DFT Frequency (4B)

- Negative Frequency
- Angular Frequency
- Fundamental Frequency
- Harmonic Frequency
- Sampling Frequency
- Normalized Frequency
- Examples of N=8 DFT Matrix

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5/16/11

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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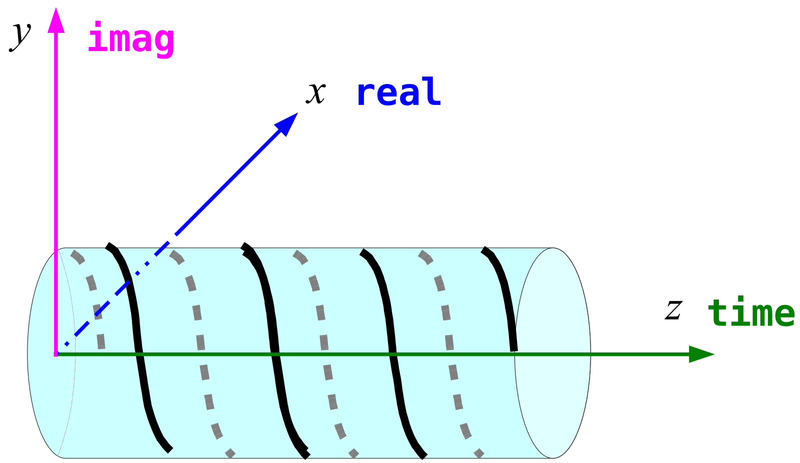
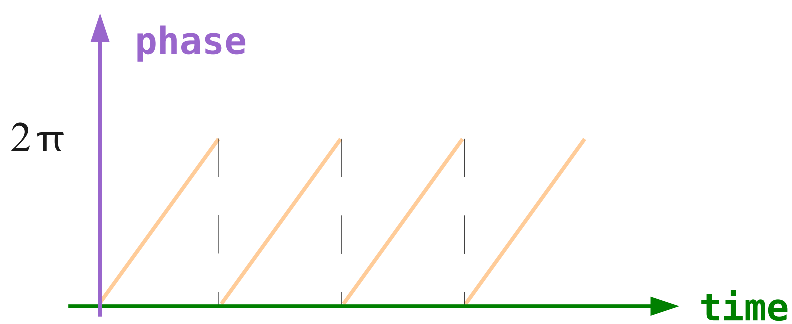
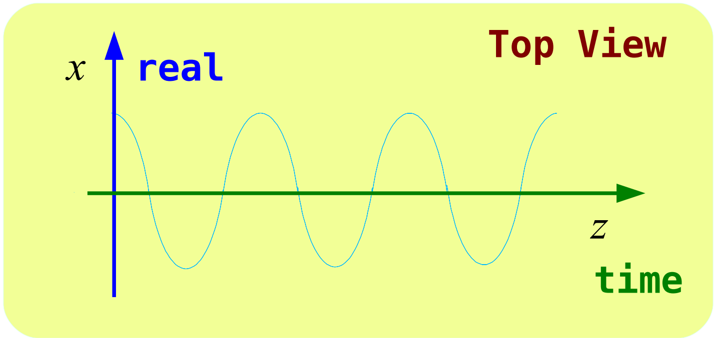
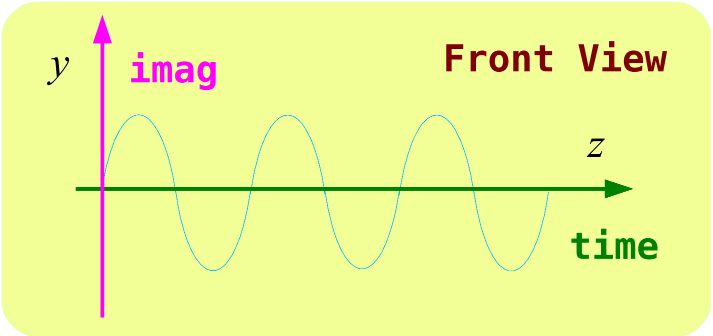
Euler Equation

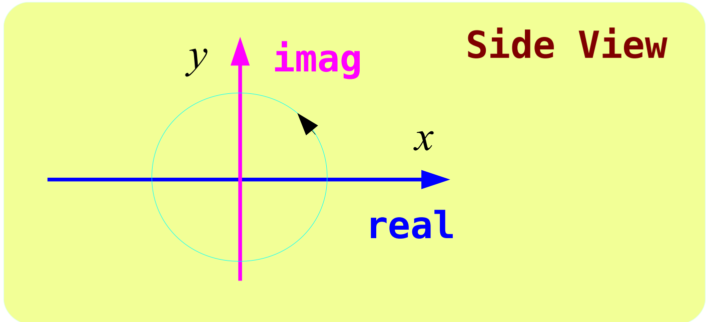
$$e^{+j\omega t} = \cos\omega t + j\sin\omega t$$







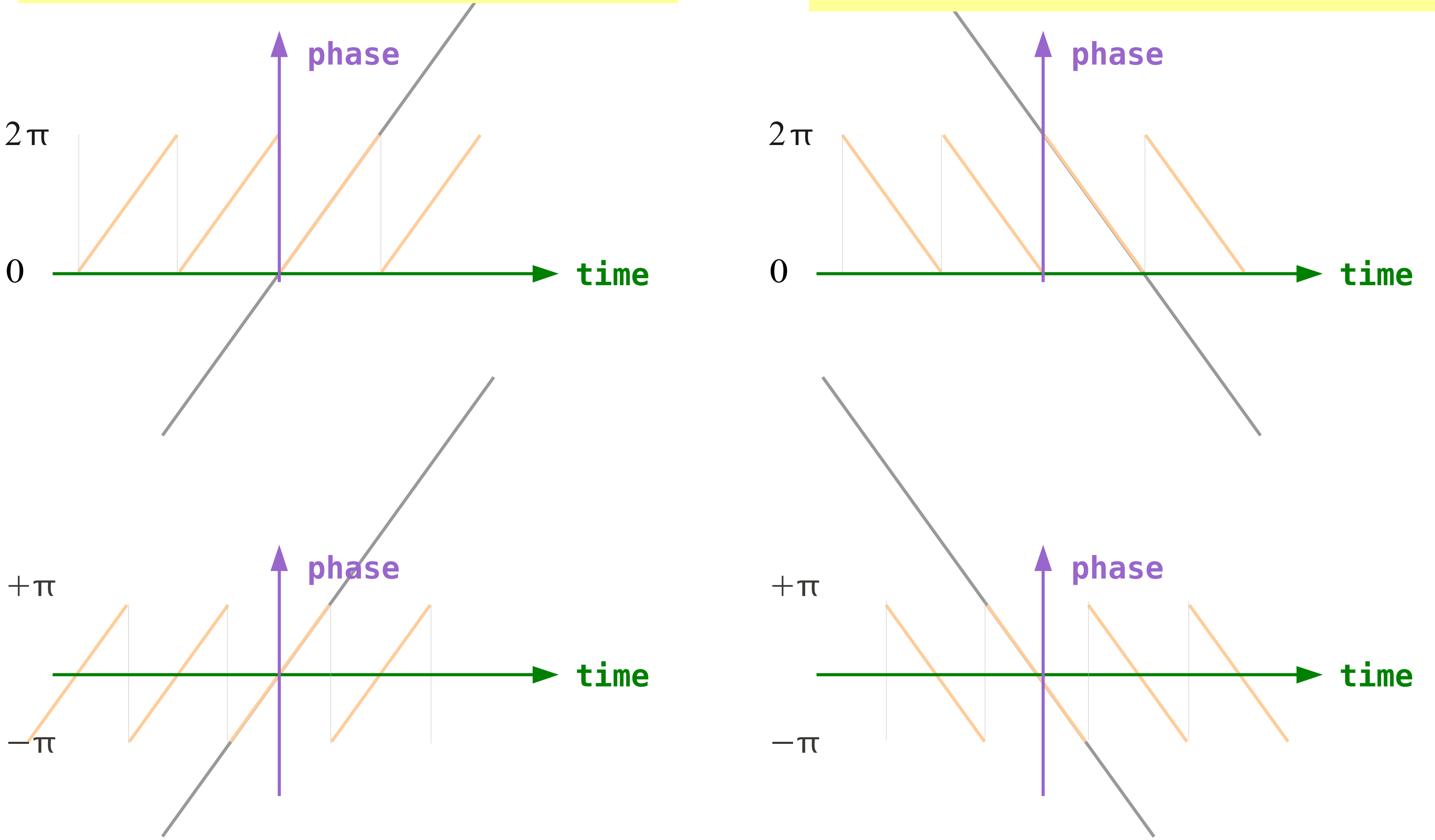




Linear Phase (1)

$$\Phi = \omega_1 t \quad (\omega_1 > 0)$$

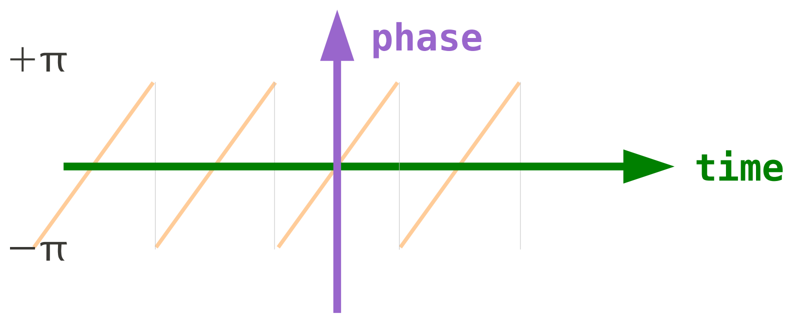
$$\Phi = \omega_2 t \quad (\omega_2 < 0)$$

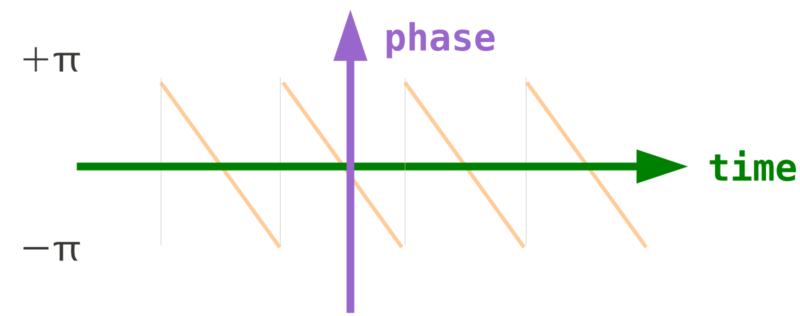


Linear Phase (2)

$$\Phi = \omega_1 t \quad (\omega_1 > 0)$$

$$\Phi = \omega_2 t \quad (\omega_2 < 0)$$





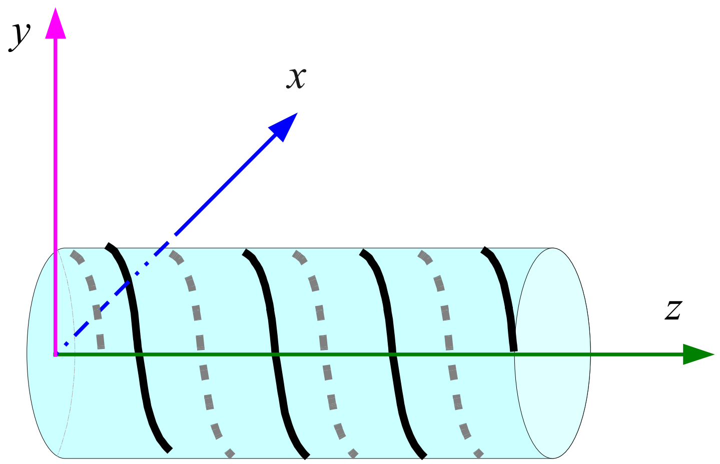
$$\omega_2 = -\omega_1$$

$$\cos(\omega_2 t) = \cos(-\omega_1 t) \Rightarrow \cos(\omega_2 t) = \cos(\omega_1 t)$$

$$\sin(\omega_2 t) = \sin(-\omega_1 t) \Rightarrow \sin(\omega_2 t) = -\sin(\omega_1 t)$$

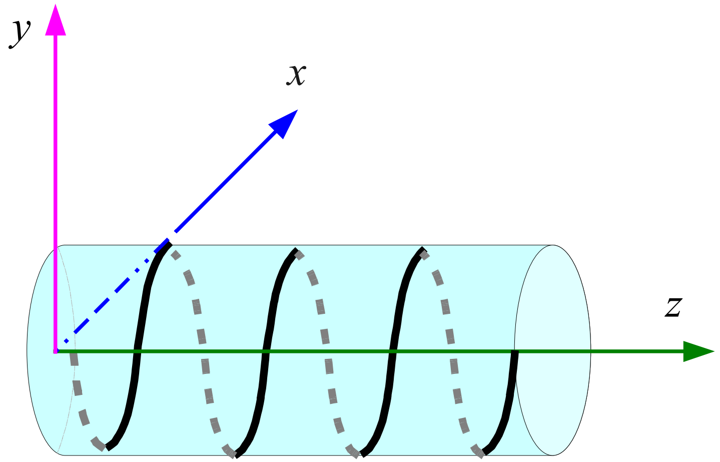
$$e^{j\omega_2 t} = e^{-j\omega_1 t} \Rightarrow e^{j\omega_2 t} = \cos(\omega_1 t) - j\sin(\omega_1 t)$$

Linear Phase (3)



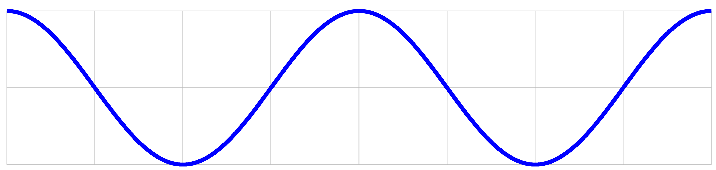
$$e^{j\omega_1 t} = \cos(\omega_1 t) + j\sin(\omega_1 t)$$

$$\omega_2 = -\omega_1$$

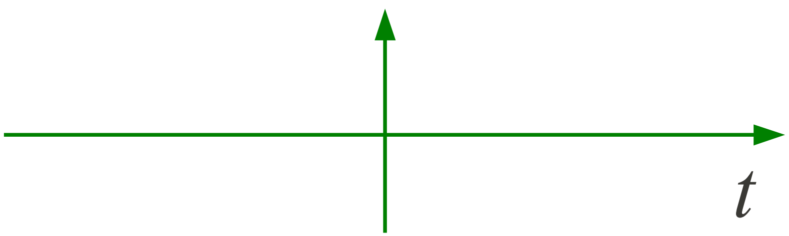


$$e^{j\omega_2 t} = \cos(\omega_1 t) - j\sin(\omega_1 t)$$

Negative Frequency (1)

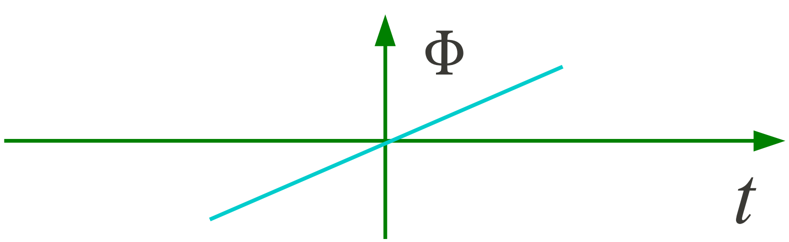


***Coordinate* (A)**

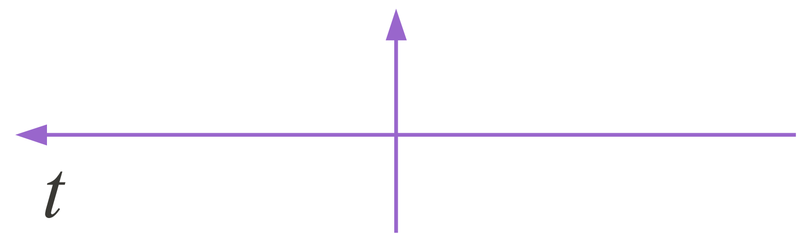


*As **t** increases, the phase increases.*

➡ *positive angular speed*

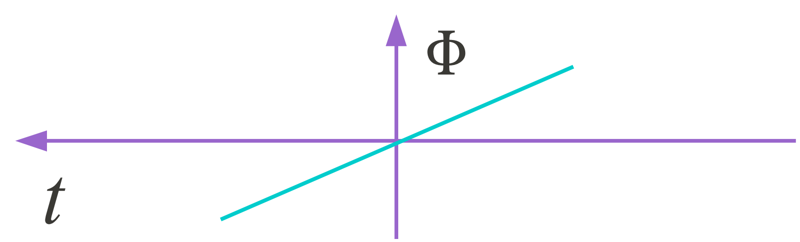


***Coordinate* (B)**

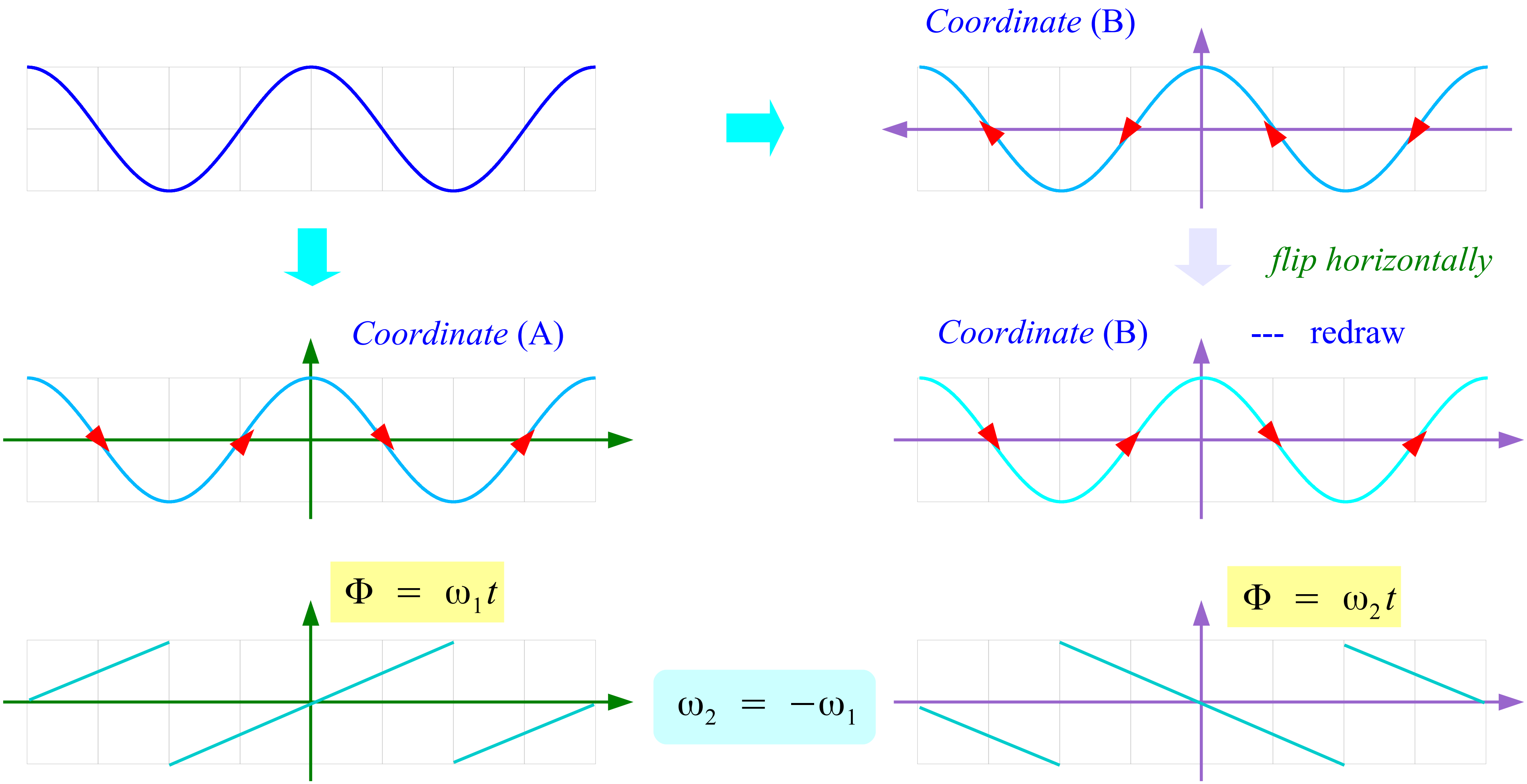


*As **t** increases, the phase decreases.*

➡ *negative angular speed*

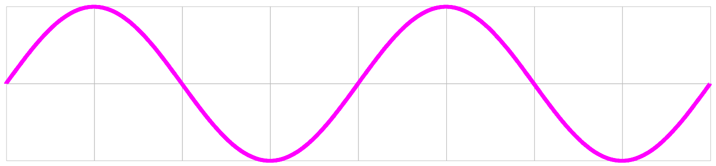


Negative Frequency (2)

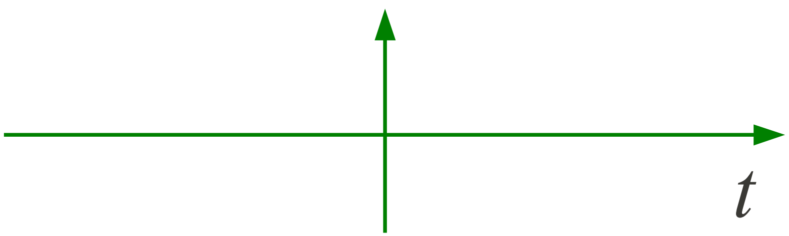


$$\cos(\omega_2 t) = \cos(\omega_1 t)$$

Negative Frequency (3)

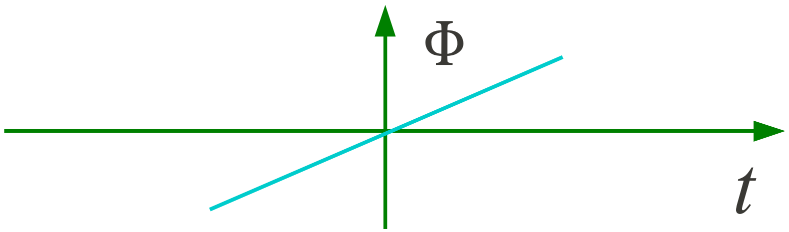


Coordinate **(A)**

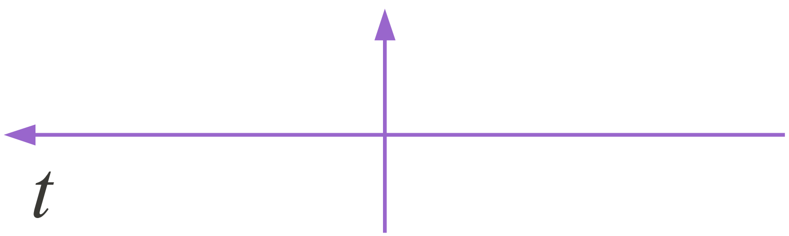


*As **t** increases, the phase increases.*

➡ *positive angular speed*

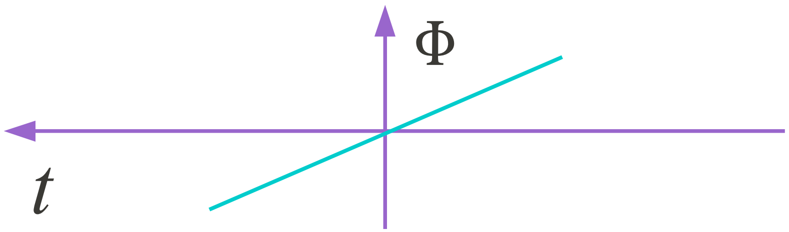


Coordinate **(B)**

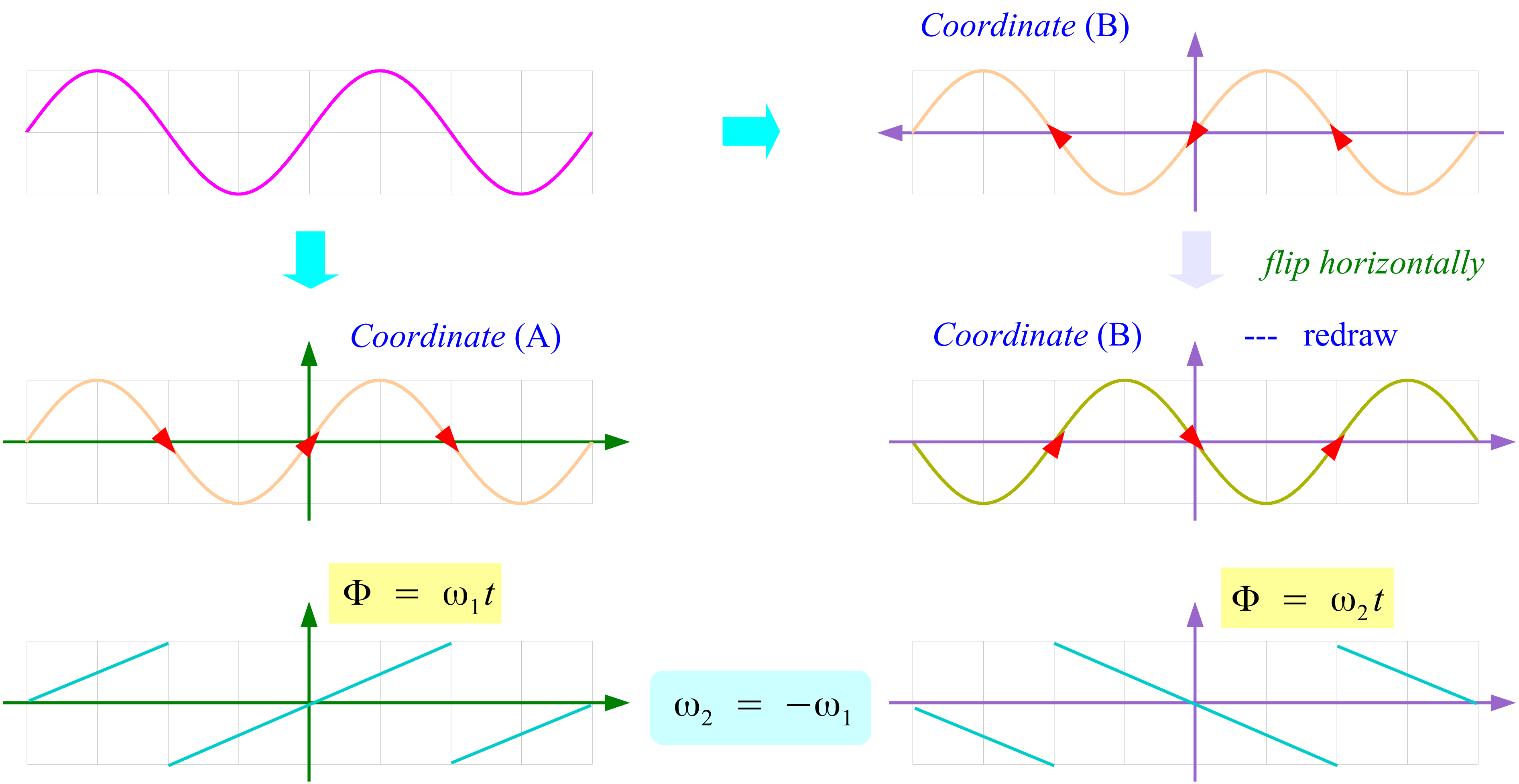


*As **t** increases, the phase decreases.*

➡ *negative angular speed*



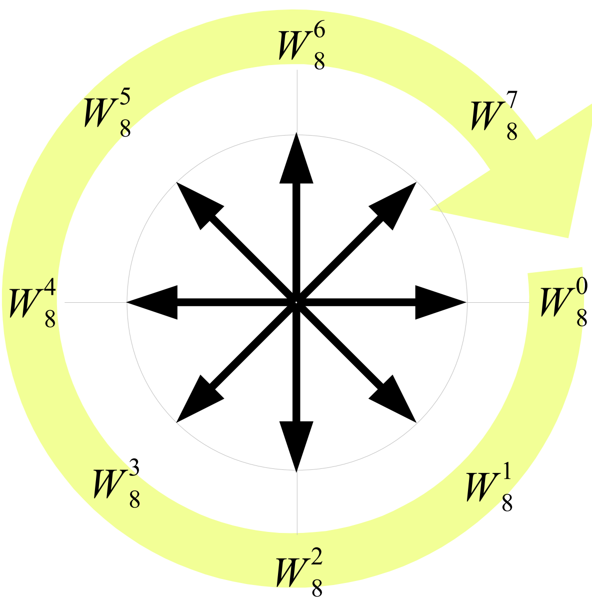
Negative Frequency (4)



$$\sin(\omega_2 t) = -\sin(\omega_1 t)$$

Complex Phase Factor

$$W_8^{k} = e^{-j(\frac{2\pi}{8})k}$$



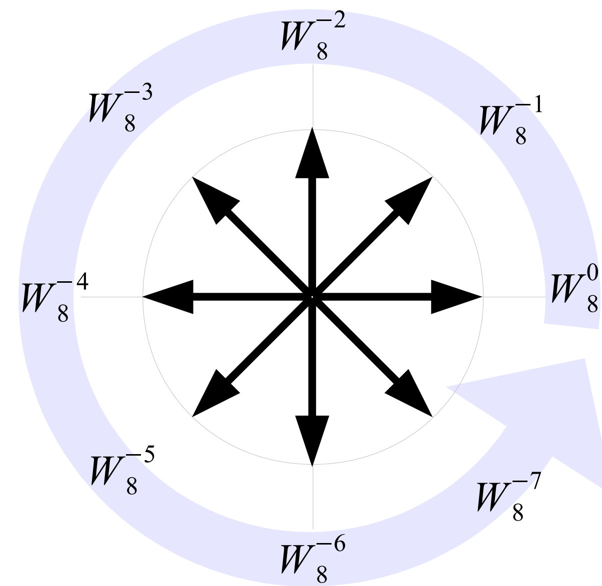
c.w (– angle)

c.c.w (+ angle)

$$W_8^{1} = W_8^{-7} = W_8^{1}$$
$$W_8^{2} = W_8^{-6} = W_8^{2}$$
$$W_8^{3} = W_8^{-5} = W_8^{3}$$
$$W_8^{4} = W_8^{-4} = W_8^{4}$$
$$W_8^{5} = W_8^{-3} = W_8^{-3}$$
$$W_8^{6} = W_8^{-2} = W_8^{-2}$$
$$W_8^{7} = W_8^{-1} = W_8^{-1}$$

$$W_N^{k \pm N} = W_N^{k}$$

$$W_8^{-k} = e^{+j(\frac{2\pi}{8})k}$$



DFT Matrix (1)

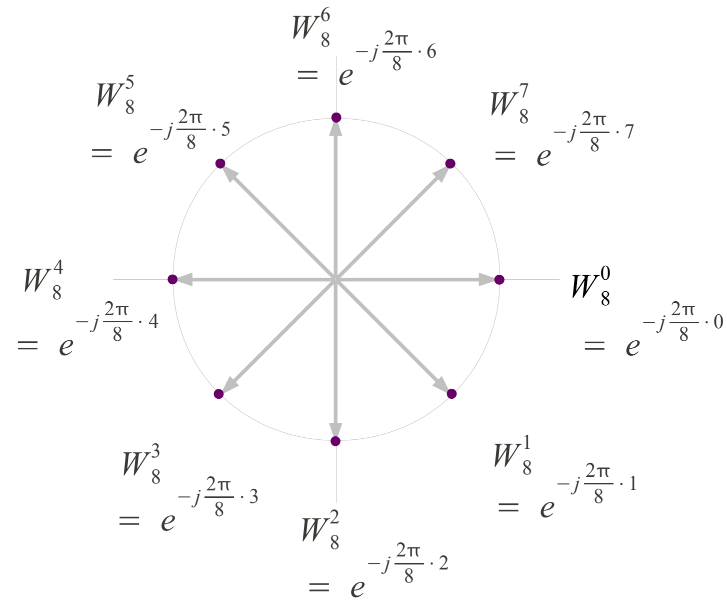
$$\begin{matrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\
W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\
W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\
W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\
W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\
W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1
\end{matrix}$$

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

	n=0		n=1		n=2		n=3		n=4		n=5		n=6		n=7
k=0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0
k=1	0	-1	-1	-1	-2	-1	-3	-1	-4	-1	-5	-1	-6	-1	-7
k=2	0	-2	-2	-2	-4	-2	-6	-2	0	-2	-2	-2	-4	-2	-6
k=3	0	-3	-3	-3	-6	-3	-1	-3	-4	-3	-7	-3	-2	-3	-5
k=4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4
k=5	0	-5	-5	-5	-2	-5	-7	-5	-4	-5	-1	-5	-6	-5	-3
k=6	0	-6	-6	-6	-4	-6	-2	-6	0	-6	-6	-6	-4	-6	-2
k=7	0	-7	-7	-7	-6	-7	-5	-7	-4	-7	-3	-7	-2	-7	-1

DFT Matrix (2)

	n=0		n=1		n=2		n=3		n=4		n=5		n=6		n=7
k=0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0
k=1	0	-1	-1	-1	-2	-1	-3	-1	-4	-1	-5	-1	-6	-1	-7
k=2	0	-2	-2	-2	-4	-2	-6	-2	0	-2	-2	-2	-4	-2	-6
k=3	0	-3	-3	-3	-6	-3	-1	-3	-4	-3	-7	-3	-2	-3	-5
k=4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4
k=5	0	-5	-5	-5	-2	-5	-7	-5	-4	-5	-1	-5	-6	-5	-3
k=6	0	-6	-6	-6	-4	-6	-2	-6	0	-6	-6	-6	-4	-6	-2
k=7	0	-7	-7	-7	-6	-7	-5	-7	-4	-7	-3	-7	-2	-7	-1



k=0	stride = 0	cw angular speed = 0
k=1	stride = -1	cw angular speed = -1ω
k=2	stride = -2	cw angular speed = -2ω
k=3	stride = -3	cw angular speed = -3ω
k=4	stride = -4	cw angular speed = -4ω
k=5	stride = -5	cw angular speed = -5ω
k=6	stride = -6	cw angular speed = -6ω
k=7	stride = -7	cw angular speed = -7ω

-7

DFT Matrix (3)

$$
\begin{matrix}
W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} \\
W_8^{0} & W_8^{1} & W_8^{2} & W_8^{3} & W_8^{4} & W_8^{5} & W_8^{6} & W_8^{7} \\
W_8^{0} & W_8^{2} & W_8^{4} & W_8^{6} & W_8^{0} & W_8^{2} & W_8^{4} & W_8^{6} \\
W_8^{0} & W_8^{3} & W_8^{6} & W_8^{1} & W_8^{4} & W_8^{7} & W_8^{2} & W_8^{5} \\
W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} & W_8^{0} & W_8^{4} \\
W_8^{0} & W_8^{-3} & W_8^{-6} & W_8^{-1} & W_8^{-4} & W_8^{-7} & W_8^{-2} & W_8^{-5} \\
W_8^{0} & W_8^{-2} & W_8^{-4} & W_8^{-6} & W_8^{0} & W_8^{-2} & W_8^{-4} & W_8^{-6} \\
W_8^{0} & W_8^{-1} & W_8^{-2} & W_8^{-3} & W_8^{-4} & W_8^{-5} & W_8^{-6} & W_8^{-7}
\end{matrix}
$$

still symmetric matrix

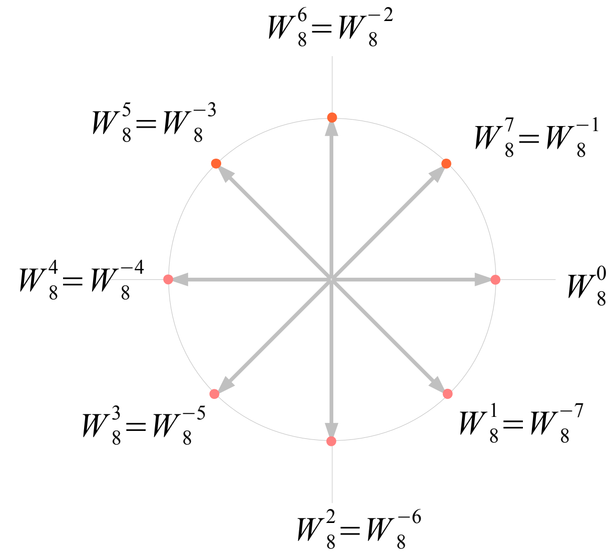
$$W_N^{nk \pm N} = W_N^{nk}$$

$$W_8^{nk} = e^{-j(\frac{2\pi}{8})nk}$$

	n=0		n=1		n=2		n=3		n=4		n=5		n=6		n=7
k=0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0
k=1	0	-1	-1	-1	-2	-1	-3	-1	-4	-1	-5	-1	-6	-1	-7
k=2	0	-2	-2	-2	-4	-2	-6	-2	0	-2	-2	-2	-4	-2	-6
k=3	0	-3	-3	-3	-6	-3	-1	-3	-4	-3	-7	-3	-2	-3	-5
k=4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4
k=5	0	+3	+3	+3	+6	+3	+1	+3	+4	+3	+7	+3	+2	+3	+5
k=6	0	+2	+2	+2	+4	+2	+6	+2	0	+2	+2	+2	+4	+2	+6
k=7	0	+1	+1	+1	+2	+1	+3	+1	+4	+1	+5	+1	+6	+1	+7

DFT Matrix (4)

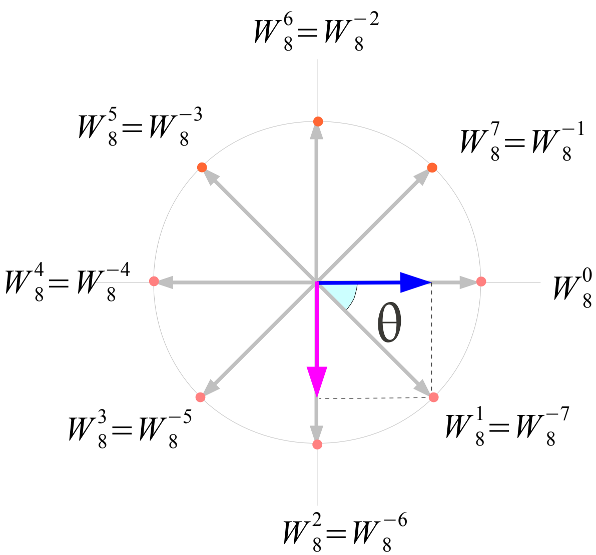
	n=0		n=1		n=2		n=3		n=4		n=5		n=6		n=7
k=0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0	-0	0
k=1	0	-1	-1	-1	-2	-1	-3	-1	-4	-1	-5	-1	-6	-1	-7
k=2	0	-2	-2	-2	-4	-2	-6	-2	0	-2	-2	-2	-4	-2	-6
k=3	0	-3	-3	-3	-6	-3	-1	-3	-4	-3	-7	-3	-2	-3	-5
k=4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4	-4	0	-4	-4
k=5	0	+3	+3	+3	+6	+3	+1	+3	+4	+3	+7	+3	+2	+3	+5
k=6	0	+2	+2	+2	+4	+2	+6	+2	0	+2	+2	+2	+4	+2	+6
k=7	0	+1	+1	+1	+2	+1	+3	+1	+4	+1	+5	+1	+6	+1	+7



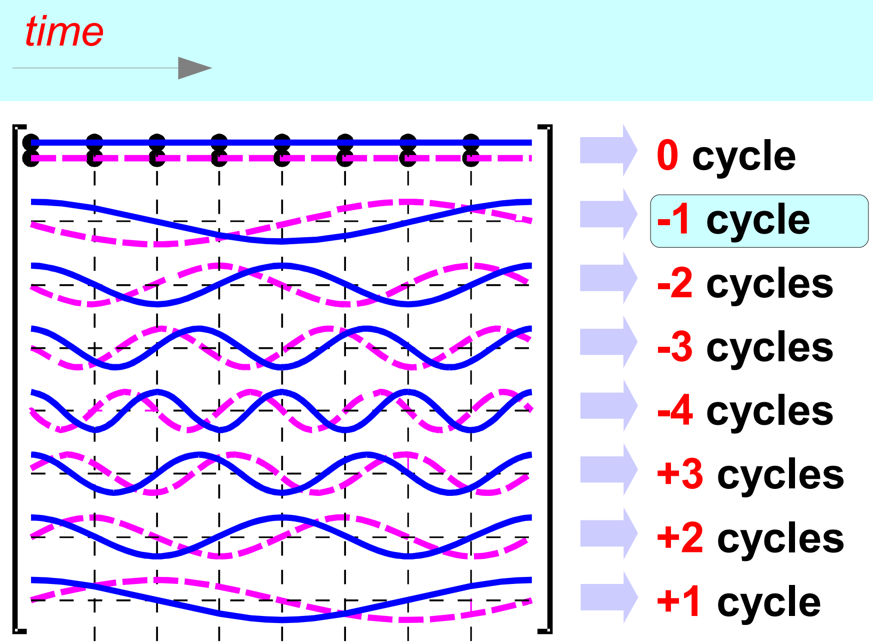
k=0	stride = 0	cw angular speed = 0			
k=1	stride = -1	cw angular speed = -1ω			
k=2	stride = -2	cw angular speed = -2ω			
k=3	stride = -3	cw angular speed = -3ω			
k=4	stride = -4	cw angular speed = -4ω			
k=5	stride = -5	cw angular speed = -5ω	⇔	stride = +3	ccw angular speed = $+3\omega$
k=6	stride = -6	cw angular speed = -6ω	⇔	stride = +2	ccw angular speed = $+2\omega$
k=7	stride = -7	cw angular speed = -7ω	⇔	stride = +1	ccw angular speed = $+1\omega$

Fundamental Frequency

N=8 DFT Matrix → 8 complex phases → View as 8 samples in time domain



$$W_N^{nk} \triangleq e^{-j(2\pi/N)nk}$$



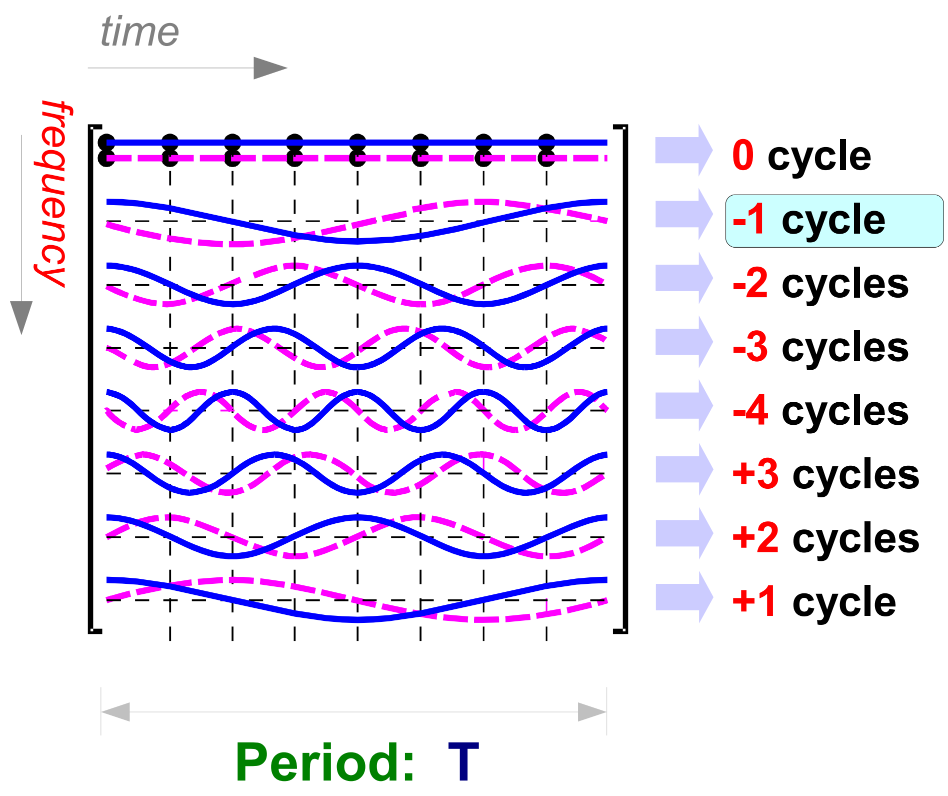
Period: T

Fundamental Frequency : f = 1 / T

Harmonic Frequency

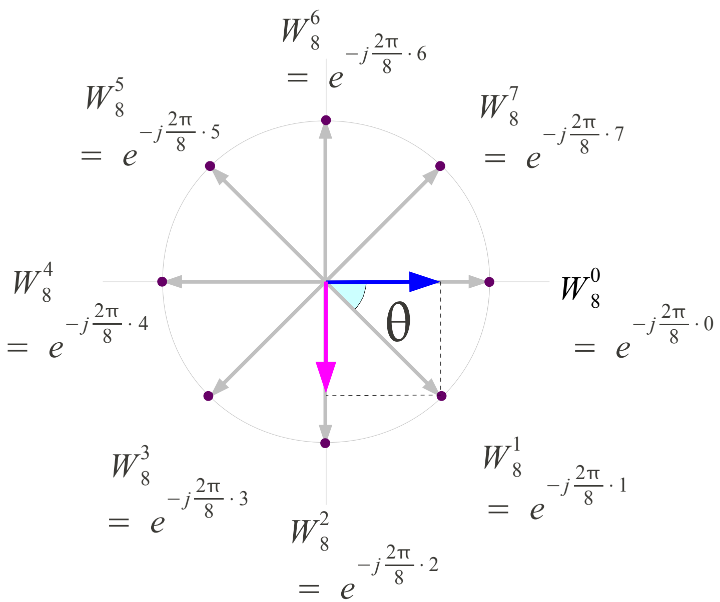
N=8 DFT ➡ 8 complex phases ➡ View as 8 samples in time domain

Measuring Frequency		*stride*	*angular speed*	
	0	0	cw	0
1st harmonic	**+f**	-1	cw	-1ω
2nd harmonic	**+2f**	-2	cw	-2ω
3rd harmonic	**+3f**	-3	cw	-3ω
4th harmonic	**+4f**	-4	cw	-4ω
5th harmonic	**-3f**	+3	ccw	+3ω
6th harmonic	**-2f**	+2	ccw	+2ω
7th harmonic	**-f**	+1	ccw	+1ω



Fundamental Frequency : f = 1 /T

Sampling Time



N=8 DFT Matrix

8 complex phases

8 samples in time domain

Sampling Time : τ

Period: **T** $T = N\tau$

	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7	
k=1	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 1}$	$e^{-j\cdot\frac{\pi}{4}\cdot 2}$	$e^{-j\cdot\frac{\pi}{4}\cdot 3}$	$e^{-j\cdot\frac{\pi}{4}\cdot 4}$	$e^{-j\cdot\frac{\pi}{4}\cdot 5}$	$e^{-j\cdot\frac{\pi}{4}\cdot 6}$	$e^{-j\cdot\frac{\pi}{4}\cdot 7}$	**-1 cycle**

τ

$T = N\tau$

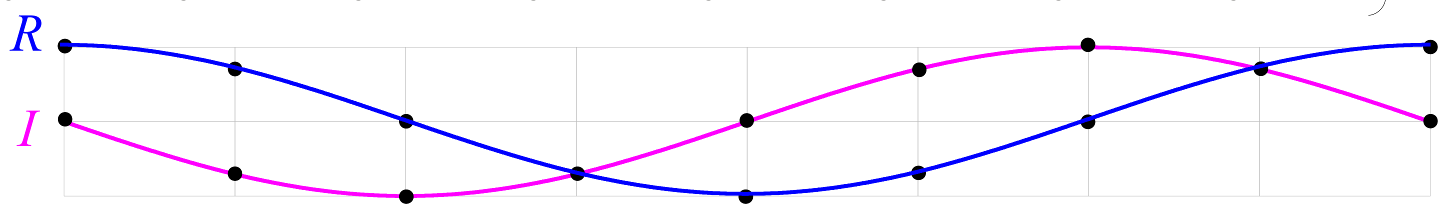
DFT Matrix and Signal

2^{nd} Row of DFT Matrix

k=1

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix}$$

➡ **-1 cycle**

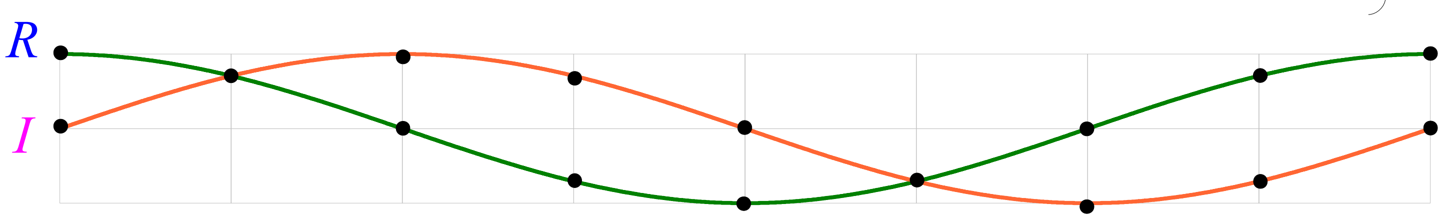


n=0 **n=1** **n=2** **n=3** **n=4** **n=5** **n=6** **n=7**

***X*[1] measures +1 cycle signal**

Time-domain Signal

$$\begin{pmatrix} e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 1} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 3} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 5} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix}$$



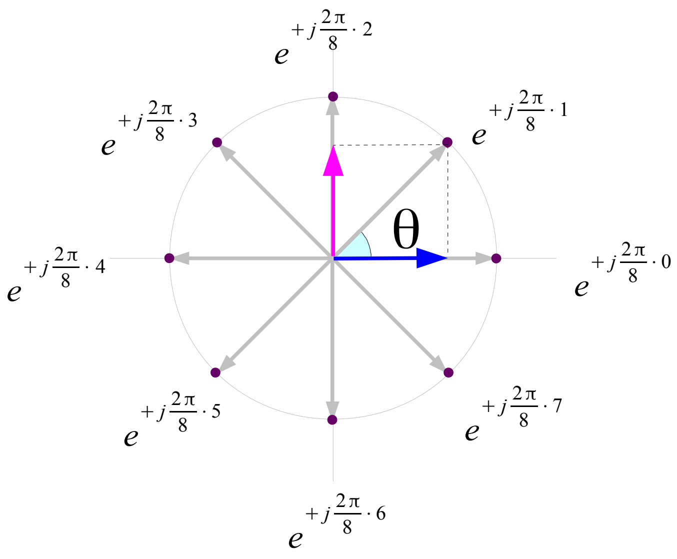
x[0] **x[1]** **x[2]** **x[3]** **x[4]** **x[5]** **x[6]** **x[7]**

This x[i]'s give maximum X[1]

$$T = N\tau$$

$$f = \frac{f_s}{N}$$

Fundamental Frequency



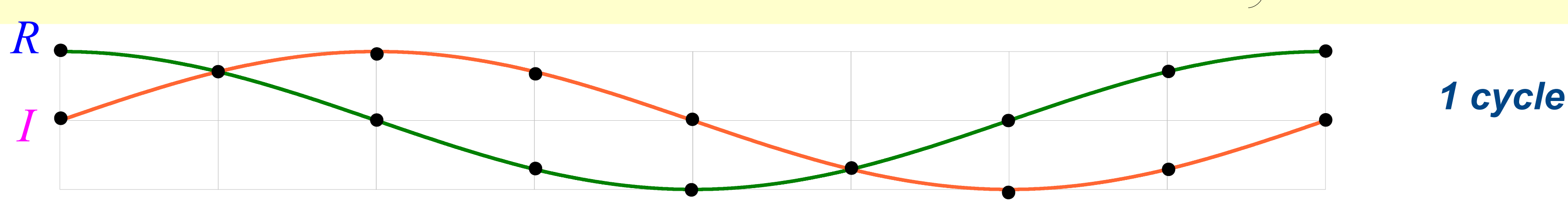
Signal's Frequency

Period	Sampling Time
$T = N\tau$	τ
Fundamental Freq $f = \frac{1}{T}$ (cycles per second)	**Sampling Frequency** $f_s = \frac{1}{\tau}$ (samples per second)

Sampling Frequency

$$f = \frac{f_s}{N} \left(= \frac{1}{N\tau} \right)$$

$$\left(e^{+j\cdot\frac{\pi}{4}\cdot 0} \quad e^{+j\cdot\frac{\pi}{4}\cdot 1} \quad e^{+j\cdot\frac{\pi}{4}\cdot 2} \quad e^{+j\cdot\frac{\pi}{4}\cdot 3} \quad e^{+j\cdot\frac{\pi}{4}\cdot 4} \quad e^{+j\cdot\frac{\pi}{4}\cdot 5} \quad e^{+j\cdot\frac{\pi}{4}\cdot 6} \quad e^{+j\cdot\frac{\pi}{4}\cdot 7} \right)$$



τ ➡ $f_s = \frac{1}{\tau}$ ➡ $f = \frac{f_s}{N}$

$T = N\tau$

Cycles / Sample

$T = N\tau$

τ *second / sample*

$1/\tau$ *sample / second*

$\frac{0}{N\tau}$	(*cycles / second*)	➡	**0 cycle**	**over N sample periods**		**= 0 / N** (*cycles / sample*)
$\frac{1}{N\tau}$	(*cycles / second*)	➡	**1 cycle**	"	"	**= 1 / N** (*cycles / sample*)
$\frac{2}{N\tau}$	(*cycles / second*)	➡	**2 cycles**	"	"	**= 2 / N** (*cycles / sample*)
$\frac{3}{N\tau}$	(*cycles / second*)	➡	**3 cycles**	"	"	**= 3 / N** (*cycles / sample*)
$\frac{4}{N\tau}$	(*cycles / second*)	➡	**4 cycles**	"	"	**= 4 / N** (*cycles / sample*)

Normalized Frequency

$$\frac{f_n}{f_s} = \frac{n}{N}$$

(cycles per sample)

$$\frac{\text{(cycles per second)}}{\text{(samples per second)}}$$

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Normalized Frequency

τ

$T = N\tau$

Sampling Time	τ	(seconds per sample)
Sequence Time Length	$T = N\tau$	
Sampling Frequency	$f_s = \frac{1}{\tau}$	(samples per second)

1st Harmonic Freq $\quad f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{1}{N} f_s$

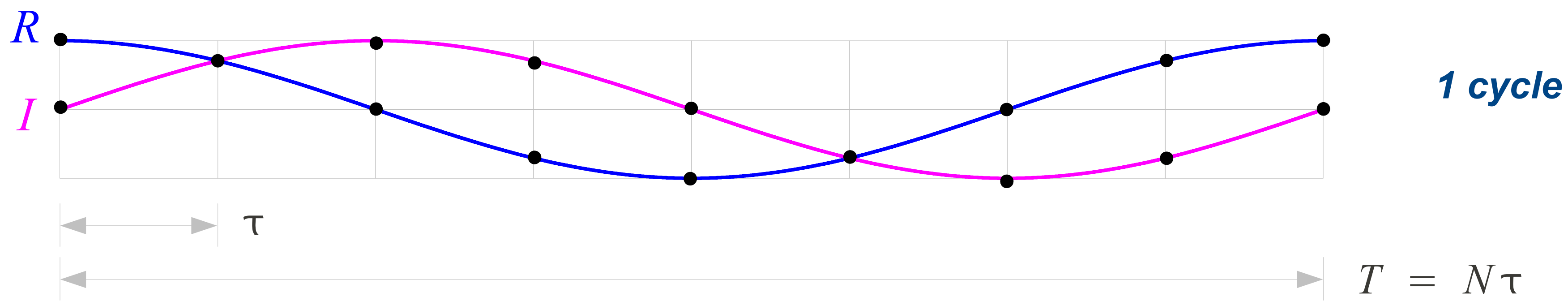
nth Harmonic Freq $\quad f_n = \frac{n}{T} = \frac{n}{N\tau} = \frac{n}{N} f_s \qquad n = 0,\ 1,\ 2,\ \ldots,\ \frac{N}{2}$

Normalized Frequency

$$\frac{f_n}{f_s} = \frac{n}{N}$$

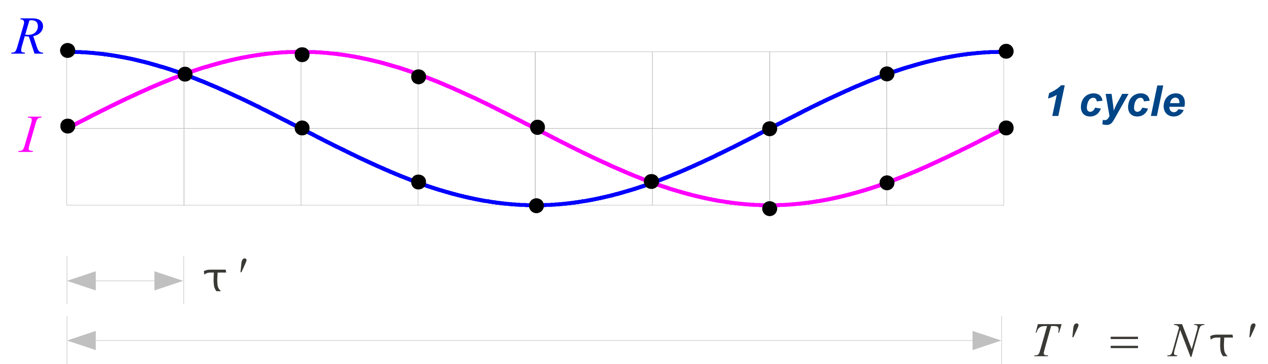
(cycles per sample) $\quad \dfrac{\text{(cycles per second)}}{\text{(samples per second)}}$

Normalized Frequency (Ex 1)



1st Harmonic Freq $f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

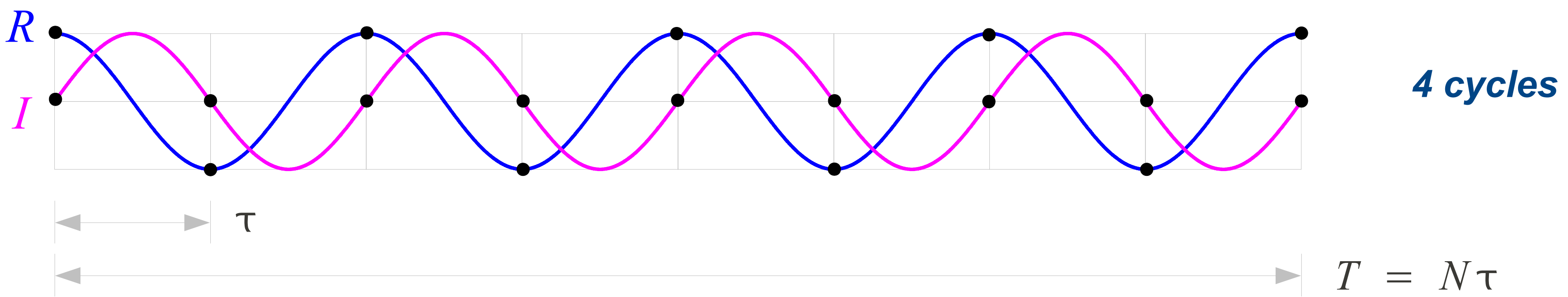
Normalized Freq $\frac{f_1}{f_s} = \frac{1}{N}$



1st Harmonic Freq $f_1' = \frac{1}{T'} = \frac{1}{N\tau'} = \frac{f_s'}{N}$

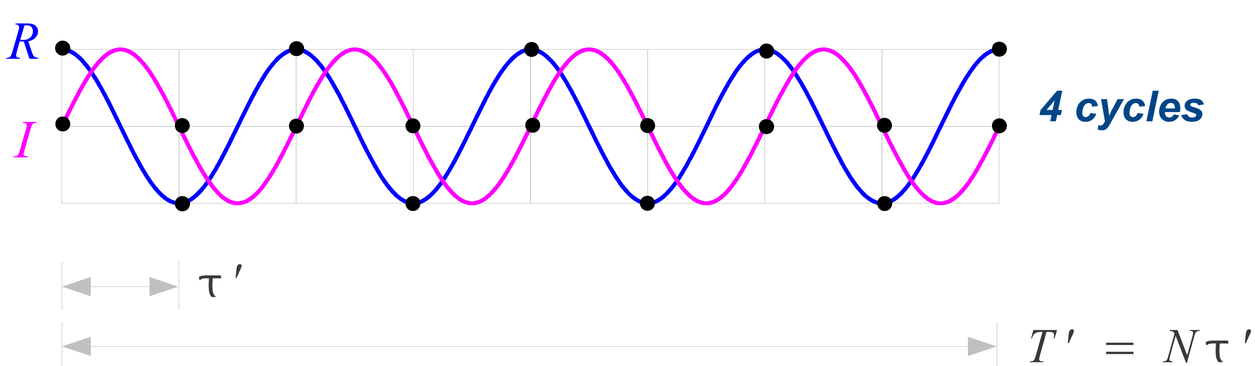
Normalized Freq $\frac{f_1'}{f_s'} = \frac{1}{N}$

Normalized Frequency (Ex 2)



4th Harmonic Freq $f_1 = \frac{4}{T} = \frac{4}{N\tau} = \frac{4f_s}{N}$

Normalized Freq $\frac{f_1}{f_s} = \frac{4}{N}$



4th Harmonic Freq $f_1' = \frac{4}{T'} = \frac{4}{N\tau'} = \frac{4f_s'}{N}$

Normalized Freq $\frac{f_1'}{f_s'} = \frac{4}{N}$

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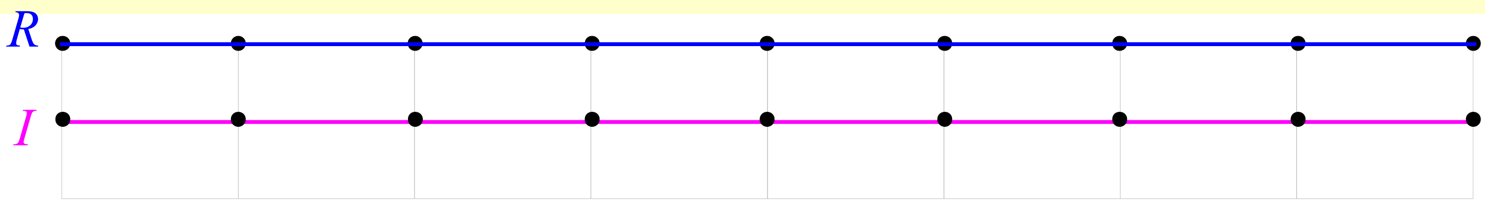
N=8 DFT

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\ W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

N=8 DFT : The 1st Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} \end{pmatrix}$$



0 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 0, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-\omega t) = \cos(\omega t)$

I ➡ *samples of* $\sin(-\omega t) = -\sin(\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi \cdot \left(\tfrac{0}{8}\right) \cdot f_s \cdot t$$

X[0] *measures how much of the* ***+0·ω*** *component is present in* ***x***.

$T = N\tau$

Sampling Time τ

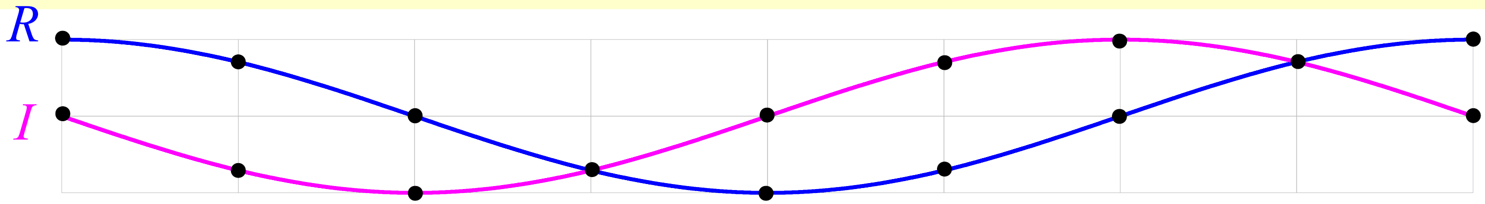
Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

zero Frequency

N=8 DFT : The 2nd Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix}$$



1 cycle

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 1, \quad n = 0, 1, \ldots, 7$$

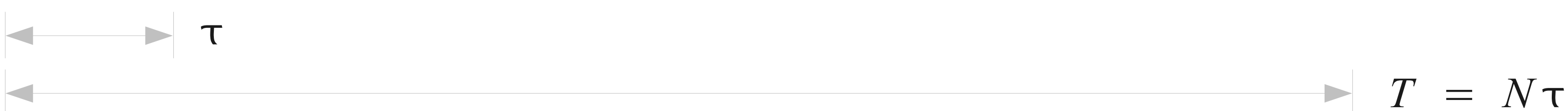
R → *samples of* $\cos(-\omega t) = \cos(\omega t)$

I → *samples of* $\sin(-\omega t) = -\sin(\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{1}{8}\right)\cdot f_s\cdot t$$

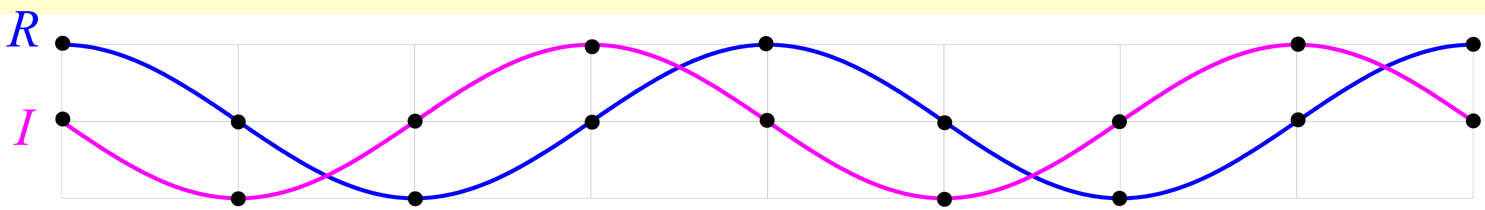
X[1] *measures how much of the* ***+1·ω*** *component is present in* ***x***.



Sampling Time	τ	*Sampling Frequency*	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	*1st Harmonic Freq*	$f_1 = \frac{1}{T} = \frac{1}{N\tau} = \frac{f_s}{N}$

N=8 DFT : The 3rd Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} \end{pmatrix}$$



2 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 2, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-2\omega t) = \cos(2\omega t)$

I ➡ *samples of* $\sin(-2\omega t) = -\sin(2\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi\cdot\left(\tfrac{2}{8}\right)\cdot f_s\cdot t$$

$X[2]$ *measures how much of the* $+2\cdot\omega$ *component is present in* x.

Sampling Time τ

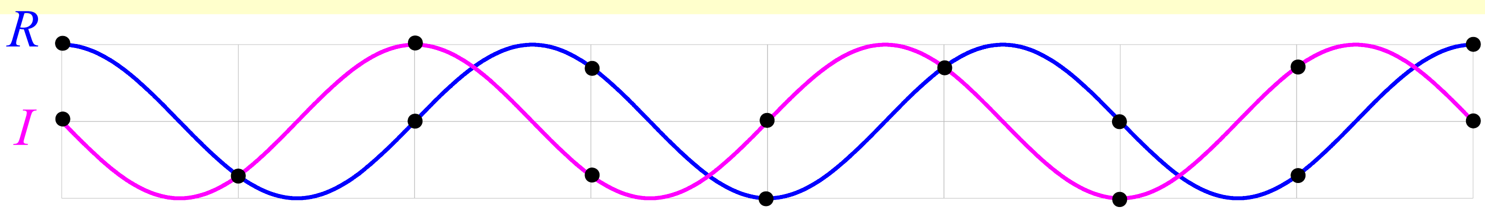
Sampling Frequency $f_s = \dfrac{1}{\tau}$

Sequence Time Length $T = N\tau$

2nd Harmonic Freq $f_2 = \dfrac{2}{T} = \dfrac{2}{N\tau} = \dfrac{2 f_s}{N}$

N=8 DFT : The 4th Row of the DFT Matrix

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 3} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \quad e^{-j\cdot\frac{\pi}{4}\cdot 1} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 7} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 5} \right)$$



3 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 3, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-3\omega t) = \cos(3\omega t)$

I ➡ *samples of* $\sin(-3\omega t) = -\sin(3\omega t)$

measure →

$$\omega t = 2\pi f t$$

$$2\pi\cdot\left(\tfrac{3}{8}\right)\cdot f_s\cdot t$$

***X*[3] *measures how much of the* $+3\cdot\omega$ *component is present in* x.**

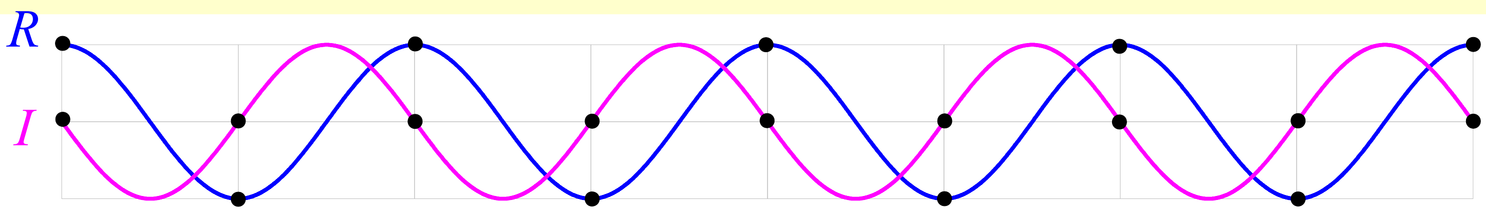
τ

$T = N\tau$

Sampling Time	τ	*Sampling Frequency*	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	*3rd Harmonic Freq*	$f_3 = \frac{3}{T} = \frac{3}{N\tau} = \frac{3 f_s}{N}$

N=8 DFT : The 5th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} \end{pmatrix}$$



4 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 4, \quad n = 0, 1, \dots, 7$$

R ➡ *samples of* $\cos(-4\omega t) = \cos(4\omega t)$

I ➡ *samples of* $\sin(-4\omega t) = -\sin(4\omega t)$

measure →

$$\omega t = 2\pi f t$$
$$2\pi \cdot \left(\tfrac{4}{8}\right) \cdot f_s \cdot t$$

X[4] *measures how much of the* ***+4·ω*** *component is present in* ***x***.

Sampling Time τ

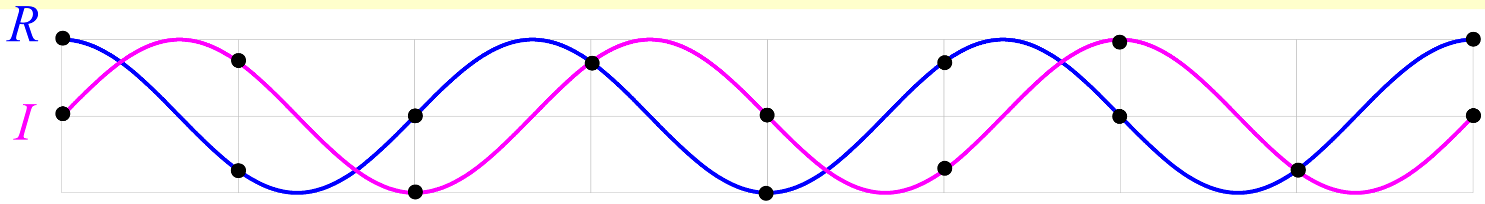
Sampling Frequency $f_s = \dfrac{1}{\tau}$

Sequence Time Length $T = N\tau$

4th Harmonic Freq $f_4 = \dfrac{4}{T} = \dfrac{4}{N\tau} = \dfrac{4 f_s}{N}$

N=8 DFT : The 6th Row of the DFT Matrix

$$\left(e^{-j\cdot\frac{\pi}{4}\cdot 0} \quad e^{-j\cdot\frac{\pi}{4}\cdot 5} \quad e^{-j\cdot\frac{\pi}{4}\cdot 2} \quad e^{-j\cdot\frac{\pi}{4}\cdot 7} \quad e^{-j\cdot\frac{\pi}{4}\cdot 4} \quad e^{-j\cdot\frac{\pi}{4}\cdot 1} \quad e^{-j\cdot\frac{\pi}{4}\cdot 6} \quad e^{-j\cdot\frac{\pi}{4}\cdot 3} \right)$$



$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 5, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-(-3\omega)t) = \cos(3\omega t)$

I ➡ *samples of* $\sin(-(-3\omega)t) = \sin(3\omega t)$

measure ➡

$$-\omega t = -2\pi f t$$

$$2\pi\cdot\left(\tfrac{-3}{8}\right)\cdot f_s\cdot t$$

***X*[5] *measures how much of the* –3·ω *component is present in* x.**

τ

$T = N\tau$

Sampling Time	τ	Sampling Frequency	$f_s = \frac{1}{\tau}$
Sequence Time Length	$T = N\tau$	-3^{rd} Harmonic Freq	$f_{-3} = \frac{-3}{T} = \frac{-3}{N\tau} = \frac{-3f_s}{N}$

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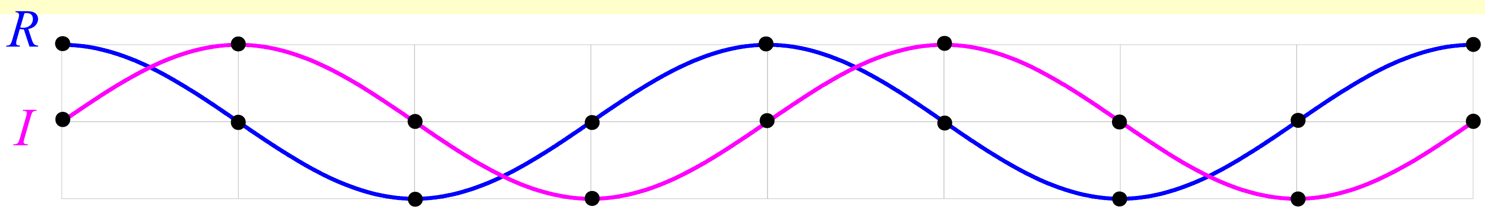
N=8 DFT : The 7th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} \end{pmatrix}$$



2 cycles

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} \qquad k = 2, \quad n = 0, 1, \dots, 7$$

R ➡ *samples of* $\cos(-(-2\omega)t) = \cos(2\omega t)$

I ➡ *samples of* $\sin(-(-2\omega)t) = \sin(2\omega t)$

measure →

$$-\omega t = -2\pi f t$$

$$2\pi \cdot \left(\tfrac{-2}{8}\right) \cdot f_s \cdot t$$

X[6] *measures how much of the* $-2\cdot\omega$ *component is present in* ***x***.

τ

$T = N\tau$

Sampling Time τ | *Sampling Frequency* $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$ | *-2nd Harmonic Freq* $f_{-2} = \frac{-2}{T} = \frac{-2}{N\tau} = \frac{-2 f_s}{N}$

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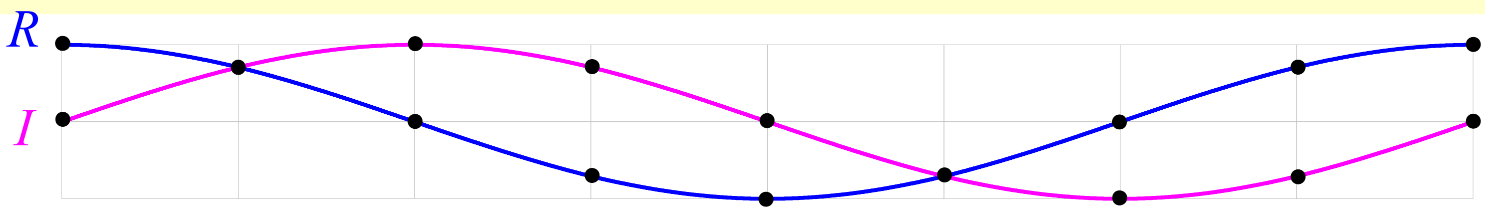
N=8 DFT : The 8th Row of the DFT Matrix

$$\begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 1} \end{pmatrix}$$



1 cycle

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \qquad k = 7, \quad n = 0, 1, \ldots, 7$$

R ➡ *samples of* $\cos(-(-\omega)t) = \cos(\omega t)$

I ➡ *samples of* $\sin(-(-\omega)t) = \sin(\omega t)$

measure →

$$-\omega t = -2\pi f t$$
$$2\pi\cdot(\tfrac{-1}{8})\cdot f_s\cdot t$$

X[7] *measures how much of the* $-1\cdot\omega$ *component is present in* ***x***.

Sampling Time τ

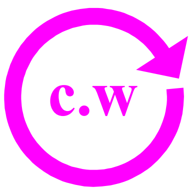
Sampling Frequency $f_s = \frac{1}{\tau}$

Sequence Time Length $T = N\tau$

-1st Harmonic Freq $f_{-1} = \frac{-1}{T} = \frac{-1}{N\tau} = \frac{f_s}{N}$

N=8 DFT : DFT Matrix in + or – Frequencies

$$\omega_0 = 2\pi \cdot \frac{f_s}{N}$$



0th row:	*samples of*	$\cos(\ \ 0\,\omega_0)t + j\cdot\sin(\ \ 0\,\omega_0)t$	(0 cycle)
1th row:	*samples of*	$\cos(-1\,\omega_0)t + j\cdot\sin(-1\,\omega_0)t$	(-1 cycle)
2th row:	*samples of*	$\cos(-2\,\omega_0)t + j\cdot\sin(-2\,\omega_0)t$	(-2 cycles)
3th row:	*samples of*	$\cos(-3\,\omega_0)t + j\cdot\sin(-3\,\omega_0)t$	(-3 cycles)
4th row:	*samples of*	$\cos(-4\,\omega_0)t + j\cdot\sin(-4\,\omega_0)t$	(-4 cycles)
5th row:	*samples of*	$\cos(-5\,\omega_0)t + j\cdot\sin(-5\,\omega_0)t$	(-5 cycles)
6th row:	*samples of*	$\cos(-6\,\omega_0)t + j\cdot\sin(-6\,\omega_0)t$	(-6 cycles)
7th row:	*samples of*	$\cos(-7\,\omega_0)t + j\cdot\sin(-7\,\omega_0)t$	(-7 cycles)

=

0th row:	*samples of*	$\cos(\ \ 0\,\omega_0)t + j\cdot\sin(\ \ 0\,\omega_0)t$	(0 cycle)
1th row:	*samples of*	$\cos(+7\,\omega_0)t + j\cdot\sin(+7\,\omega_0)t$	(7 cycles)
2th row:	*samples of*	$\cos(+6\,\omega_0)t + j\cdot\sin(+6\,\omega_0)t$	(6 cycles)
3th row:	*samples of*	$\cos(+5\,\omega_0)t + j\cdot\sin(+5\,\omega_0)t$	(5 cycles)
4th row:	*samples of*	$\cos(+4\,\omega_0)t + j\cdot\sin(+4\,\omega_0)t$	(4 cycles)
5th row:	*samples of*	$\cos(+3\,\omega_0)t + j\cdot\sin(+3\,\omega_0)t$	(3 cycles)
6th row:	*samples of*	$\cos(+2\,\omega_0)t + j\cdot\sin(+2\,\omega_0)t$	(2 cycles)
7th row:	*samples of*	$\cos(+1\,\omega_0)t + j\cdot\sin(+1\,\omega_0)t$	(1 cycles)

N=8 DFT : DFT Matrix in Both Frequencies

$$\omega_0 = 2\pi \cdot \frac{f_s}{N}$$

0th row: *samples of*	$\cos(\ 0\,\omega_0)t + j\cdot\sin(\ 0\,\omega_0)t$	(0 cycle)
1th row: *samples of*	$\cos(-1\,\omega_0)t + j\cdot\sin(-1\,\omega_0)t$	(-1 cycle)
2th row: *samples of*	$\cos(-2\,\omega_0)t + j\cdot\sin(-2\,\omega_0)t$	(-2 cycles)
3th row: *samples of*	$\cos(-3\,\omega_0)t + j\cdot\sin(-3\,\omega_0)t$	(-3 cycles)
4th row: *samples of*	$\cos(-4\,\omega_0)t + j\cdot\sin(-4\,\omega_0)t$	(-4 cycles)
5th row: *samples of*	$\cos(-5\,\omega_0)t + j\cdot\sin(-5\,\omega_0)t$	(-5 cycles)
6th row: *samples of*	$\cos(-6\,\omega_0)t + j\cdot\sin(-6\,\omega_0)t$	(-6 cycles)
7th row: *samples of*	$\cos(-7\,\omega_0)t + j\cdot\sin(-7\,\omega_0)t$	(-7 cycles)

=

0th row: *samples of*	$\cos(\ 0\,\omega_0)t + j\cdot\sin(\ 0\,\omega_0)t$	(0 cycle)
1th row: *samples of*	$\cos(-1\,\omega_0)t + j\cdot\sin(-1\,\omega_0)t$	(-1 cycle)
2th row: *samples of*	$\cos(-2\,\omega_0)t + j\cdot\sin(-2\,\omega_0)t$	(-2 cycles)
3th row: *samples of*	$\cos(-3\,\omega_0)t + j\cdot\sin(-3\,\omega_0)t$	(-3 cycles)
4th row: *samples of*	$\cos(-4\,\omega_0)t + j\cdot\sin(-4\,\omega_0)t$	(-4 cycles)
5th row: *samples of*	$\cos(+3\,\omega_0)t + j\cdot\sin(+3\,\omega_0)t$	(3 cycles)
6th row: *samples of*	$\cos(+2\,\omega_0)t + j\cdot\sin(+2\,\omega_0)t$	(2 cycles)
7th row: *samples of*	$\cos(+1\,\omega_0)t + j\cdot\sin(+1\,\omega_0)t$	(1 cycles)

Frequency View of a DFT Matrix

$row\ \ 0$	$f = 0$	$(0\,cycle)$	0
$row\ \ 1$	$f = -1 * f_o$	$(-1\,cycle)$	$\frac{-1}{N}$
$row\ \ 2$	$f = -2 * f_o$	$(-2\,cycles)$	$\frac{-2}{N}$
$row\ \ (\frac{N}{2}-1)$	$f = -(\frac{N}{2} - 1) * f_o$	$(-(\frac{N}{2}-1)\,cycles)$	$\frac{-1}{2} + \frac{1}{N}$
$row\ \ (\frac{N}{2})$	$f = -(\frac{N}{2}) * f_o$	$(\frac{N}{2}\ \ cycles)$	$\frac{-1}{2}$
$row\ \ (\frac{N}{2} + 1)$	$f = +(\frac{N}{2} - 1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{1}{2} - \frac{1}{N}$
$row\ \ N-2$	$f = +2 * f_o$	$(2\,cycles)$	$\frac{2}{N}$
$row\ \ N-1$	$f = +1 * f_o$	$(1\,cycle)$	$\frac{1}{N}$

$\frac{N}{2} - 1$ (rows 1 to $\frac{N}{2}-1$)

$\frac{N}{2} - 1$ (rows $\frac{N}{2}+1$ to $N-1$)

Normalized Frequency

$$f_o = \frac{f_s}{N}$$

Young Won Lim
5/16/11

Frequency View of a X[i] Vector

$\frac{N}{2} - 1$ (rows $X[1]$ to $X[\frac{N}{2}-1]$); $\frac{N}{2} - 1$ (rows $X[\frac{N}{2}+1]$ to $X[N-1]$)

			Normalized Frequency	
$X[0]$	$f = 0$	0	0	$(0\,cycle)$
$X[1]$	$f = +1 * f_o$	$\frac{1}{N} f_s$	$\frac{1}{N}$	$(1\,cycle)$
$X[2]$	$f = +2 * f_o$	$\frac{2}{N} f_s$	$\frac{2}{N}$	$(2\,cycles)$
$X[\frac{N}{2}-1]$	$f = +(\frac{N}{2} - 1) * f_o$	$\frac{(\frac{N}{2}-1)}{N} f_s$	$\frac{1}{2} - \frac{1}{N}$	$(\frac{N}{2} - 1\,cycles)$
$X[\frac{N}{2}]$	$f = +(\frac{N}{2}) * f_o$	$\frac{1}{2} f_s$	$\frac{1}{2}$	$(\frac{N}{2}\ cycles)$
$X[\frac{N}{2}+1]$	$f = -(\frac{N}{2} - 1) * f_o$	$-\frac{(\frac{N}{2}-1)}{N} f_s$	$\frac{-1}{2} + \frac{1}{N}$	$(-(\frac{N}{2} - 1)\,cycles)$
$X[N-2]$	$f = -2 * f_o$	$-\frac{2}{N} f_s$	$\frac{-2}{N}$	$(-2\,cycles)$
$X[N-1]$	$f = -1 * f_o$	$-\frac{1}{N} f_s$	$\frac{-1}{N}$	$(-1\,cycle)$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A "graphical interpretation" of the DFT and FFT, by Steve Mann