Sets (6A)

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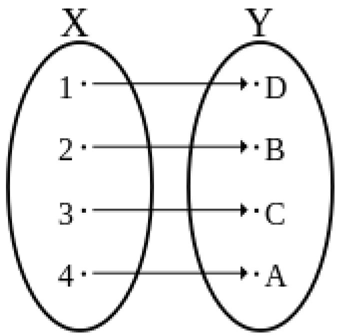
The Same Cardinality

Definition 1: |*A*| = |*B*| [edit]

Two sets *A* and *B* have the same cardinality if there exists a bijection, that is, an injective and surjective function, from *A* to *B*. Such sets are said to be *equipotent*, *equipollent*, or *equinumerous*. This relationship can also be denoted $A \approx B$ or $A \sim B$.

For example, the set $E = \{0, 2, 4, 6, ...\}$ of non-negative even numbers has the same cardinality as the set $\mathbf{N} = \{0, 1, 2, 3, ...\}$ of natural numbers, since the function $f(n) = 2n$ is a bijection from **N** to *E*.



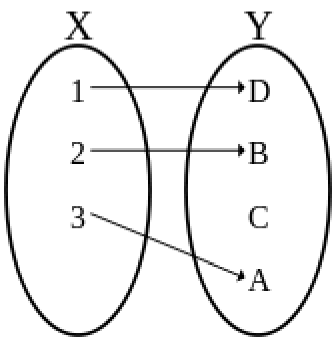
If there exists a **bijection** mapping from the set X to the set then |X| = |Y|

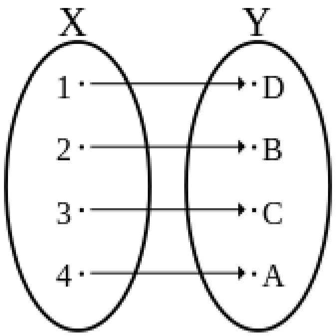
https://en.wikipedia.org/wiki/Cardinality

A less than equal cardinality

Definition 2: |*A*| ≤ |*B*| [edit]

A has cardinality less than or equal to the cardinality of *B* if there exists an injective function from *A* into *B*.





If there exists a **injective** mapping from the set X to the set then |X| <= |Y|

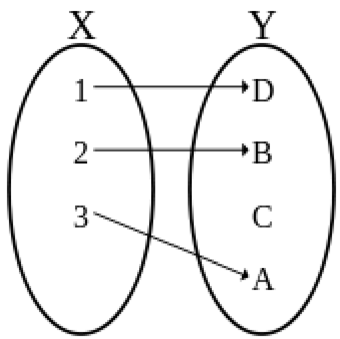
https://en.wikipedia.org/wiki/Cardinality

A less than cardinality

Definition 3: |*A*| < |*B*| [edit]

A has cardinality strictly less than the cardinality of *B* if there is an injective function, but no bijective function, from *A* to *B*.

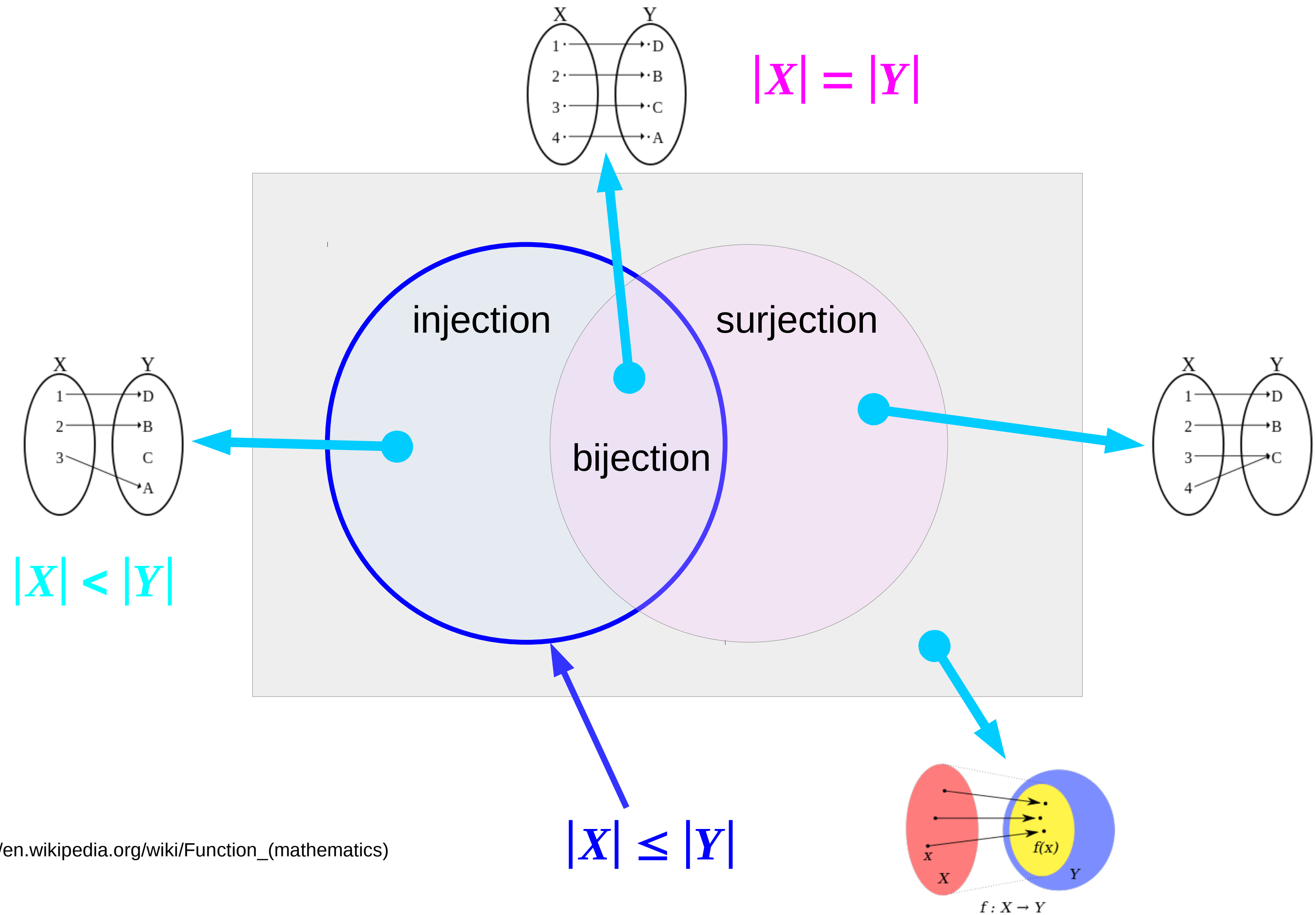
For example, the set **N** of all natural numbers has cardinality strictly less than the cardinality of the set **R** of all real numbers , because the inclusion map *i* : **N** → **R** is injective, but it can be shown that there does not exist a bijective function from **N** to **R** (see Cantor's diagonal argument or Cantor's first uncountability proof).



If there exists a **injective** but not a **surjective** mapping (thus not a **bijective** mapping) from the set X to the set then |X| < |Y|

https://en.wikipedia.org/wiki/Cardinality

Types of Functions and Cardinalities



https://en.wikipedia.org/wiki/Function_(mathematics)

The Cardinality of a Power Set

If S is a finite set with $|S| = n$ elements, then the number of subsets of S is $|\mathcal{P}(S)| = 2^n$. This fact, which is the motivation for the notation 2^S, may be demonstrated simply as follows,

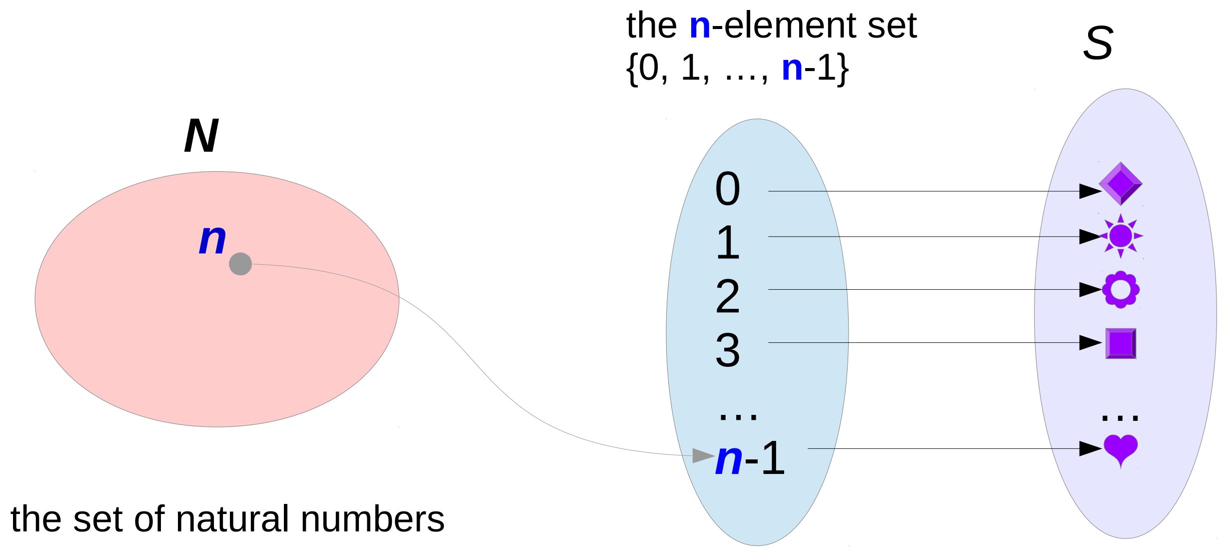
> First, order the elements of S in any manner. We write any subset of S in the format $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ where γ_i , $1 \leq i \leq n$, can take the value of 0 or 1. If $\gamma_i = 1$, the i-th element of S is in the subset; otherwise, the i-th element is not in the subset. Clearly the number of distinct subsets that can be constructed this way is 2^n as $\gamma_i \in \{0, 1\}$.

https://en.wikipedia.org/wiki/Function_(mathematics)

A Finite Set

A set S is **finite** wit cardinality $n \in N$
If there is a **bijection** from the set {0, 1, …, n-1} to S.

A set is **infinite** if it is not finite



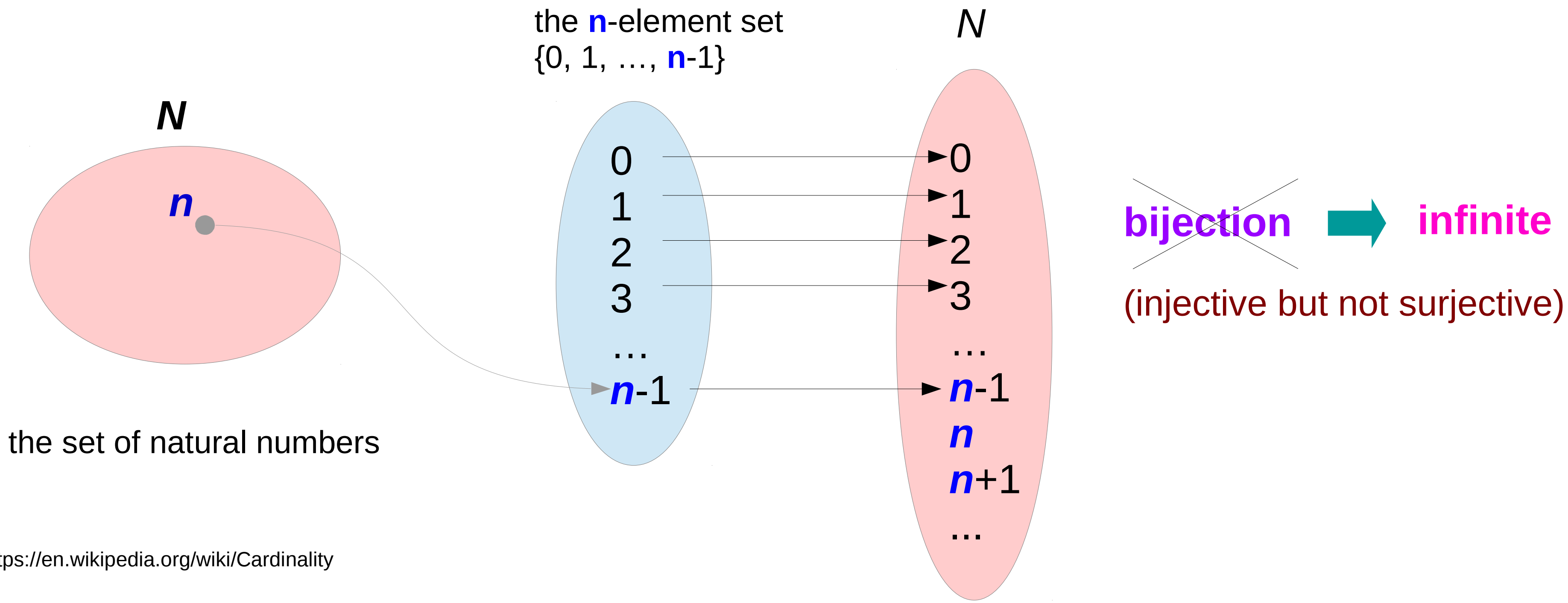
bijection → **infinite**

https://en.wikipedia.org/wiki/Cardinality

The set of natural numbers

The set of natural numbers is an **infinite** set.

N is an infinite set.



https://en.wikipedia.org/wiki/Cardinality

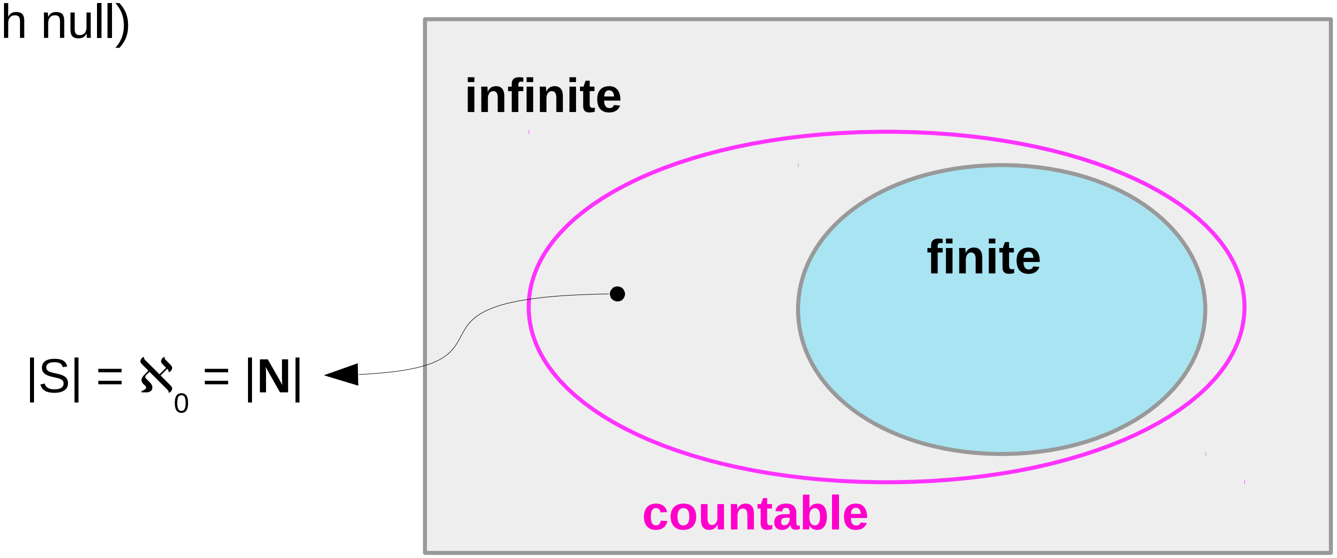
A Countable Set

A set that is either **finite** or has the same cardinality as the set of positive integers (natural numbers **N**) is **countable**.

A set that is not **countable** is called **uncountable**.

When an **infinite** set S is countable,
We denote the cardinality of S by $\aleph_0$.

We write $|S| = \aleph_0$ (aleph null)

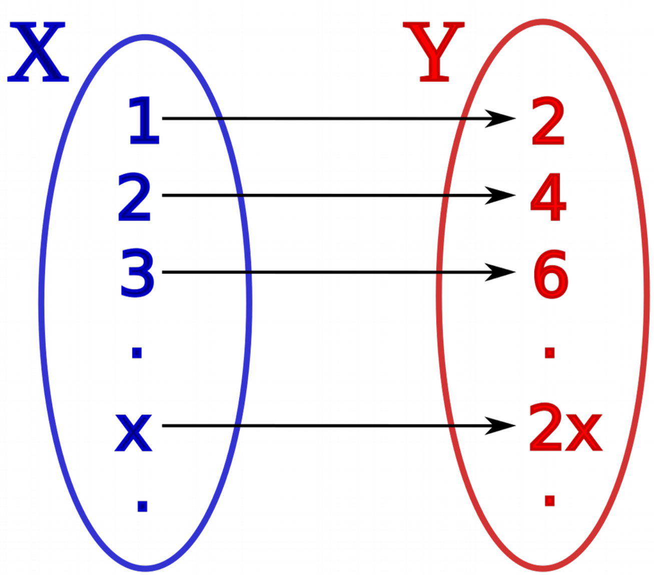


https://en.wikipedia.org/wiki/Cardinality

The Cardinality $\aleph_0$

the set of natural numbers

the set of even numbers



$\aleph_0 = |N|$

Bijective mapping
from integer to even numbers

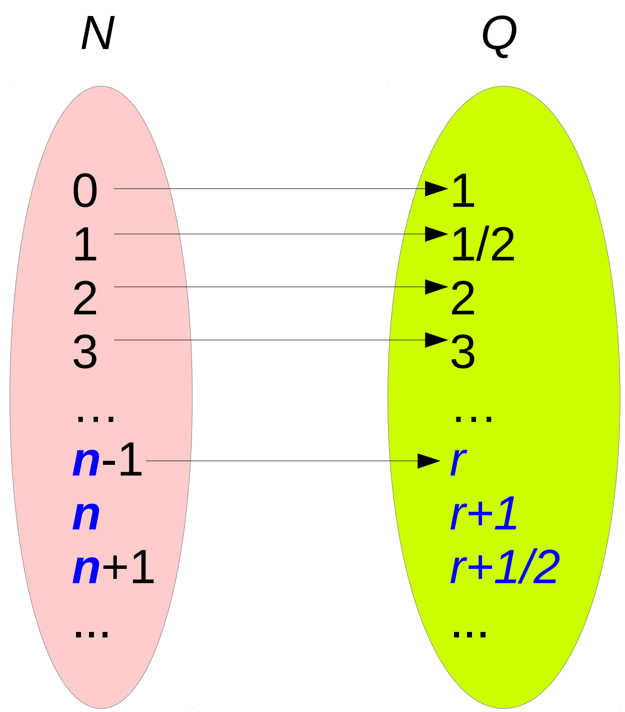
➡ the same cardinality
as the set of natural numbers

➡ **countable**

https://en.wikipedia.org/wiki/Cardinality

The set of rational numbers

The set of rational numbers is an **infinite** but **countable** set.



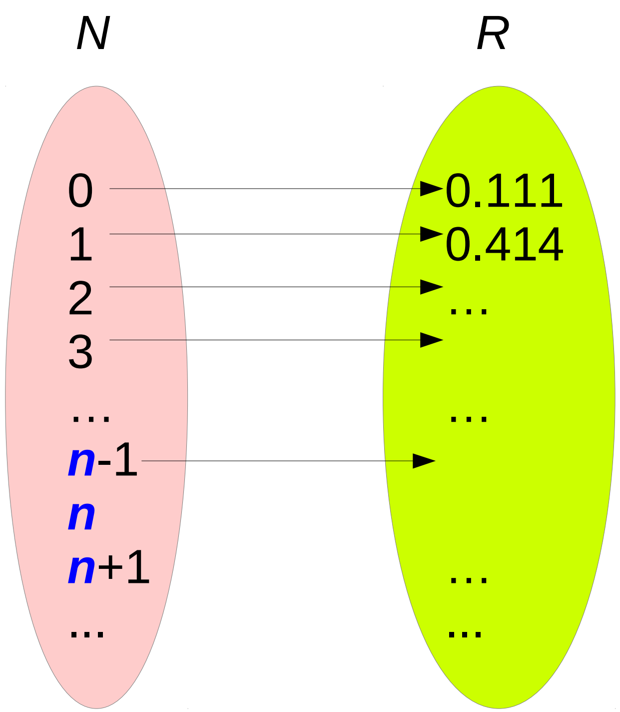
bijection ➡ **countable**

(injective and surjective)

https://en.wikipedia.org/wiki/Cardinality

The set of real numbers

The set of real numbers is an **infinite** and **uncountable** set.



~~bijection~~ ➡ **uncountable**

Cantor's diagonalization argument

https://en.wikipedia.org/wiki/Cardinality

Finite, Infinite, Countable Sets

Any set X with cardinality less than that of the natural numbers, $|X| < |\mathbf{N}|$, is said to be a finite set

Any set X with cardinality equal to that of the natural numbers, $|X| = |\mathbf{N}|$, is said to be a countably infinite set

Any set X with cardinality greater than that of the natural numbers, $|X| > |\mathbf{N}|$, is said to be a uncountable set

If the axiom of choice holds, the law of trichotomy holds for cardinality. Thus we can make the following definitions:

- Any set X with cardinality less than that of the natural numbers, or $|X| < |\mathbf{N}|$, is said to be a finite set.
- Any set X that has the same cardinality as the set of the natural numbers, or $|X| = |\mathbf{N}| = \aleph_0$, is said to be a countably infinite set.
- Any set X with cardinality greater than that of the natural numbers, or $|X| > |\mathbf{N}|$, for example $|\mathbf{R}| = \mathfrak{c} > |\mathbf{N}|$, is said to be uncountable.

https://en.wikipedia.org/wiki/Cardinality

Function

Assuming AC, the cardinalities of the infinite sets are denoted

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots .$$

For each ordinal α, $\aleph_{\alpha+1}$ is the least cardinal number greater than $\aleph_\alpha$.

The cardinality of the natural numbers is denoted aleph-null ($\aleph_0$), while the cardinality of the real numbers is denoted by "$\mathfrak{c}$" (a lowercase fraktur script "c"), and is also referred to as the cardinality of the continuum. Cantor showed, using the diagonal argument, that $\mathfrak{c} > \aleph_0$. We can show that $\mathfrak{c} = 2^{\aleph_0}$, this also being the cardinality of the set of all subsets of the natural numbers. The continuum hypothesis says that $\aleph_1 = 2^{\aleph_0}$, i.e. $2^{\aleph_0}$ is the smallest cardinal number bigger than $\aleph_0$, i.e. there is no set whose cardinality is strictly between that of the integers and that of the real numbers. The continuum hypothesis is independent of ZFC, a standard axiomatization of set theory; that is, it is impossible to prove the continuum hypothesis or its negation from ZFC (provided ZFC is consistent). See below for more details on the cardinality of the continuum.[5][6][7]

https://en.wikipedia.org/wiki/Cardinality

Cantor’s diagonal argument

$$
\begin{array}{lcl}
s_1 & = & 0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;\ldots \\
s_2 & = & 1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;1\;\ldots \\
s_3 & = & 0\;1\;0\;1\;0\;1\;0\;1\;0\;1\;0\;\ldots \\
s_4 & = & 1\;0\;1\;0\;1\;0\;1\;0\;1\;0\;1\;\ldots \\
s_5 & = & 1\;1\;0\;1\;0\;1\;1\;0\;1\;0\;1\;\ldots \\
s_6 & = & 0\;0\;1\;1\;0\;1\;1\;0\;1\;1\;0\;\ldots \\
s_7 & = & 1\;0\;0\;0\;1\;0\;0\;0\;1\;0\;0\;\ldots \\
s_8 & = & 0\;0\;1\;1\;0\;0\;1\;1\;0\;0\;1\;\ldots \\
s_9 & = & 1\;1\;0\;0\;1\;1\;0\;0\;1\;1\;0\;\ldots \\
s_{10} & = & 1\;1\;0\;1\;1\;1\;0\;0\;1\;0\;1\;\ldots \\
s_{11} & = & 1\;1\;0\;1\;0\;1\;0\;0\;1\;0\;0\;\ldots \\
\vdots & & \vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\vdots\;\ddots
\end{array}
$$

$$
s = 1\;0\;1\;1\;1\;0\;1\;0\;0\;1\;1\;\ldots
$$

An illustration of Cantor's diagonal argument (in base 2) for the existence of uncountable sets. The sequence at the bottom cannot occur anywhere in the enumeration of sequences above.

https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

Function

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https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

Function

the set T of all infinite sequences of binary digits

If s1, s2, … , sn, … is any enumeration of elements from T,
then there is always an element s of T
which corresponds to no sn in the enumeration.

To prove this, given an enumeration of elements from T, like e.g.

s1 = (0, 0, 0, 0, 0, 0, 0, ...)
s2 = (1, 1, 1, 1, 1, 1, 1, ...)
s3 = (0, 1, 0, 1, 0, 1, 0, ...)
s4 = (1, 0, 1, 0, 1, 0, 1, ...)
s5 = (1, 1, 0, 1, 0, 1, 1, ...)
s6 = (0, 0, 1, 1, 0, 1, 1, ...)
s7 = (1, 0, 0, 0, 1, 0, 0, ...)
...

https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

Function

construct the sequence s by choosing
the 1st digit as complementary to the 1st digit of s1,
the 2nd digit as complementary to the 2nd digit of s2,
the 3rd digit as complementary to the 3rd digit of s3,
and generally for every n,
the nth digit as complementary to the nth digit of sn.

In the example, this yields:

s1	=	(**0**,	0,	0,	0,	0,	0,	0,	...)
s2	=	(1,	**1**,	1,	1,	1,	1,	1,	...)
s3	=	(0,	1,	**0**,	1,	0,	1,	0,	...)
s4	=	(1,	0,	1,	**0**,	1,	0,	1,	...)
s5	=	(1,	1,	0,	1,	**0**,	1,	1,	...)
s6	=	(0,	0,	1,	1,	0,	**1**,	1,	...)
s7	=	(1,	0,	0,	0,	1,	0,	**0**,	...)
...									
s	=	(1,	0,	1,	1,	1,	0,	1,	...)

https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

Function

By construction,
s differs from each sn,
since their nth digits differ.
Hence, s cannot occur in the enumeration.

Based on this theorem,
Cantor then uses a proof by contradiction to show that:

The set T is uncountable.

https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

Function

The set T is uncountable.

He assumes for contradiction that T was countable.
Then all its elements could be written
as an enumeration s1, s2, … , sn, … .

Applying the previous theorem to this enumeration
would produce a sequence s not belonging to the enumeration.

However, s was an element of T
and should therefore be in the enumeration.

This contradicts the original assumption,
so T must be uncountable.

https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

Function

https://en.wikipedia.org/wiki/Cantor%27s_diagonal_argument

References

[1] http://en.wikipedia.org/
[2]