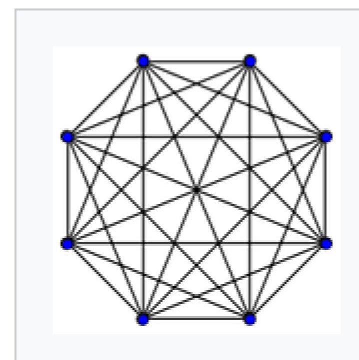
K_5



K_6



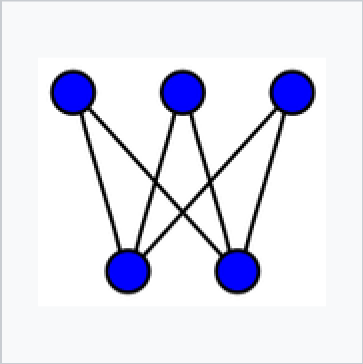
K_7



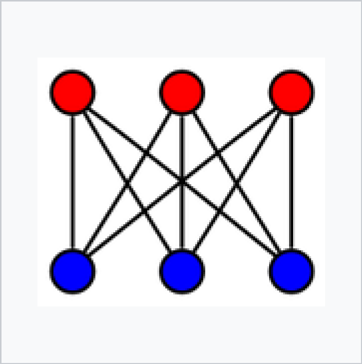
K_8

https://en.wikipedia.org/wiki/Gallery_of_named_graphs

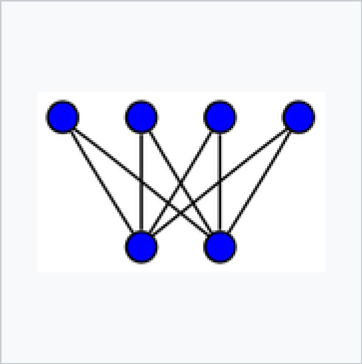
Complete Bipartite Graphs



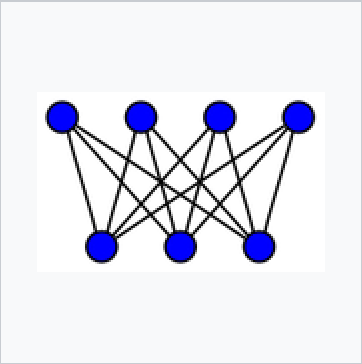
$K_{2,3}$



$K_{3,3}$, the utility graph



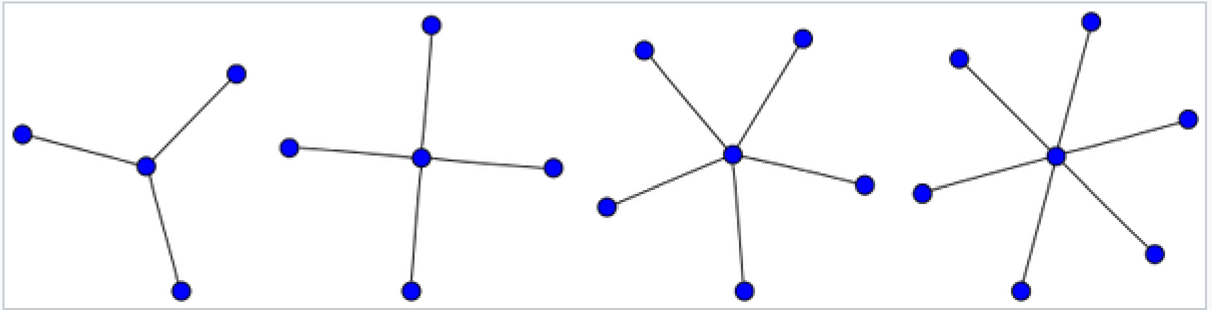
$K_{2,4}$



$K_{3,4}$

https://en.wikipedia.org/wiki/Gallery_of_named_graphs

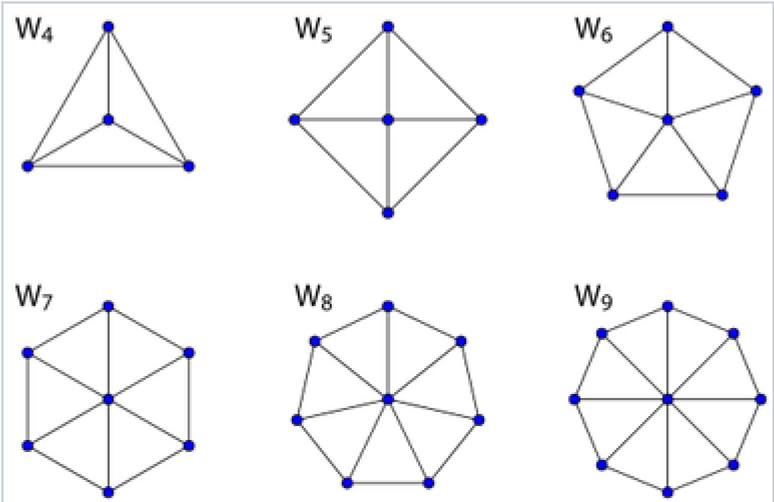
Star Graphs



The star graphs S_3, S_4, S_5 and S_6.

https://en.wikipedia.org/wiki/Gallery_of_named_graphs

Wheel Graphs



Wheels W_4 - W_9.

https://en.wikipedia.org/wiki/Gallery_of_named_graphs

Some class of graphs (2)

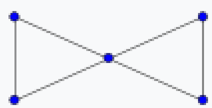
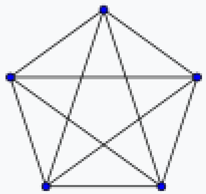
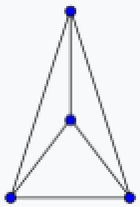
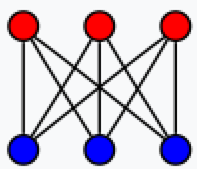
Planar graph
A planar graph is a graph whose vertices and edges can be drawn in a plane such that no two of the edges intersect.

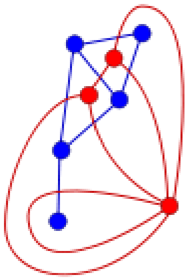
Cycle graph
A cycle graph or circular graph of order $n \geq 3$ is a graph in which the vertices can be listed in an order $v1, v2, \ldots, vn$ such that the edges are the $\{vi, vi+1\}$ where $i = 1, 2, \ldots, n - 1$, plus the edge $\{vn, v1\}$.
Cycle graphs can be characterized as connected graphs in which the degree of all vertices is 2.
If a cycle graph occurs as a subgraph of another graph, it is a cycle or circuit in that graph.

Tree
A tree is a connected graph with no cycles.

https://en.wikipedia.org/wiki/Graph_(discrete_mathematics)

Planar Graphs

Example graphs	
Planar	**Nonplanar**
 Butterfly graph	 Complete graph K_5
 Complete graph K_4	 Utility graph $K_{3,3}$

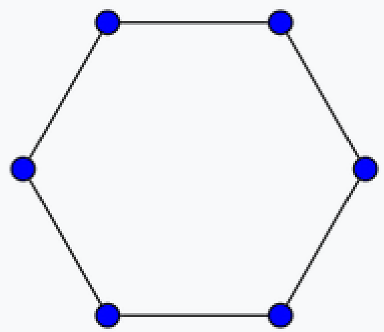


A planar graph and its dual

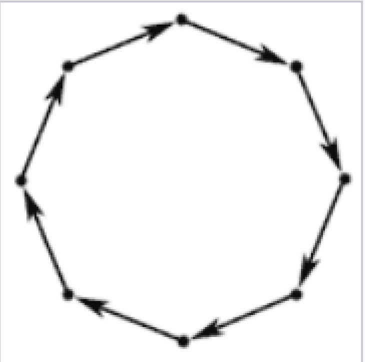
https://en.wikipedia.org/wiki/Planar_graph

Cycle Graphs

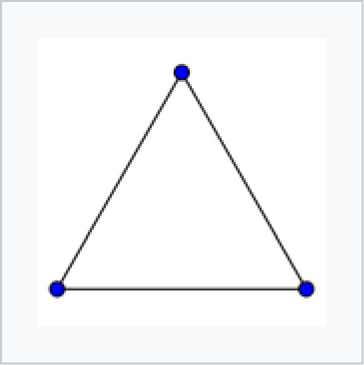
Cycle graph



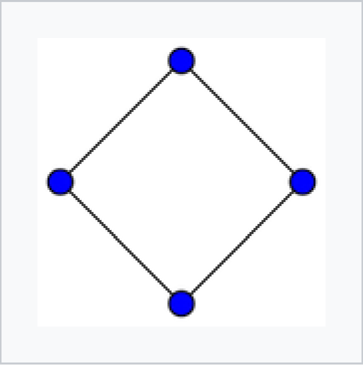
A cycle graph of length 6



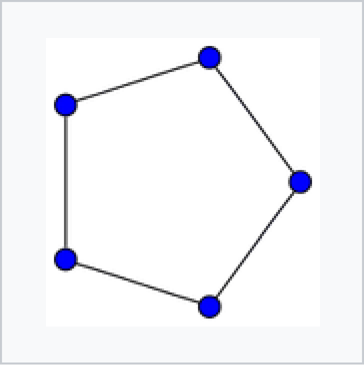
A directed cycle graph of length 8



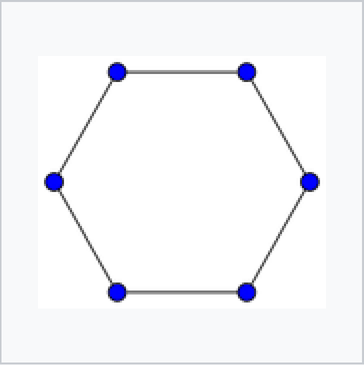
C_3



C_4



C_5

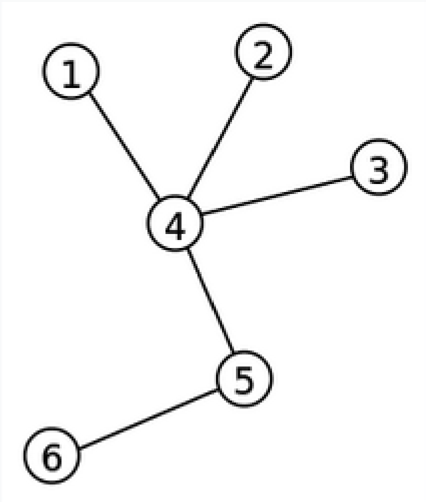


C_6

https://en.wikipedia.org/wiki/Cycle_graph
https://en.wikipedia.org/wiki/Gallery_of_named_graphs

Tree Graphs

Trees

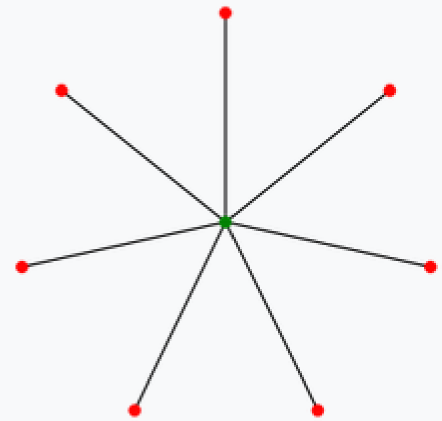


A labeled tree with 6 vertices and 5 edges.

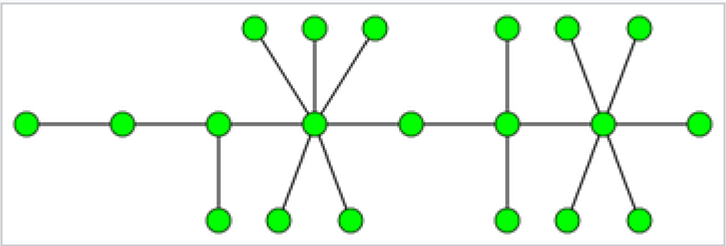
Path graph

A path graph on 6 vertices

Star



The star S_7. (Some authors index this as S_8.)



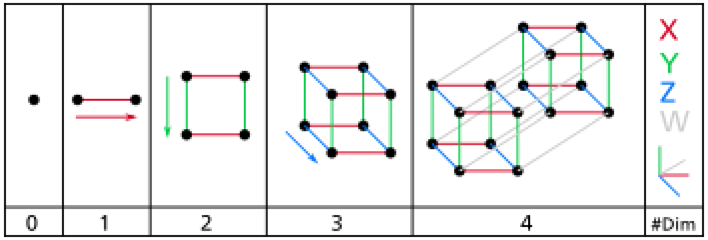
A caterpillar

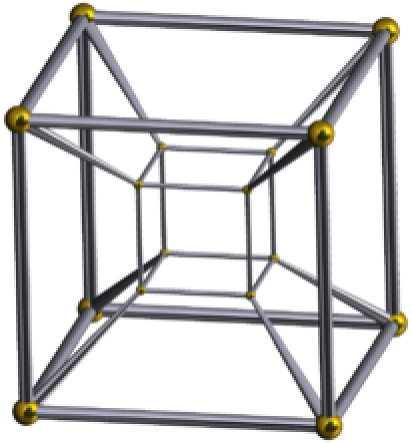
https://en.wikipedia.org/wiki/Cycle_graph

Hypercube

A hypercube can be defined by increasing the numbers of dimensions of a shape:

0 – A point is a hypercube of dimension zero.
1 – If one moves this point one unit length, it will sweep out a line segment, which is a unit hypercube of dimension one.
2 – If one moves this line segment its length in a perpendicular direction from itself; it sweeps out a 2-dimensional square.
3 – If one moves the square one unit length in the direction perpendicular to the plane it lies on, it will generate a 3-dimensional cube.
4 – If one moves the cube one unit length into the fourth dimension, it generates a 4-dimensional unit hypercube (a unit tesseract).

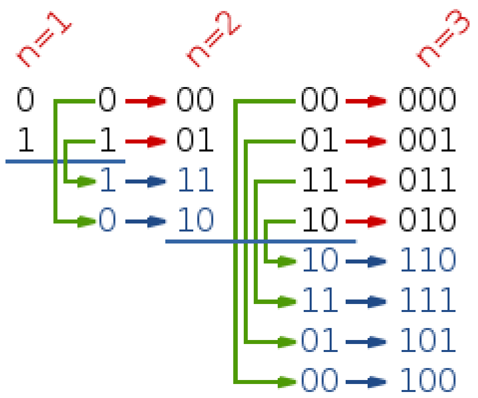


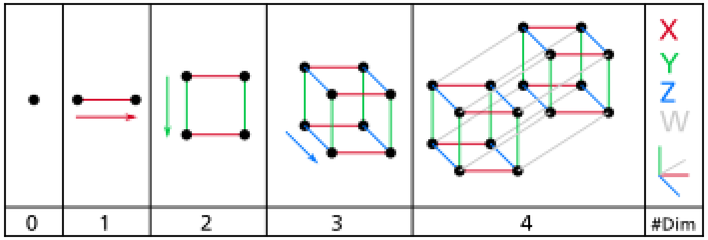


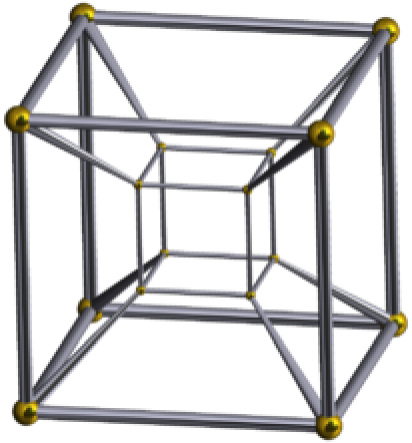
Tesseract

https://en.wikipedia.org/wiki/Hypercube

Gray Code



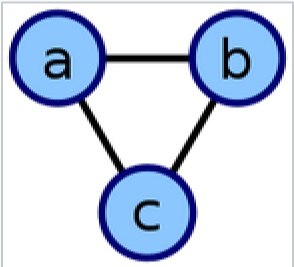




Tesseract

https://en.wikipedia.org/wiki/Gray_code

Adjacency Lists

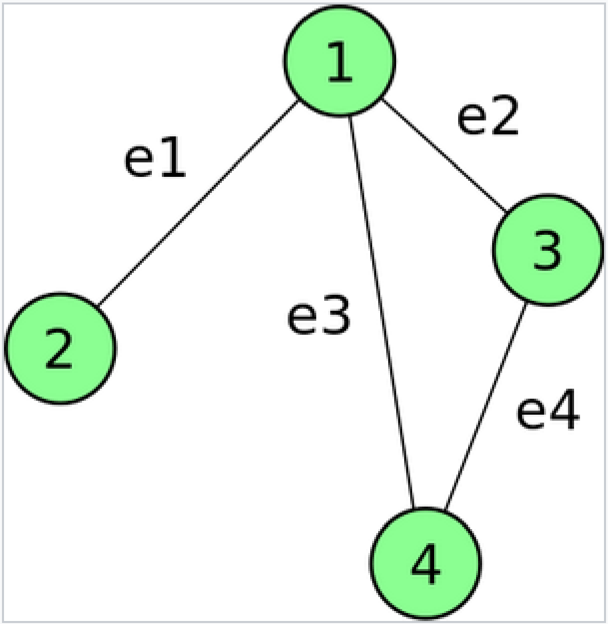


This undirected cyclic graph can be described by the three unordered lists {b, c}, {a, c}, {a, b}.

The graph pictured above has this adjacency list representation:		
a	adjacent to	b, c
b	adjacent to	a,c
c	adjacent to	a,b

https://en.wikipedia.org/wiki/Adjacency_list

Incidence Matrix



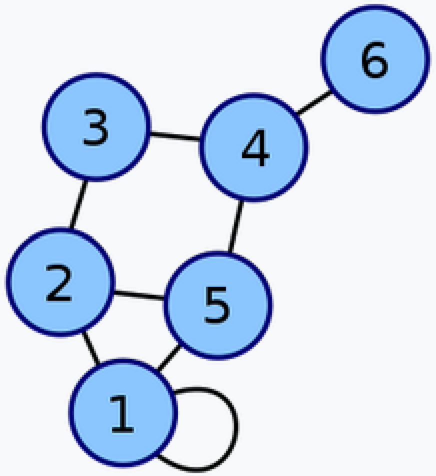
An undirected graph.

	e_1	e_2	e_3	e_4
1	1	1	1	0
2	1	0	0	0
3	0	1	0	1
4	0	0	1	1

$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

https://en.wikipedia.org/wiki/Incidence_matrix

Adjacency Matrix

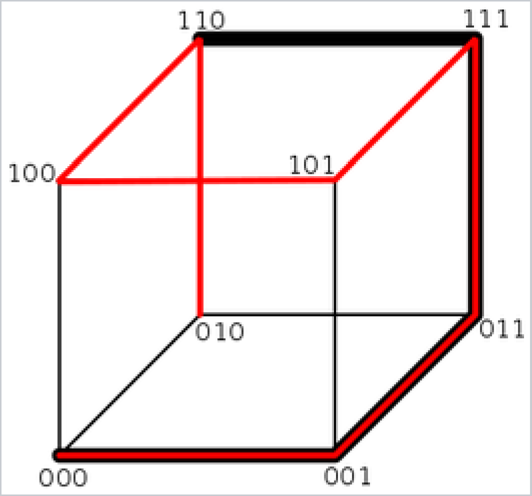


$$\begin{pmatrix} 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

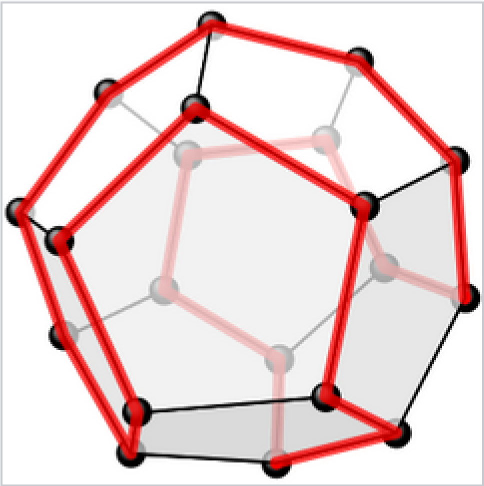
Coordinates are 1–6.

https://en.wikipedia.org/wiki/Adjacency_matrix

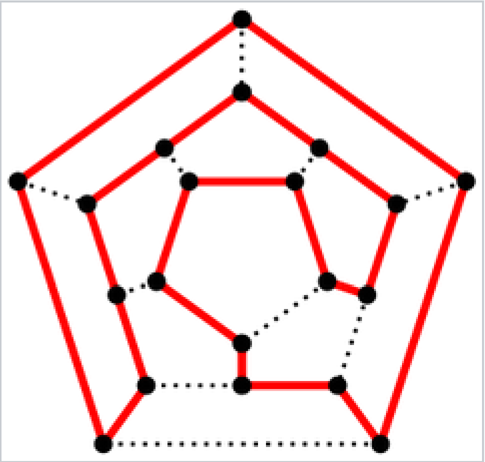
Hamiltonian Path



A hypercube graph showing a Hamiltonian path in red, and a longest induced path in bold black.



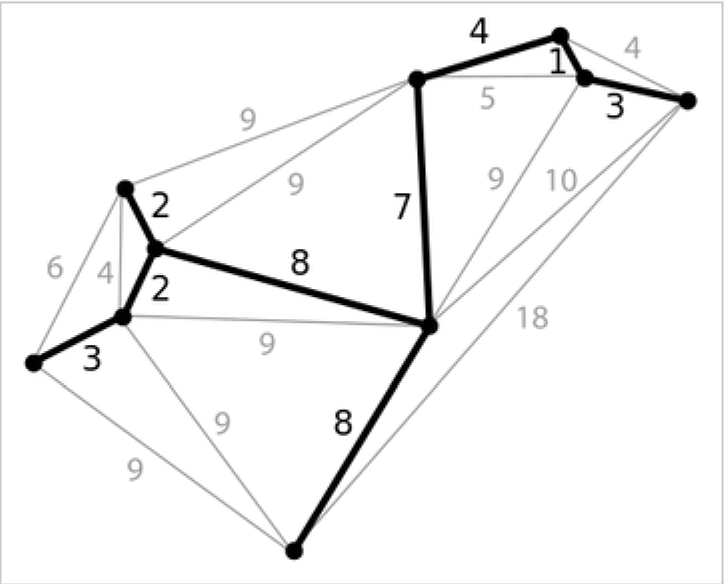
One possible Hamiltonian cycle through every vertex of a dodecahedron is shown in red – like all platonic solids, the dodecahedron is Hamiltonian



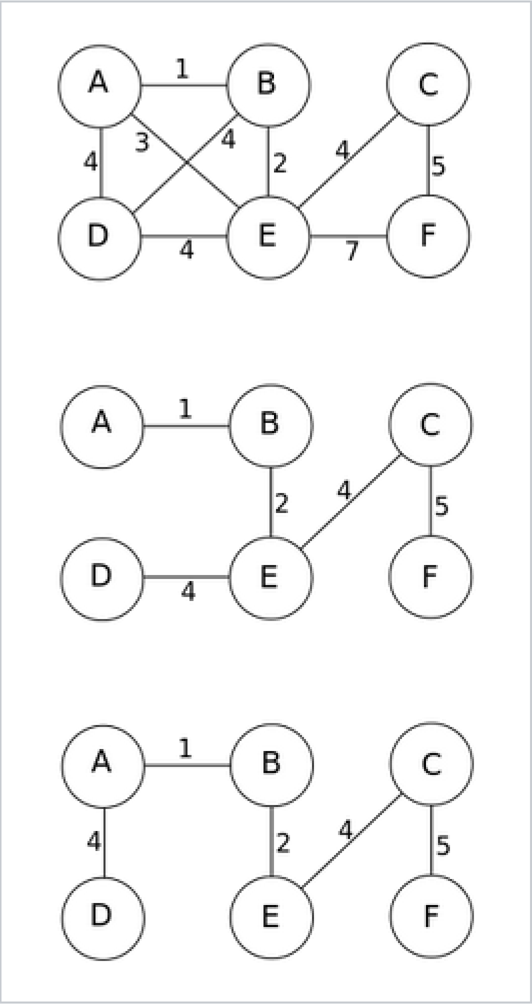
The above as a two-dimensional planar graph

https://en.wikipedia.org/wiki/Path_(graph_theory))

Minimum Spanning Tree



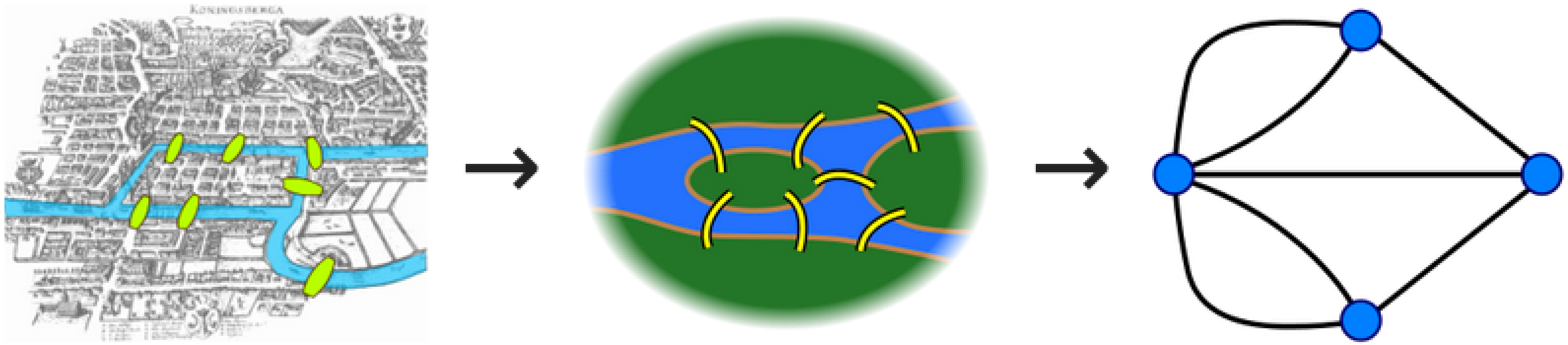
A planar graph and its minimum spanning tree. Each edge is labeled with its weight, which here is roughly proportional to its length.



This figure shows there may be more than one minimum spanning tree in a graph. In the figure, the two trees below the graph are two possibilities of minimum spanning tree of the given graph.

https://en.wikipedia.org/wiki/Minimum_spanning_tree

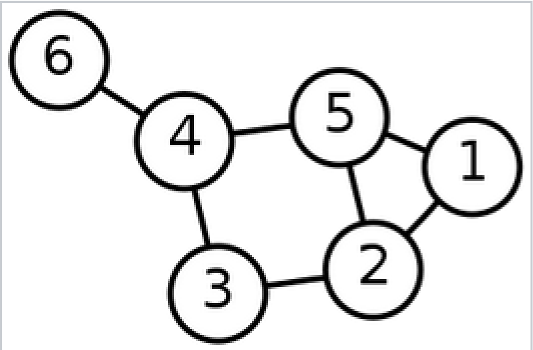
Seven Bridges of Königsberg



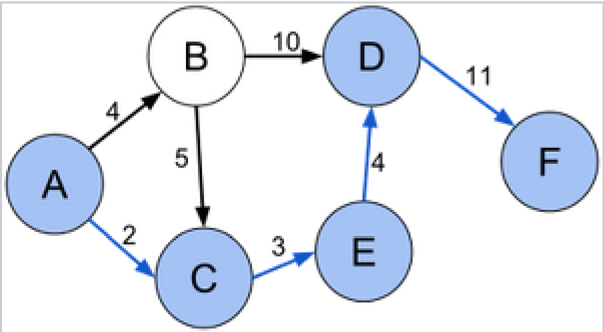
The problem was to devise a walk through the city that would cross each of those bridges once and only once.

https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg

Shortest path problem



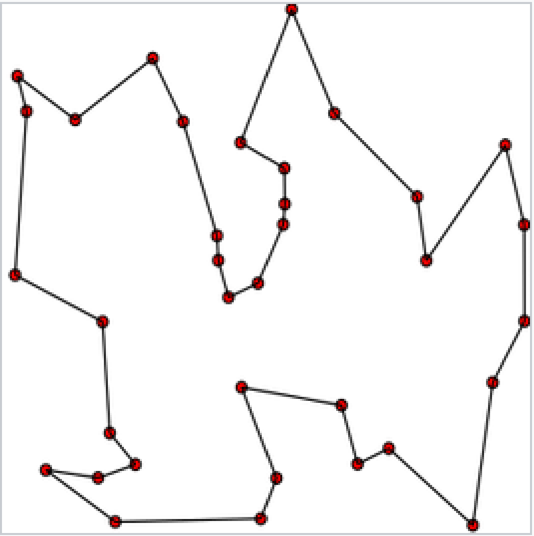
(6, 4, 5, 1) and (6, 4, 3, 2, 1) are both paths between vertices 6 and 1



Shortest path (A, C, E, D, F) between vertices A and F in the weighted directed graph

https://en.wikipedia.org/wiki/Shortest_path_problem

Traveling salesman problem



Solution of a travelling salesman problem: the black line shows the shortest possible loop that connects every red dot

https://en.wikipedia.org/wiki/Travelling_salesman_problem

Simple Graph

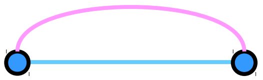
A simple graph is an undirected graph
without multiple edges or **loops**.

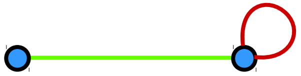
the edges form a set (rather than a multiset)
each edge is an unordered pair of distinct vertices.

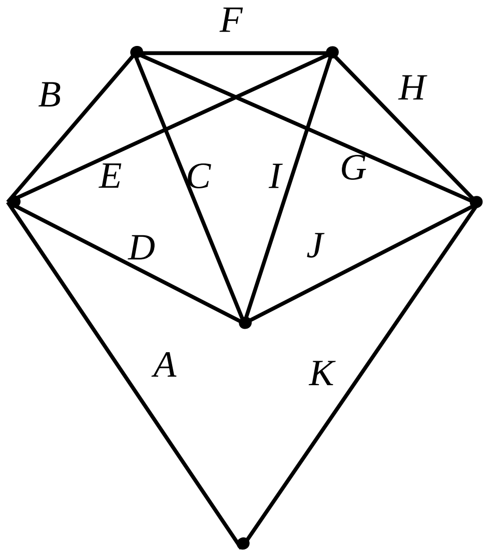
can define a simple graph to be a **set V** of vertices
together with a **set E** of edges,

E are 2-element subsets of **V**

with **n** vertices,
the **degree** of every vertex is at most **n − 1**



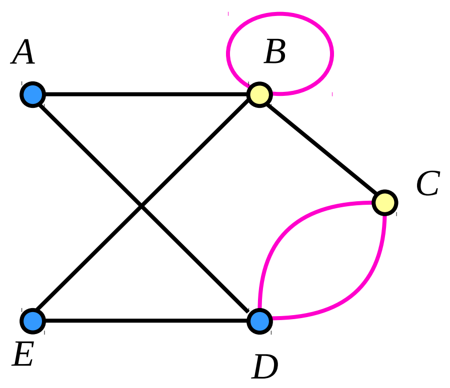


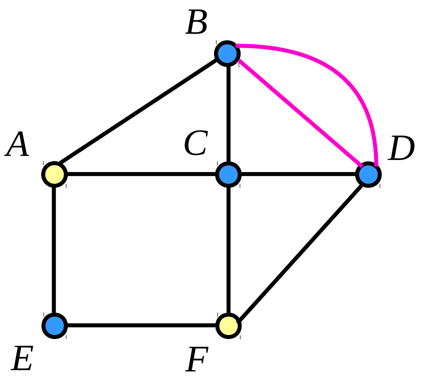


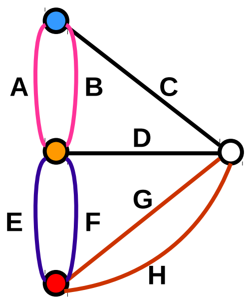
https://en.wikipedia.org/wiki/Travelling_salesman_problem

Multi-Graph

A **multigraph**, as opposed to a **simple graph**, is an undirected graph in which **multiple edges** (and sometimes **loops**) are allowed.







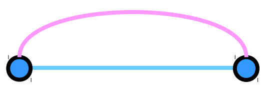
https://en.wikipedia.org/wiki/Travelling_salesman_problem

Multiple Edges

- multiple edges
- parallel edges
- Multi-edges

are <u>two or more</u> edges
that are <u>incident</u> to the same two vertices

A **simple graph** has <u>no</u> multiple edges.



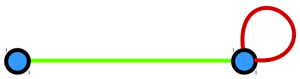
https://en.wikipedia.org/wiki/Travelling_salesman_problem

Loop

- a loop
- a self-loop
- a buckle

is an edge that connects a vertex to itself.

A **simple graph** contains no loops.

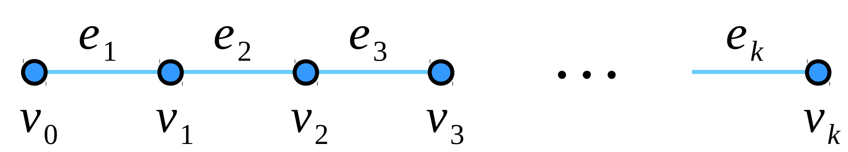


https://en.wikipedia.org/wiki/Travelling_salesman_problem

Walks

For a graph G= (V, E), a **walk** is defined as a sequence of alternating **vertices** and **edges** such as $v_{0,}\ e_{1,}\ v_{1,}\ e_{2,}\ \cdots,\ e_k,\ v_k$

where each edge $e_i = \{v_{i-1},\ v_i\}$

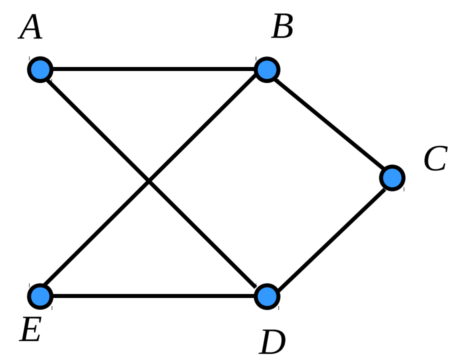


The length of this walk is k

Edges are allowed to be repeated $e_i = e_j$ *for some* i, j

ABCDE

ABCDCBE

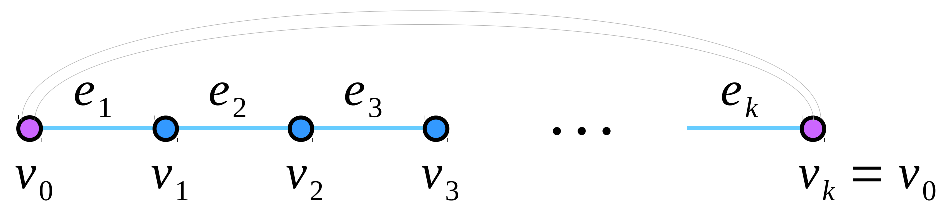


http://mathonline.wikidot.com/walks-trails-paths-cycles-and-circuits

Open / Closed Walks

A walk is considered to be **closed** if the **starting** vertex is the <u>same</u> as the **ending** vertex.

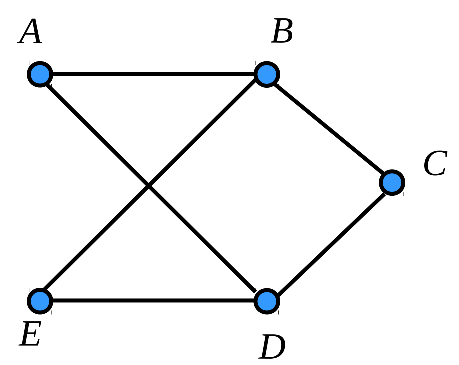
Otherwise **open**



ABCDA

ABCDE

ABCDCBE



http://mathonline.wikidot.com/walks-trails-paths-cycles-and-circuits

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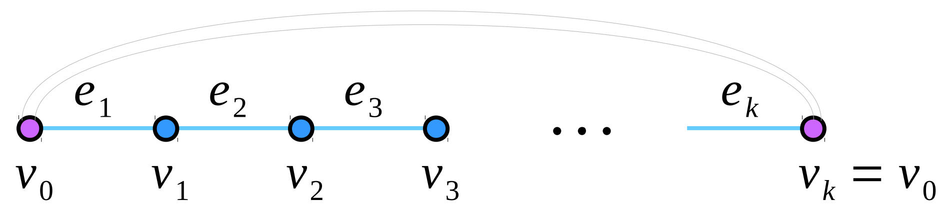
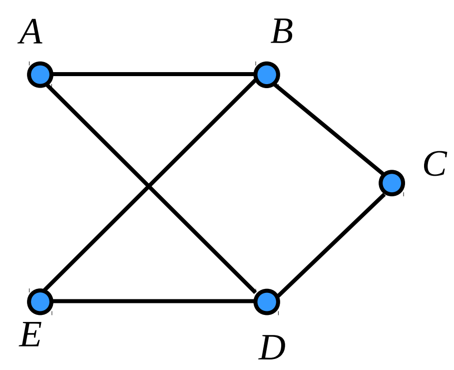


closed walk $ABCDA$

open walk $ABCDE$

open walk $ABCDCBE$

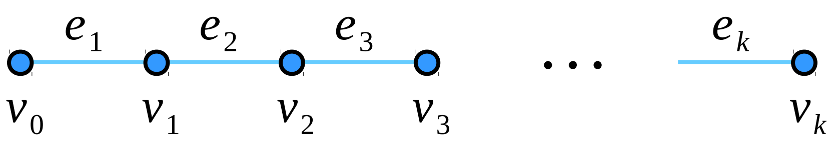


http://mathonline.wikidot.com/walks-trails-paths-cycles-and-circuits

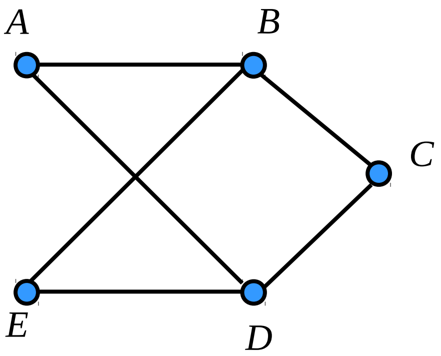
Trails

A **trail** is defined as a **walk** with <u>no</u> <u>repeated</u> **edges**.

$e_i \neq e_j \quad for\ all\ i,\ j$



closed trail	closed walk	*ABCDA*
open trail	open walk	*ABCDE*
~~open trail~~	open walk	*ABCDCBE*



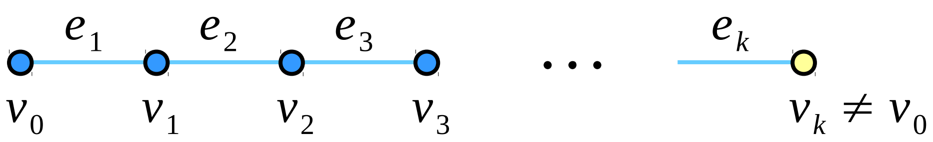
http://mathonline.wikidot.com/walks-trails-paths-cycles-and-circuits

Paths

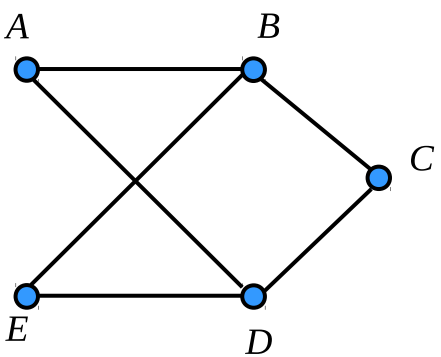
A **path** is defined as a **open trail** with no repeated **vertices**.

$$e_i \neq e_j \quad for\ all\ i,\ j$$

$$v_i \neq v_j \quad for\ all\ i,\ j$$



~~path~~	closed trail	closed walk	*ABCDA*
path	open trail	open walk	*ABCDE*
~~path~~	~~open trail~~	open walk	*ABCDCBE*
~~path~~	open trail	open walk	*BEDABC*



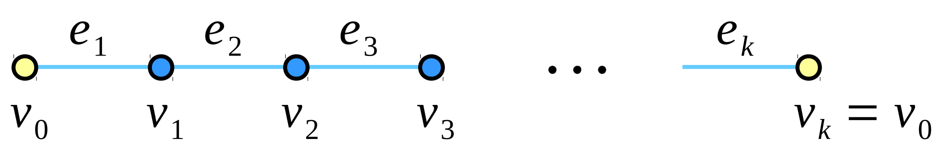
http://mathonline.wikidot.com/walks-trails-paths-cycles-and-circuits

Cycles

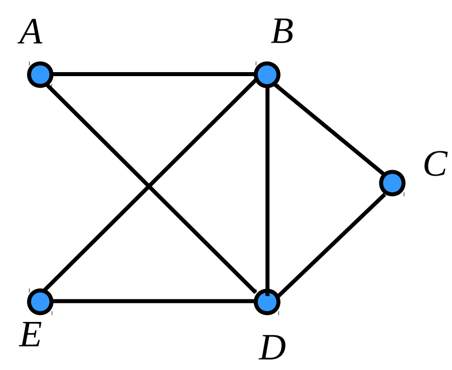
A **cycle** is defined as a **closed trail** with <u>no repeated</u> **vertices** except the **start/end vertex**

$$e_i \neq e_j \quad for\ all\ i,\ j$$

$$v_i \neq v_j \quad for\ all\ i,\ j$$



cycle	circuit	closed walk	*ABCDA*
~~cycle~~	circuit	closed walk	*ABCDEBDA*

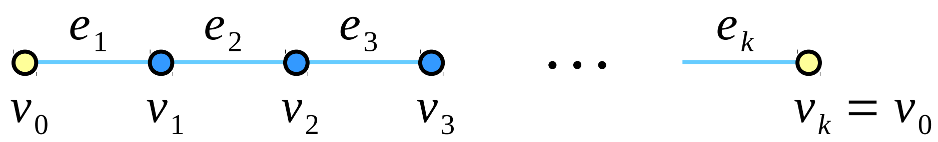


http://mathonline.wikidot.com/walks-trails-paths-cycles-and-circuits

Circuits

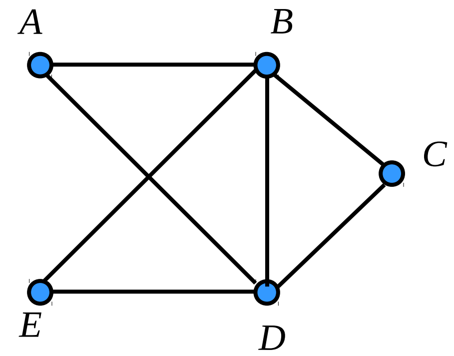
A **circuit** is defined as a **closed trail** with possibly repeated **vertices** but with no repeated **edges**

$$e_i \neq e_j \quad for\ all\ i,\ j$$
$$v_i = v_j \quad for\ some\ i,\ j$$



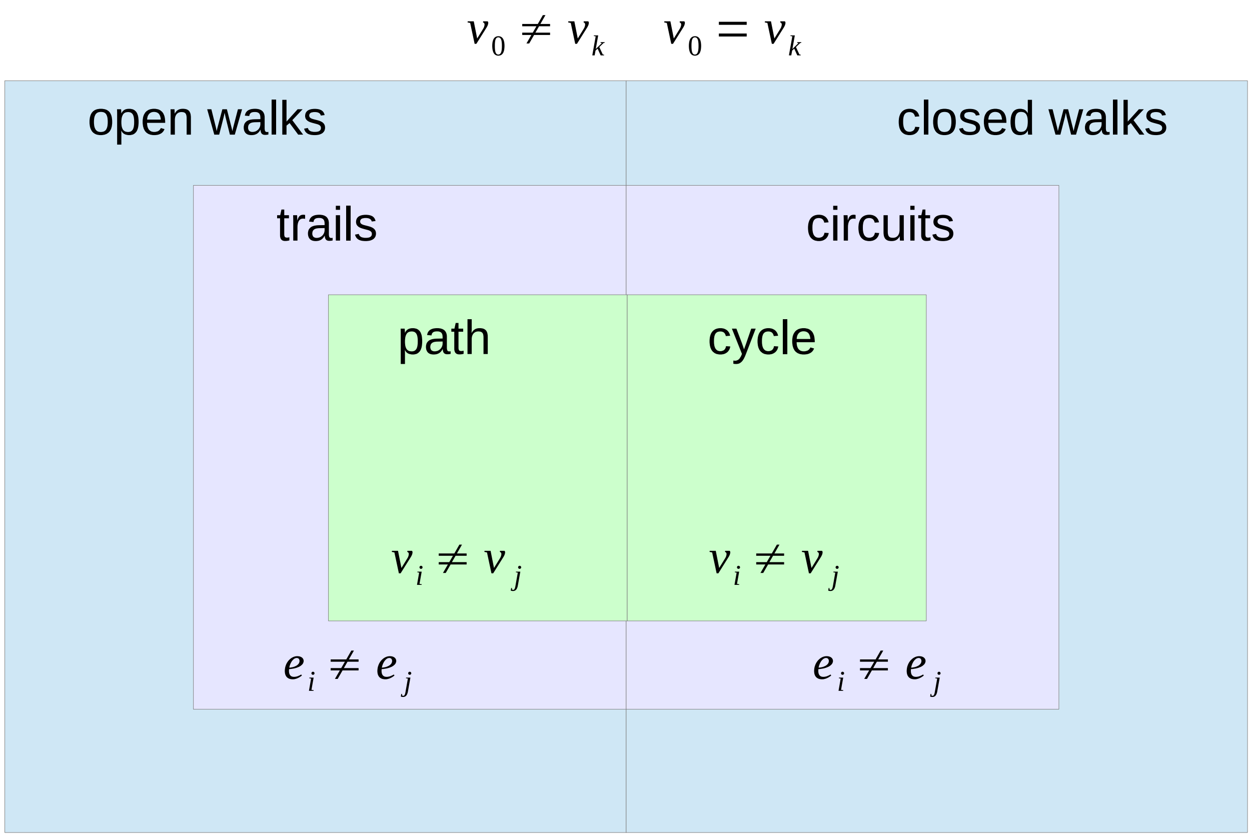
circuit closed walk *ABCDA*

circuit closed walk *ABCDEBDA*

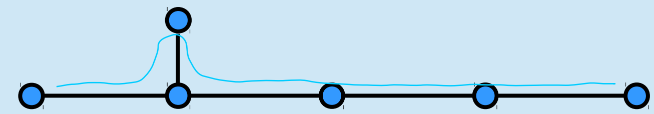
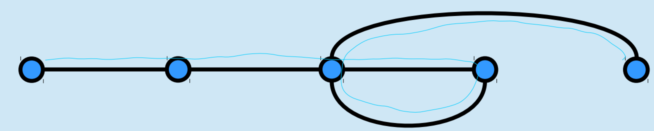
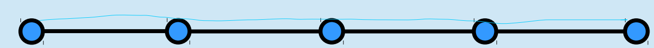
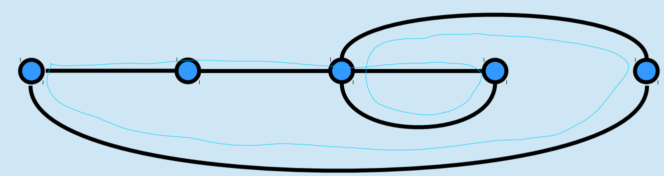
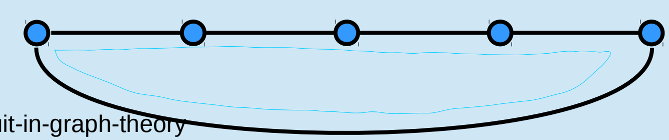


http://mathonline.wikidot.com/walks-trails-paths-cycles-and-circuits

Walk, Trail, Path, Circuit, Cycle



Walk, Trail, Path, Circuit, Cycle

	Vertices	Edges		
Walk	may repeat	may repeat	(Closed/Open)	
Trail	may repeat	<u>cannot</u> repeat	(Open)	
Path	<u>cannot</u> repeat	<u>cannot</u> repeat	(Open)	
Circuit	may repeat	<u>cannot</u> repeat	(Closed)	
Cycle	<u>cannot</u> repeat	<u>cannot</u> repeat	(Closed)	

https://math.stackexchange.com/questions/655589/what-is-difference-between-cycle-path-and-circuit-in-graph-theory

References

[1] http://en.wikipedia.org/
[2]