First Order Logic (4A)

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Predicate

In mathematical logic, a **predicate** is commonly understood to be a Boolean-valued function P: $X \to$ {true, false}, called the predicate on X. However, predicates have many different uses and interpretations in mathematics and logic, and their precise definition, meaning and use will vary from theory to theory. So, for example, when a theory defines the concept of a relation, then a predicate is simply the characteristic function or the indicator function of a relation. However, not all theories have relations, or are founded on set theory, and so one must be careful with the proper definition and semantic interpretation of a predicate.

https://en.wikipedia.org/wiki/Predicate_(mathematical_logic)

First Order Logic

First-order logic—also known as **first-order predicate calculus** and **predicate logic**—is a collection of formal systems used in mathematics, philosophy, linguistics, and computer science. First-order logic uses quantified variables over non-logical objects and allows the use of sentences that contain variables, so that rather than propositions such as *Socrates is a man* one can have expressions in the form "there exists X such that X is Socrates and X is a man" and *there exists* is a quantifier while *X* is a variable.[1] This distinguishes it from propositional logic, which does not use quantifiers or relations.[2]

https://en.wikipedia.org/wiki/First-order_logic

Formulas

The set of **formulas** (also called **well-formed formulas**[9] or **WFFs**) is inductively defined by the following rules:

1. **Predicate symbols.** If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms then $P(t_1,\ldots,t_n)$ is a formula.
2. **Equality.** If the equality symbol is considered part of logic, and t_1 and t_2 are terms, then $t_1 = t_2$ is a formula.
3. **Negation.** If φ is a formula, then $\neg\varphi$ is a formula.
4. **Binary connectives.** If φ and ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula. Similar rules apply to other binary logical connectives.
5. **Quantifiers.** If φ is a formula and x is a variable, then $\forall x\varphi$ (for all x, φ holds) and $\exists x\varphi$ (there exists x such that φ) are formulas.

https://en.wikipedia.org/wiki/First-order_logic

Quantifiers (1)

In logic, **quantification** specifies the quantity of specimens in the domain of discourse that satisfy an open formula. The two most common quantifiers mean "for all" and "there exists". For example, in arithmetic, quantifiers allow one to say that the natural numbers go on for ever, by writing that *for all* n (where n is a natural number), there is another number (say, the successor of n) which is one bigger than n.

https://en.wikipedia.org/wiki/First-order_logic

Quantifiers (2)

A language element which generates a quantification (such as "every") is called a **quantifier**. The resulting expression is a quantified expression, it is said to be **quantified** over the predicate (such as "the natural number *x* has a successor") whose free variable is bound by the quantifier. In formal languages, quantification is a formula constructor that produces new formulas from old ones. The semantics of the language specifies how the constructor is interpreted. Two fundamental kinds of quantification in predicate logic are universal quantification and existential quantification. The traditional symbol for the universal quantifier "all" is "∀", a rotated letter "A", and for the existential quantifier "exists" is "∃", a rotated letter "E". These quantifiers have been generalized beginning with the work of Mostowski and Lindström.

https://en.wikipedia.org/wiki/First-order_logic

Valid, Satisfiable, and Unsatisfiable Formulas

A **formula** is **valid** iff
Its truth value is **T** in all interpretations (**tautology**: $\top$)

A **formula** is **satisfiable** iff
Its truth value is **T** in at least one interpretation

A **formula** is **unsatisfiable** iff
Its truth value is **F** in all interpretations (**contradiction**: $\perp$)

https://en.wikipedia.org/wiki/Propositional_calculus

Valid and Sound Arguments

a valid argument

$H_1 = \boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{F}$
$H_2 = \boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$
$H_2 = \boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{F}$	$\boldsymbol{F}$
$C = \boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$	$\boldsymbol{T}$

a sound argument

a invvalid argument

$H_1 = \boldsymbol{T}$
$H_2 = \boldsymbol{T}$
$H_2 = \boldsymbol{T}$
$C = \boldsymbol{F}$

https://en.wikipedia.org/wiki/Soundness

De Morgan's Law for Quantifiers

$$\neg \exists x \; P(x)$$

$$\forall x \; \neg P(x)$$

¬(There is an x for which P(x) is true)

For every x, P(x) is false.

$$\neg \forall x \; P(x)$$

$$\exists x \; \neg P(x)$$

¬(for every x, P(x) is true)

There is an x, for which P(x) is false.

Discrete Mathematics and Its Applications, Rosen

Example 1-a

Every student in this class has studied calculus.

For every student in this class, that student has studied calculus.

For every student x in this class, x has studied calculus.

C(x) : x has studied calculus

$$\forall x \; C(x)$$

Discrete Mathematics and Its Applications, Rosen

Example 1-b

Every student in this class has studied calculus.

For every person x, if person x is a student in this class then x has studied calculus.

S(x) : person x is in this class

C(x) : x has studied calculus

$$\forall x \; (S(x) \rightarrow C(x))$$

different domains of discourse
a wider group of people

Discrete Mathematics and Its Applications, Rosen

Example 1-c

Every student in this class has studied calculus.

$\neq$ All people are students in this class and have studied calculus.

S(x) : person x is in this class

C(x) : x has studied calculus

$$\forall x \left(S(x) \land C(x) \right)$$

Discrete Mathematics and Its Applications, Rosen

Example 1-d

Every student in this class has studied calculus.

For every person x, if person x is a student in this class then x has studied calculus.

Q(x,y) : student x has studied subject y

$$\forall x \; Q(x, calculus)$$

$$\forall x \; (S(x) \rightarrow Q(x, calculus))$$

Discrete Mathematics and Its Applications, Rosen

Example 2-a

Some student in this class has visited Mexico

There is a student in this class with the property that the student has visited Mexico.

There is a student x in this class having the property that x has visited Mexico.

M(x) : x has visited Mexico

$$\exists x \ M(x)$$

Discrete Mathematics and Its Applications, Rosen

Example 2-b

Some student in this class has visited Mexico

There is a person x having the properties
that x is a student in this class and x has visited Mexico.

S(x) : x is a student in this class

M(x) : x has visited Mexico

$$\exists x \left(S(x) \wedge M(x) \right)$$

Discrete Mathematics and Its Applications, Rosen

Example 2-c

Some student in this class has visited Mexico

$\neq$ There is a person x having the properties
that **if** x is a student in this class **then** x has visited Mexico
or **if** x is <u>not</u> a student in this class, **then** x has visited Mexico
or **if** x is <u>not</u> a student in this class, **then** x has <u>not</u> visited Mexico

S(x) : x is a student in this class
M(x) : x has visited Mexico

$$\exists x \left(S\left(x\right) \rightarrow M\left(x\right)\right)$$

Discrete Mathematics and Its Applications, Rosen

Example 3-a

Every student in this class has visited either Canada or Mexico.

For every x in this class, x has the property
That x has visited Mexico or x has visited Canada.

C(x) : x has visited Canada
M(x) : x has visited Mexico

$$\forall x \, (C(x) \vee M(x))$$

Discrete Mathematics and Its Applications, Rosen

Example 3-b

Every student in this class has visited either Canada or Mexico.

For every person x,
if x is a student in this class, then x has visited Mexico or x has visited Canada.

S(x) : x is a student in this class
C(x) : x has visited Canada
M(x) : x has visited Mexico

$$\forall x \left(S\left(x\right) \rightarrow \left(C\left(x\right) \vee M\left(x\right)\right)\right)$$

Discrete Mathematics and Its Applications, Rosen

Example 3-c

Every student in this class has visited either Canada or Mexico.

For every person x,
if x is a student in this class, then x has visited Mexico or x has visited Canada.

V(x,y) : x has visited country y

$$\forall x \, (S(x) \rightarrow (V(x, Canada) \vee V(x, Mexico)))$$

Discrete Mathematics and Its Applications, Rosen

Quantifications of 2 variables (1)

$$\forall x \forall y \; P(x, y)$$
$$\forall y \forall x \; P(x, y)$$

T when P(x,y) is true for every pair x, y.

F when there is a pair x, y for which P(x,y) is **F**

Discrete Mathematics and Its Applications, Rosen

Quantifications of 2 variables (2)

$$\forall x \exists y \ P(x,y)$$

T when for every x there is a y for which P(x,y) is **T**.

F when there is an x for which P(x,y) is **F** for every y.

Discrete Mathematics and Its Applications, Rosen

Quantifications of 2 variables (3)

$$\exists x \forall y \; P(x, y)$$

T when for an x for which P(x,y) is **T** for every y.

F when for every x there is an y for which P(x,y) is **F**.

Discrete Mathematics and Its Applications, Rosen

Quantifications of 2 variables (4)

$$\exists x \exists y \; P(x, y)$$
$$\exists y \exists x \; P(x, y)$$

T when there is a pair x, y for which P(x,y) is **T**.

F when P(x,y) is **F** for every pairs x, y.

Discrete Mathematics and Its Applications, Rosen

References

[1] http://en.wikipedia.org/
[2]