DFT Matrix Properties (3B)

- X[1]
- X[2]
- X[3]
- X[4]
- X[5]
- X[6]
- X[7]

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N=8 DFT

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

N=8 IDFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{7} W_8^{-kn}\, X[k] \qquad W_8^{-kn} = e^{+j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} & W_8^{0} \\ W_8^{0} & W_8^{-1} & W_8^{-2} & W_8^{-3} & W_8^{-4} & W_8^{-5} & W_8^{-6} & W_8^{-7} \\ W_8^{0} & W_8^{-2} & W_8^{-4} & W_8^{-6} & W_8^{-8} & W_8^{-10} & W_8^{-12} & W_8^{-14} \\ W_8^{0} & W_8^{-3} & W_8^{-6} & W_8^{-9} & W_8^{-12} & W_8^{-15} & W_8^{-18} & W_8^{-21} \\ W_8^{0} & W_8^{-4} & W_8^{-8} & W_8^{-12} & W_8^{-16} & W_8^{-20} & W_8^{-24} & W_8^{-28} \\ W_8^{0} & W_8^{-5} & W_8^{-10} & W_8^{-15} & W_8^{-20} & W_8^{-25} & W_8^{-30} & W_8^{-35} \\ W_8^{0} & W_8^{-6} & W_8^{-12} & W_8^{-18} & W_8^{-24} & W_8^{-30} & W_8^{-36} & W_8^{-42} \\ W_8^{0} & W_8^{-7} & W_8^{-14} & W_8^{-21} & W_8^{-28} & W_8^{-35} & W_8^{-42} & W_8^{-49} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix}$$

N=8 DFT Matrix (1)

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad\qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix}
e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} \\
e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \\
e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} \\
e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 5} \\
e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} \\
e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 3} \\
e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} \\
e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 1}
\end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

A

N=8 IDFT Matrix (1)

$$x[n] = \frac{1}{N}\sum_{k=0}^{7} W_8^{-kn}\, X[k] \qquad W_8^{-kn} = e^{+j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix} = \begin{bmatrix}
e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 0} \\
e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 1} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 3} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 5} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 7} \\
e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 6} \\
e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 3} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 1} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 7} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 5} \\
e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 4} \\
e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 5} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 7} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 1} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 3} \\
e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 2} \\
e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 7} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 5} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 3} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 1}
\end{bmatrix} \begin{bmatrix} \frac{X[0]}{N} \\ \frac{X[1]}{N} \\ \frac{X[2]}{N} \\ \frac{X[3]}{N} \\ \frac{X[4]}{N} \\ \frac{X[5]}{N} \\ \frac{X[6]}{N} \\ \frac{X[7]}{N} \end{bmatrix}$$

B

Symmetric DFT Matrix – Index (1)

$$X[k] = \sum_{n=0}^{N-1} x[n] \, e^{-j(2\pi/N)kn} \qquad X[k] = \sum_{n=0}^{N-1} x[n] \, W_N^{kn}$$

$$\begin{array}{c|ccccc}
 & n=0 & n=1 & n=2 & \cdots & n=N-1 \\
\hline
k=0 & 0\cdot 0 & 0\cdot 1 & 0\cdot 2 & \cdots & 0\cdot(N-1) \\
k=1 & 1\cdot 0 & 1\cdot 1 & 1\cdot 2 & \cdots & 1\cdot(N-1) \\
k=2 & 2\cdot 0 & 2\cdot 1 & 2\cdot 2 & \cdots & 2\cdot(N-1) \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
k=N-1 & (N-1)\cdot 0 & (N-1)\cdot 1 & (N-1)\cdot 2 & \cdots & (N-1)\cdot(N-1)
\end{array}$$

Exponents in **DFT** matrix **A** and **IDFT** matrix **B**

$$A = A^T$$

$$B = B^T$$

Symmetric DFT Matrix – Index (2)

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \qquad X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

$$\begin{array}{c|ccccccccc|l}
 & n=0 & & n=1 & & n=2 & & \cdots & & n=N-1 & \\
\hline
k=0 & 0\cdot0 & \xrightarrow[+0]{} & 0\cdot1 & \xrightarrow[+0]{} & 0\cdot2 & \xrightarrow[+0]{} & \cdots & \xrightarrow[+0]{} & 0\cdot(N-1) & +\mathbf{0} \pmod N \\
k=1 & 1\cdot0 & \xrightarrow[+1]{} & 1\cdot1 & \xrightarrow[+1]{} & 1\cdot2 & \xrightarrow[+1]{} & \cdots & \xrightarrow[+1]{} & 1\cdot(N-1) & +\mathbf{1} \pmod N \\
k=2 & 2\cdot0 & \xrightarrow[+2]{} & 2\cdot1 & \xrightarrow[+2]{} & 2\cdot2 & \xrightarrow[+2]{} & \cdots & \xrightarrow[+2]{} & 2\cdot(N-1) & +\mathbf{2} \pmod N \\
\vdots & \vdots & & \vdots & & \vdots & & & & \vdots & \\
k=N-1 & (N-1)\cdot0 & \xrightarrow[+(N-1)]{} & (N-1)\cdot1 & \xrightarrow[+(N-1)]{} & (N-1)\cdot2 & \xrightarrow[+(N-1)]{} & \cdots & \xrightarrow[+(N-1)]{} & (N-1)\cdot(N-1) & +\mathbf{N-1} \pmod N
\end{array}$$

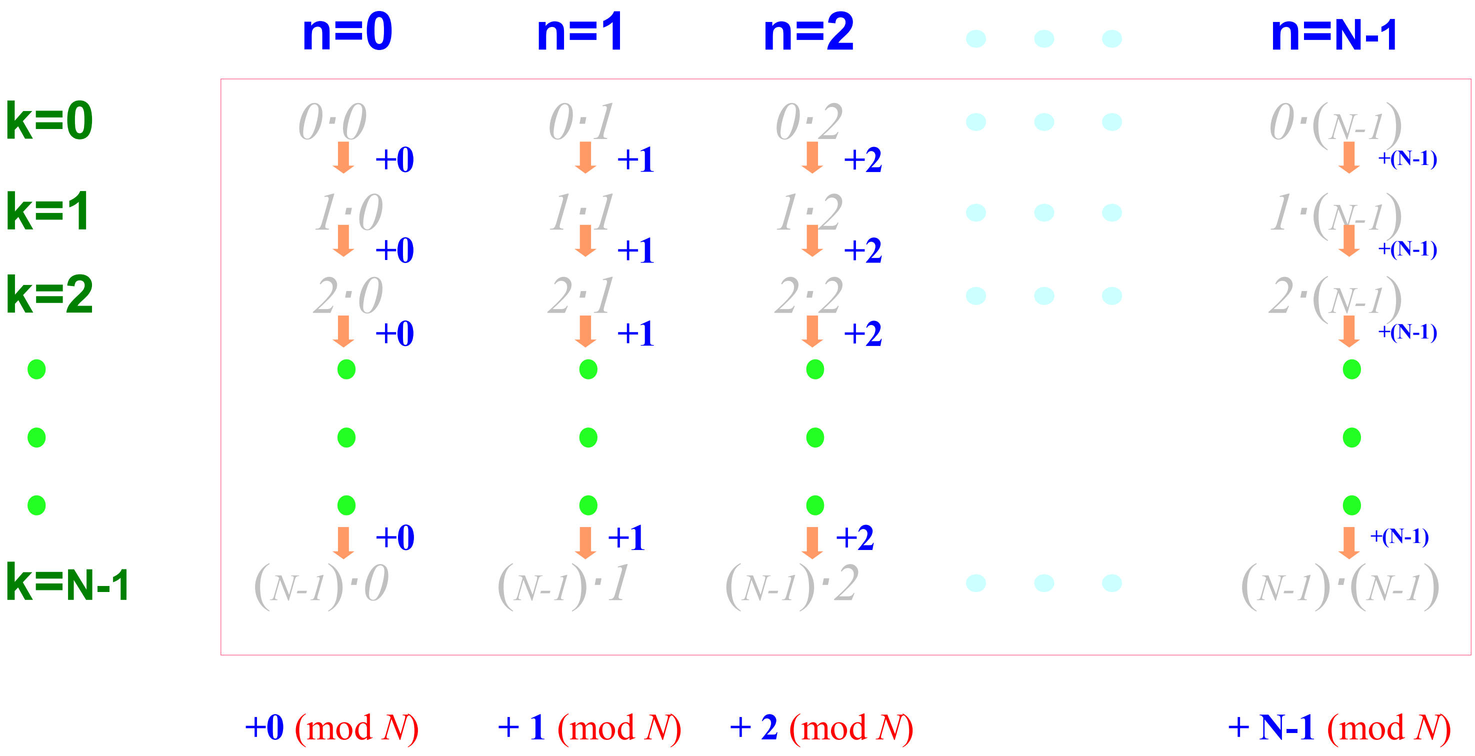
$$\boldsymbol{A} = \boldsymbol{A}^T$$

$$\boldsymbol{B} = \boldsymbol{B}^T$$

Exponents in
DFT matrix **A** and
IDFT matrix **B**

Symmetric DFT Matrix – Index (3)

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \qquad X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$



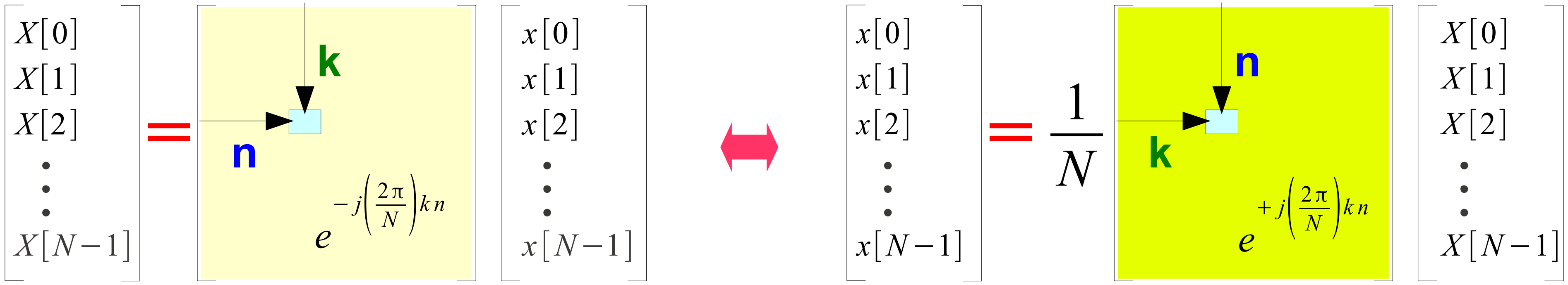
$$A = A^T$$

$$B = B^T$$

Exponents in
DFT matrix **A** and
IDFT matrix **B**

Conjugate Transpose DFT Matrix

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$



$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} e^{-j\left(\frac{2\pi}{N}\right)kn} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} \quad \Longleftrightarrow \quad \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} e^{+j\left(\frac{2\pi}{N}\right)kn} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\begin{cases} \boldsymbol{A} = \boldsymbol{A}^T \\ \boldsymbol{B} = \boldsymbol{B}^T \end{cases} \quad \begin{cases} \boldsymbol{A}^* = \boldsymbol{B} \\ \boldsymbol{B}^* = \boldsymbol{A} \end{cases} \quad \Rightarrow \quad \begin{cases} \boldsymbol{A}^H = \boldsymbol{B} \\ \boldsymbol{B}^H = \boldsymbol{A} \end{cases}$$

Normal, Unitary, Orthogonal Matrix

Normal Matrix

$$Q^H Q = Q\,Q^H$$

Unitary Matrix

$$Q^H Q = Q\,Q^H = I$$

$$\Leftrightarrow \quad Q^H = Q^{-1}$$

Orthogonal Matrix

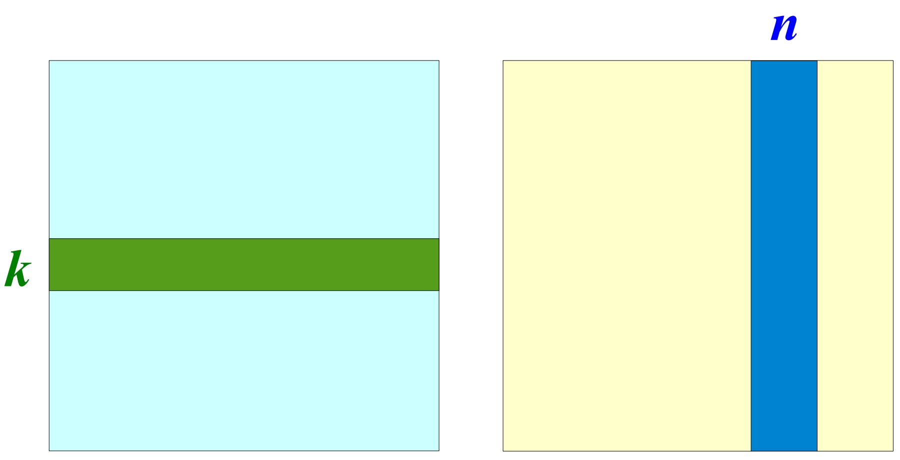
$$Q^T Q = Q\,Q^T = I$$

$$\Leftrightarrow \quad Q^T = Q^{-1}$$

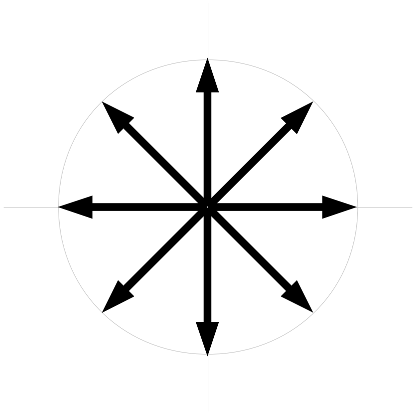
$$\begin{cases} A^H = B \\ B^H = A \end{cases} \qquad \begin{cases} AB = N\,I \\ BA = N\,I \end{cases} \quad \Rightarrow \quad \begin{cases} A^H A = A\,A^H = N\,I \\ B^H B = B\,B^H = N\,I \end{cases}$$

Product of DFT & IDFT Matrix

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \qquad X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$



$$\begin{Bmatrix} e^{+j\left(\frac{2\pi}{N}\right)0\cdot n} \\ e^{+j\left(\frac{2\pi}{N}\right)1\cdot n} \\ \vdots \\ e^{+j\left(\frac{2\pi}{N}\right)(N-1)\cdot n} \end{Bmatrix}$$



$$\underline{\left\{e^{-j\left(\frac{2\pi}{N}\right)k\cdot 0},\quad e^{-j\left(\frac{2\pi}{N}\right)k\cdot 1},\quad \cdots\quad ,\quad e^{-j\left(\frac{2\pi}{N}\right)k\cdot(N-1)}\right\}}$$

Inner product

$$e^{-j\left(\frac{2\pi}{N}\right)(k-n)\cdot 0} + e^{-j\left(\frac{2\pi}{N}\right)(k-n)\cdot 1} + \cdots + e^{-j\left(\frac{2\pi}{N}\right)(k-n)\cdot(N-1)} = \begin{cases} 0 & (n \neq k) \\ N & (n = k) \end{cases}$$

N=8 DFT & IDFT Matrix (1)

	0	1	2	3	4	5	6	7	
Row 0	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	**DFT**
	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	**IDFT**
Row 1	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 1}$	$e^{-j\cdot\frac{\pi}{4}\cdot 2}$	$e^{-j\cdot\frac{\pi}{4}\cdot 3}$	$e^{-j\cdot\frac{\pi}{4}\cdot 4}$	$e^{-j\cdot\frac{\pi}{4}\cdot 5}$	$e^{-j\cdot\frac{\pi}{4}\cdot 6}$	$e^{-j\cdot\frac{\pi}{4}\cdot 7}$	**DFT**
	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 1}$	$e^{+j\cdot\frac{\pi}{4}\cdot 2}$	$e^{+j\cdot\frac{\pi}{4}\cdot 3}$	$e^{+j\cdot\frac{\pi}{4}\cdot 4}$	$e^{+j\cdot\frac{\pi}{4}\cdot 5}$	$e^{+j\cdot\frac{\pi}{4}\cdot 6}$	$e^{+j\cdot\frac{\pi}{4}\cdot 7}$	**IDFT**
Row 2	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 2}$	$e^{-j\cdot\frac{\pi}{4}\cdot 4}$	$e^{-j\cdot\frac{\pi}{4}\cdot 6}$	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 2}$	$e^{-j\cdot\frac{\pi}{4}\cdot 4}$	$e^{-j\cdot\frac{\pi}{4}\cdot 6}$	**DFT**
	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 2}$	$e^{+j\cdot\frac{\pi}{4}\cdot 4}$	$e^{+j\cdot\frac{\pi}{4}\cdot 6}$	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 2}$	$e^{+j\cdot\frac{\pi}{4}\cdot 4}$	$e^{+j\cdot\frac{\pi}{4}\cdot 6}$	**IDFT**
Row 3	$e^{-j\cdot\frac{\pi}{4}\cdot 0}$	$e^{-j\cdot\frac{\pi}{4}\cdot 3}$	$e^{-j\cdot\frac{\pi}{4}\cdot 6}$	$e^{-j\cdot\frac{\pi}{4}\cdot 1}$	$e^{-j\cdot\frac{\pi}{4}\cdot 4}$	$e^{-j\cdot\frac{\pi}{4}\cdot 7}$	$e^{-j\cdot\frac{\pi}{4}\cdot 2}$	$e^{-j\cdot\frac{\pi}{4}\cdot 5}$	**DFT**
	$e^{+j\cdot\frac{\pi}{4}\cdot 0}$	$e^{+j\cdot\frac{\pi}{4}\cdot 3}$	$e^{+j\cdot\frac{\pi}{4}\cdot 6}$	$e^{+j\cdot\frac{\pi}{4}\cdot 1}$	$e^{+j\cdot\frac{\pi}{4}\cdot 4}$	$e^{+j\cdot\frac{\pi}{4}\cdot 7}$	$e^{+j\cdot\frac{\pi}{4}\cdot 2}$	$e^{+j\cdot\frac{\pi}{4}\cdot 5}$	**IDFT**

N=8 DFT & IDFT Matrix (2)

$$
\begin{array}{c|cccccccc|c}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
\hline
\textbf{Row 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & \text{DFT} \\
 & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & \text{IDFT} \\
\textbf{Row 5} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & \text{DFT} \\
 & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 5} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 7} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 1} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 3} & \text{IDFT} \\
\textbf{Row 6} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & \text{DFT} \\
 & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & \text{IDFT} \\
\textbf{Row 7} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & \text{DFT} \\
 & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 7} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 5} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 3} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 1} & \text{IDFT}
\end{array}
$$

Product AB – Diagonal Elements

$C = A \cdot B$

$$[\boldsymbol{C}]_{(i,j)} = [\boldsymbol{A}]_{(row\ \ i)} \cdot [\boldsymbol{B}]_{(col\ \ j)}$$

$C_{(i,\, i)}$ = N

$$C_{(1,\,1)} \quad \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix} \bullet \begin{pmatrix} e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 1} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 3} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 5} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix}^T$$

+1 +1 +1 +1 +1 +1 +1 +1 = N

Product AB – Off-Diagonal Elements

$$C = A \cdot B \qquad [C]_{(i,j)} = [A]_{(row\ \ i)} \cdot [B]_{(col\ \ j)}$$

$$C_{(i,j)} = 0$$

$$C_{(1,2)} \quad \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix} \bullet$$

$$\begin{pmatrix} e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 6} & e^{+j\cdot\frac{\pi}{4}\cdot 0} & e^{+j\cdot\frac{\pi}{4}\cdot 2} & e^{+j\cdot\frac{\pi}{4}\cdot 4} & e^{+j\cdot\frac{\pi}{4}\cdot 6} \end{pmatrix}^{T}$$

$$e^{+j\cdot\frac{\pi}{4}\cdot 0} + e^{+j\cdot\frac{\pi}{4}\cdot 1} + e^{+j\cdot\frac{\pi}{4}\cdot 2} + e^{+j\cdot\frac{\pi}{4}\cdot 3} + e^{+j\cdot\frac{\pi}{4}\cdot 4} + e^{+j\cdot\frac{\pi}{4}\cdot 5} + e^{+j\cdot\frac{\pi}{4}\cdot 6} + e^{+j\cdot\frac{\pi}{4}\cdot 7} = 0$$

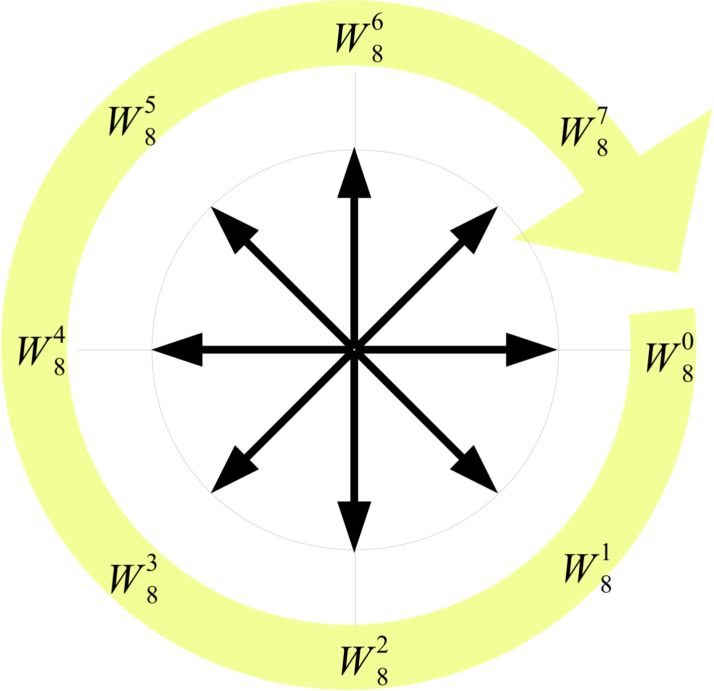
Root of Unity

$$\boxed{\sum_{k=0}^{N-1} W_N^k = \sum_{k=0}^{N-1} e^{-j\left(\frac{2\pi}{N}\right)k} = 0}$$

$$z \equiv e^{-j\left(\frac{2\pi}{N}\right)}$$

$$z^N = e^{-j\left(\frac{2\pi}{N}\right)N} = 1$$

$$\sum_{k=0}^{N-1} e^{-j\left(\frac{2\pi}{N}\right)k} = \frac{z^N - 1}{z - 1} = 0$$



$$W_8^0 + W_8^1 + W_8^2 + W_8^3 + W_8^4 + W_8^5 + W_8^6 + W_8^7 = 0$$

Complex Vector Inner Product

Hermitian inner product

$$\langle \boldsymbol{x} , \boldsymbol{y} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} = \sum x_i^* \, y_i$$

$\boldsymbol{x}^H$: conjugate transpose

Norm of Hermitian inner products

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x} , \boldsymbol{x} \rangle} = \sqrt{\boldsymbol{x}^H \cdot \boldsymbol{x}} = \sqrt{\sum x_i^* \, x_i}$$

the length of a vector

$$\boldsymbol{x} = \begin{pmatrix} a_1 + j\,b_1 \\ a_2 + j\,b_2 \\ \vdots \\ a_n + j\,b_n \end{pmatrix}$$

$$\langle \boldsymbol{x} , \boldsymbol{x} \rangle = \boldsymbol{x}^H \cdot \boldsymbol{x} = \sum x_i^* \, x_i$$

$$\begin{pmatrix} a_1 - j\,b_1 & a_2 - j\,b_2 & \cdots & a_n - j\,b_n \end{pmatrix} \begin{pmatrix} a_1 + j\,b_1 \\ a_2 + j\,b_2 \\ \vdots \\ a_n + j\,b_n \end{pmatrix} = \sum_{i=1}^{n} a_i^2 + b_i^2$$

Cauchy-Schwartz Inequality

For all vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of an inner product space

$$\left|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right|^2 \le \langle \boldsymbol{x}, \boldsymbol{x}\rangle \cdot \langle \boldsymbol{y}, \boldsymbol{y}\rangle$$

$$\left|\langle \boldsymbol{x}, \boldsymbol{y}\rangle\right| \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|$$

The equality holds if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent ⇨ maximum

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^H \cdot \boldsymbol{y} \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\| \qquad \boldsymbol{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} \qquad \boldsymbol{y} = k \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

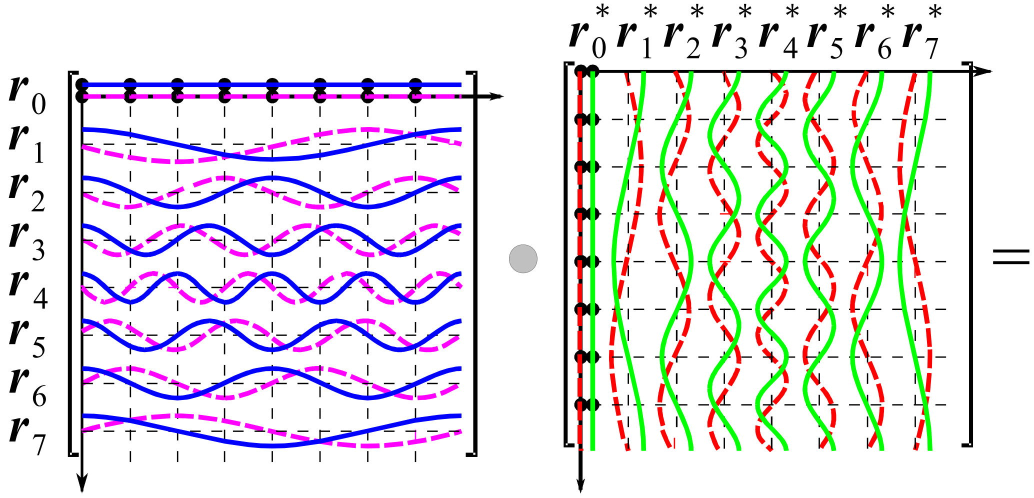
Inner product is maximum
when $\boldsymbol{y} = k\,\boldsymbol{x}$

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle \le k\left(\sum_{i=1}^{n} a_i^2 + b_i^2\right)$$

Orthogonality

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad\qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$$\begin{cases} \boldsymbol{A}^H = \boldsymbol{B} \\ \boldsymbol{B}^H = \boldsymbol{A} \end{cases} \qquad \begin{cases} \boldsymbol{AB} = N\,\boldsymbol{I} \\ \boldsymbol{BA} = N\,\boldsymbol{I} \end{cases} \qquad \Rightarrow \qquad \begin{cases} \boldsymbol{A}^H\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{A}^H = N\,\boldsymbol{I} \\ \boldsymbol{B}^H\boldsymbol{B} = \boldsymbol{B}\,\boldsymbol{B}^H = N\,\boldsymbol{I} \end{cases}$$

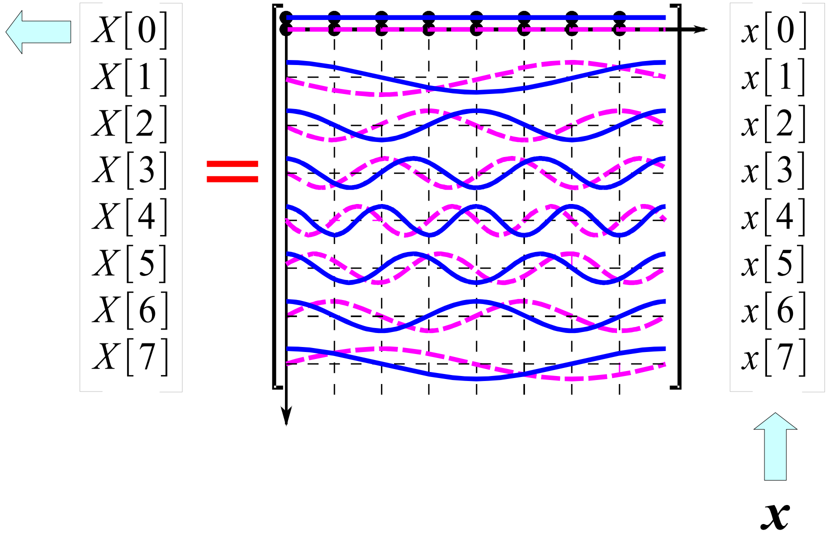


$$\begin{bmatrix}
r_0\cdot r_0^* & r_0\cdot r_1^* & r_0\cdot r_2^* & r_0\cdot r_3^* & r_0\cdot r_4^* & r_0\cdot r_5^* & r_0\cdot r_6^* & r_0\cdot r_7^* \\
r_1\cdot r_0^* & r_1\cdot r_1^* & r_1\cdot r_2^* & r_1\cdot r_3^* & r_1\cdot r_4^* & r_1\cdot r_5^* & r_1\cdot r_6^* & r_1\cdot r_7^* \\
r_2\cdot r_0^* & r_2\cdot r_1^* & r_2\cdot r_2^* & r_2\cdot r_3^* & r_2\cdot r_4^* & r_2\cdot r_5^* & r_2\cdot r_6^* & r_2\cdot r_7^* \\
r_3\cdot r_0^* & r_3\cdot r_1^* & r_3\cdot r_2^* & r_3\cdot r_3^* & r_3\cdot r_4^* & r_3\cdot r_5^* & r_3\cdot r_6^* & r_3\cdot r_7^* \\
r_4\cdot r_0^* & r_4\cdot r_1^* & r_4\cdot r_2^* & r_4\cdot r_3^* & r_4\cdot r_4^* & r_4\cdot r_5^* & r_4\cdot r_6^* & r_4\cdot r_7^* \\
r_5\cdot r_0^* & r_5\cdot r_1^* & r_5\cdot r_2^* & r_5\cdot r_3^* & r_5\cdot r_4^* & r_5\cdot r_5^* & r_5\cdot r_6^* & r_5\cdot r_7^* \\
r_6\cdot r_0^* & r_6\cdot r_1^* & r_6\cdot r_2^* & r_6\cdot r_3^* & r_6\cdot r_4^* & r_6\cdot r_5^* & r_6\cdot r_6^* & r_6\cdot r_7^* \\
r_7\cdot r_0^* & r_7\cdot r_1^* & r_7\cdot r_2^* & r_7\cdot r_3^* & r_7\cdot r_4^* & r_7\cdot r_5^* & r_7\cdot r_6^* & r_7\cdot r_7^*
\end{bmatrix}$$

$$\langle \boldsymbol{r}_i^H , \boldsymbol{r}_i^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_i^* = N$$

$$\langle \boldsymbol{r}_i^H , \boldsymbol{r}_j^H \rangle = \boldsymbol{r}_i \cdot \boldsymbol{r}_j^* = 0 \;\; (i \neq j)$$

N=8 DFT : Inner Product X[0]

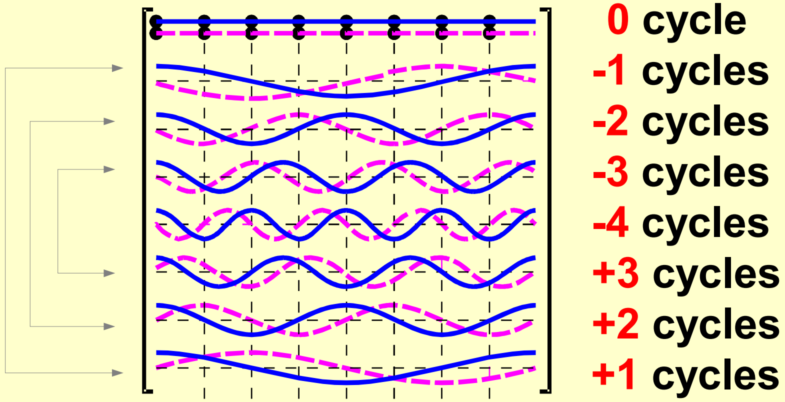


X[0] measures "0 cycle" component in x

$$\langle \boldsymbol{r}_0^H, \boldsymbol{x} \rangle = \boldsymbol{r}_0 \cdot \boldsymbol{x} \le \|\boldsymbol{r}_0^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_0^H$

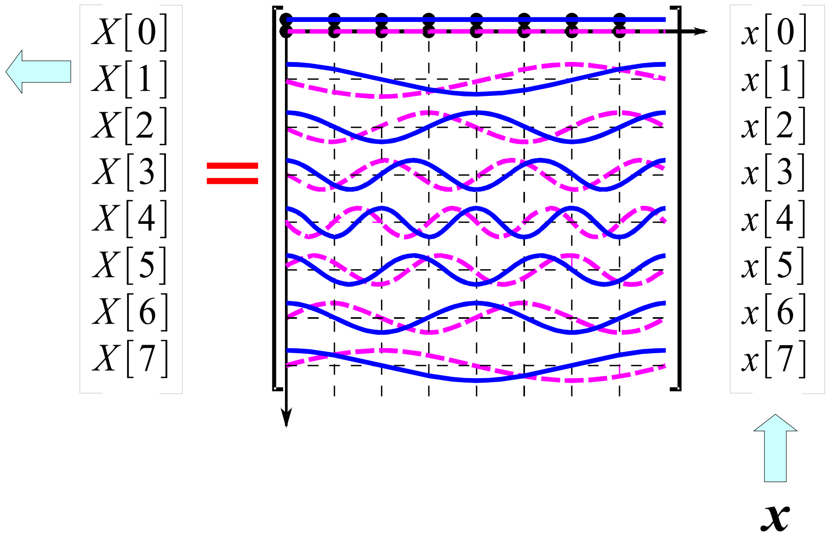
*When **x** looks like this, **X**[0] is max.*



$$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[1]

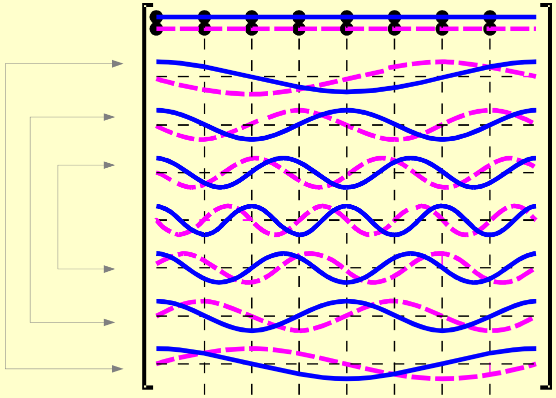


X[1] measures "+1 cycle" component in x

$$\langle \boldsymbol{r}_1^H, \boldsymbol{x} \rangle = \boldsymbol{r}_1 \cdot \boldsymbol{x} \ \leq \ \|\boldsymbol{r}_1^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_1^H$

*When **x** looks like this, **X**[1] is max.*

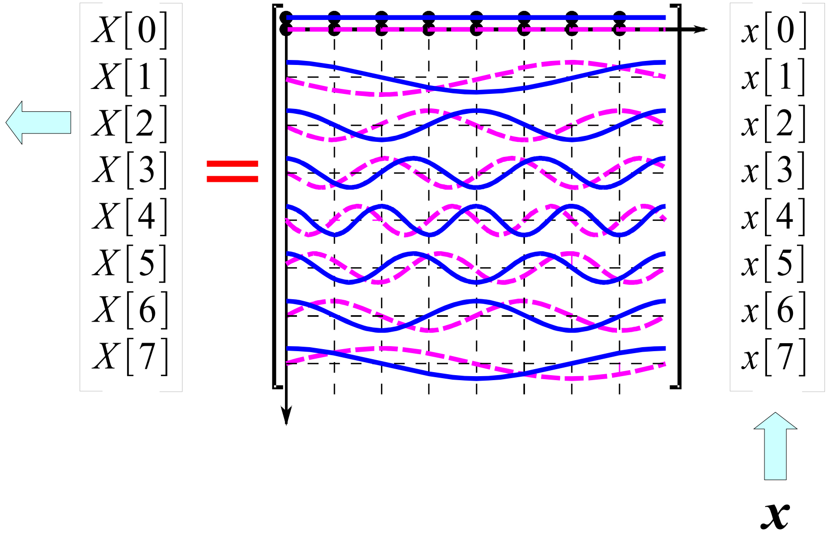


0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

— $Re\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

- - - $Im\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[2]

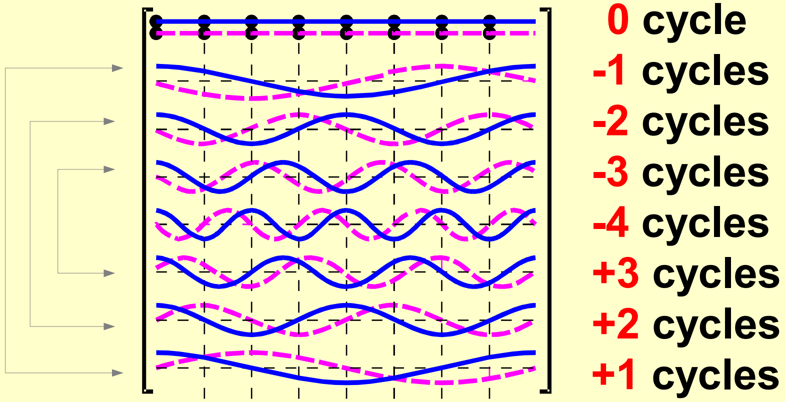


X[2] measures "+2 cycle" component in x

$$\langle \boldsymbol{r}_2^H , \boldsymbol{x} \rangle = \boldsymbol{r}_2 \cdot \boldsymbol{x} \le \| \boldsymbol{r}_2^H \| \cdot \| \boldsymbol{x} \|$$

maximum when $\boldsymbol{x} = k \, \boldsymbol{r}_2^H$

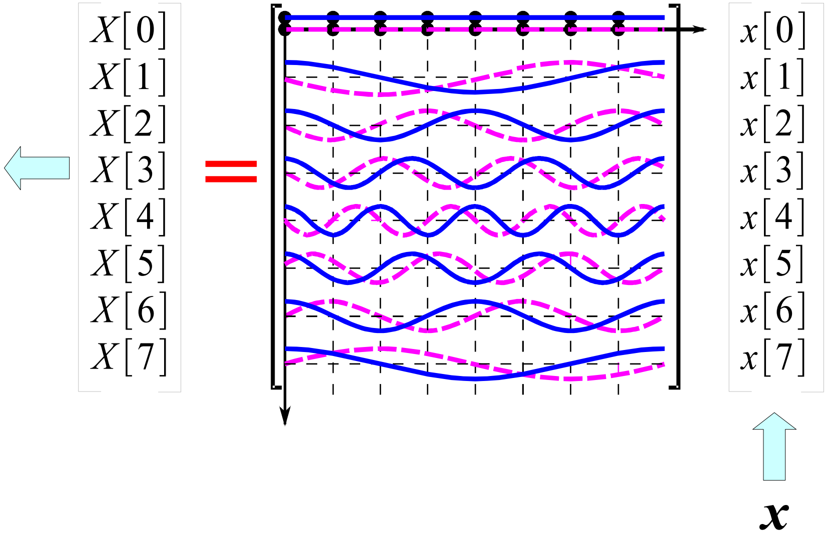
When ***x*** *looks like this,* ***X***[2] *is max.*



—— $$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn \right)$$

- - - - $$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn \right)$$

N=8 DFT : Inner Product X[3]

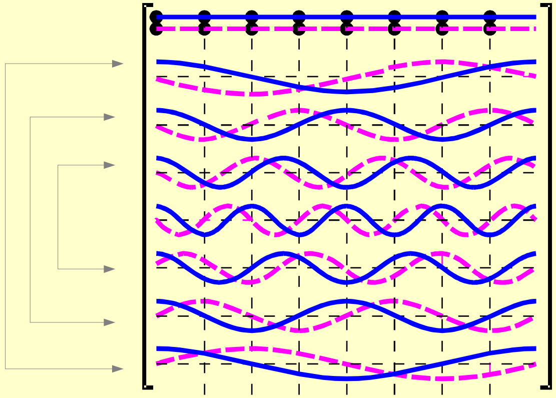


X[3] measures "**+3** cycle" component in **x**

$$\langle \boldsymbol{r}_3^H, \boldsymbol{x} \rangle = \boldsymbol{r}_3 \cdot \boldsymbol{x} \le \|\boldsymbol{r}_3^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_3^H$

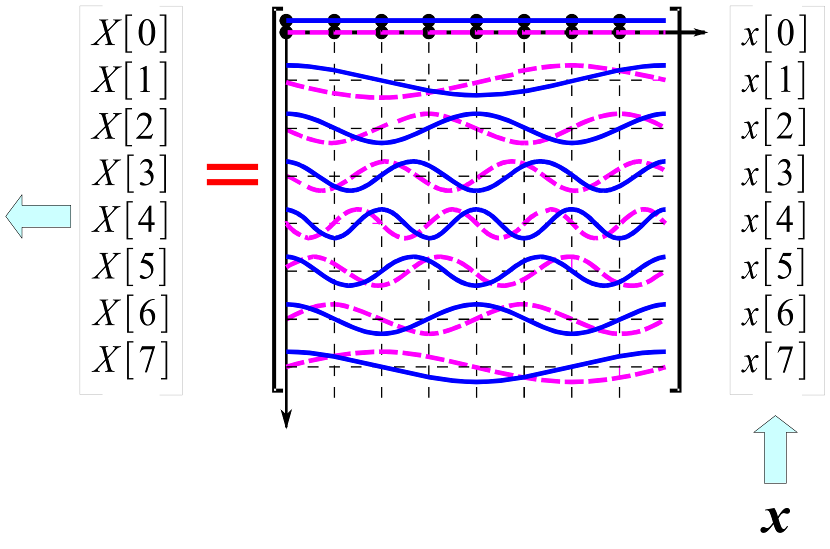
*When **x** looks like this, **X**[3] is max.*



—— $Re\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

- - - $Im\left\{e^{-j\frac{2\pi}{8}kn}\right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[4]

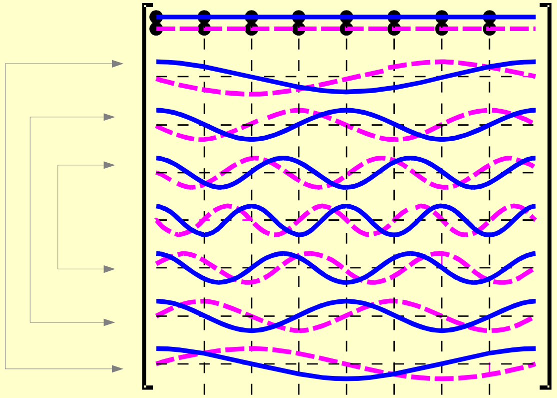


X[4] measures "+4 cycle" component in x

$$\langle \boldsymbol{r}_4^H , \boldsymbol{x} \rangle = \boldsymbol{r}_4 \cdot \boldsymbol{x} \le \|\boldsymbol{r}_4^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_4^H$

*When **x** looks like this, **X**[4] is max.*

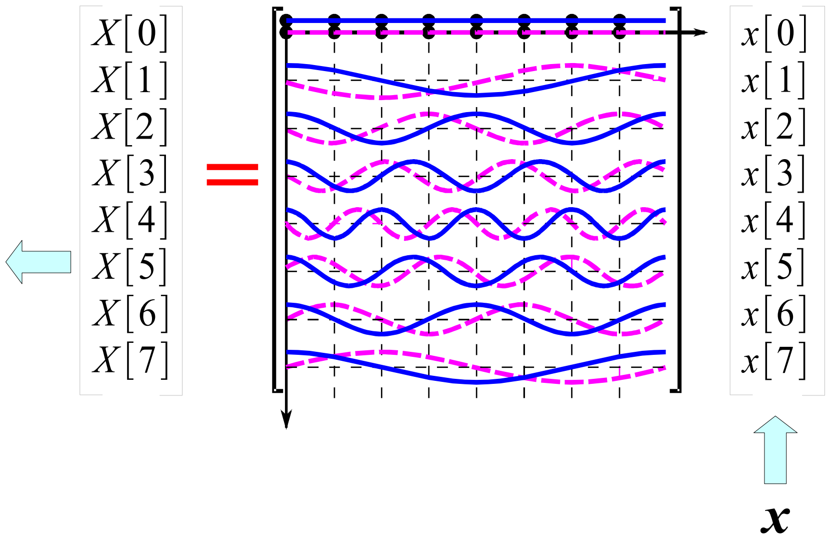


0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

——— $Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

- - - - $Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[5]

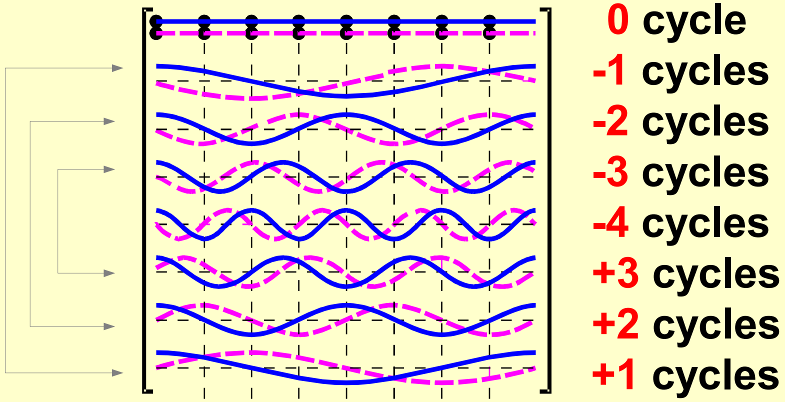


X[5] measures "-3 cycle" component in x

$$\langle \boldsymbol{r}_5^H , \boldsymbol{x} \rangle = \boldsymbol{r}_5 \cdot \boldsymbol{x} \;\leq\; \|\boldsymbol{r}_5^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_5^H$

*When **x** looks like this, **X**[5] is max.*

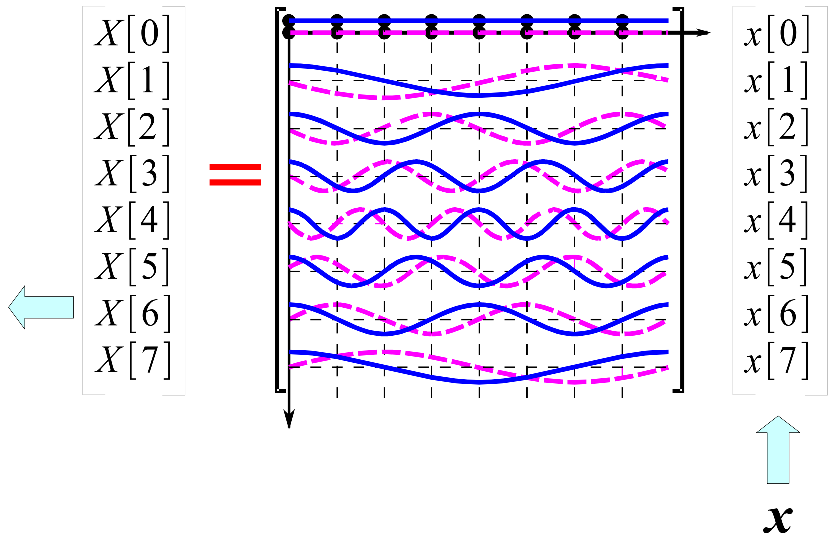


0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

— $Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

┄ $Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[6]

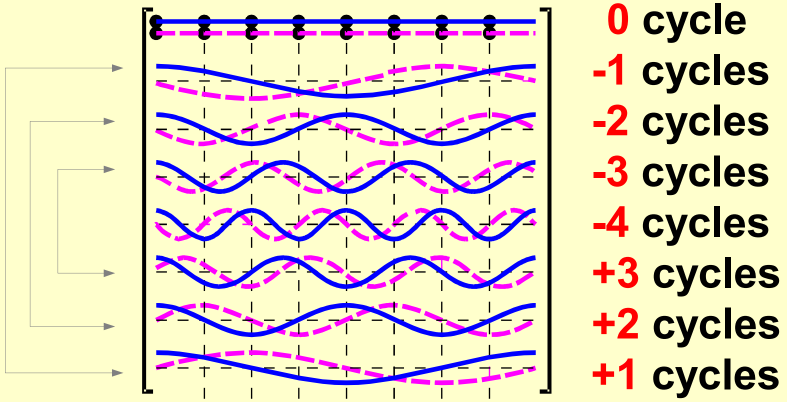


X[6] **measures "-2 cycle" component in x**

$$\langle \boldsymbol{r}_6^H , \boldsymbol{x} \rangle = \boldsymbol{r}_6 \cdot \boldsymbol{x} \ \le \ \|\boldsymbol{r}_6^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_6^H$

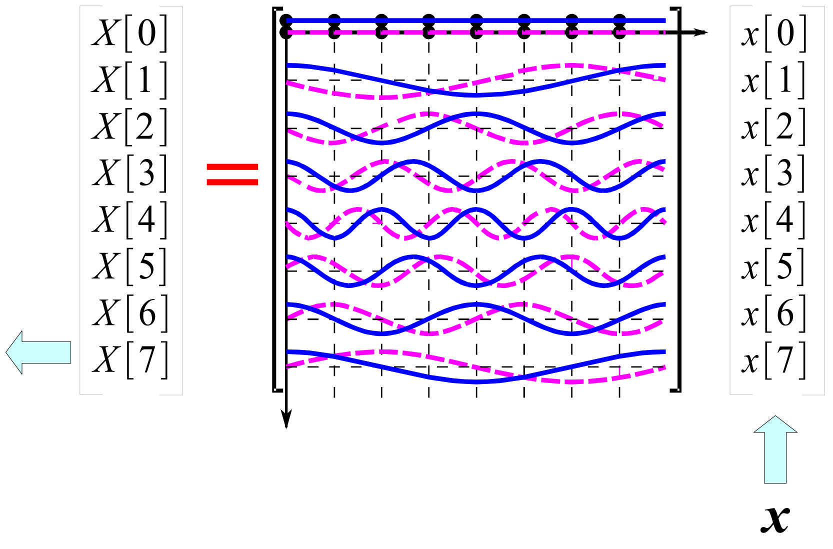
When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}[6]$ *is max.*



—— $$Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

- - - - $$Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : Inner Product X[7]

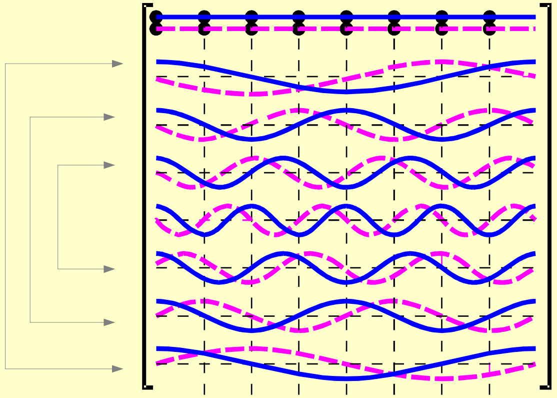


X[7] measures "-1 cycle" component in x

$$\langle \boldsymbol{r}_7^H, \boldsymbol{x} \rangle = \boldsymbol{r}_7 \cdot \boldsymbol{x} \ \le\ \|\boldsymbol{r}_7^H\| \cdot \|\boldsymbol{x}\|$$

maximum when $\boldsymbol{x} = k\,\boldsymbol{r}_7^H$

When $\boldsymbol{x}$ *looks like this,* $\boldsymbol{X}[7]$ *is max.*



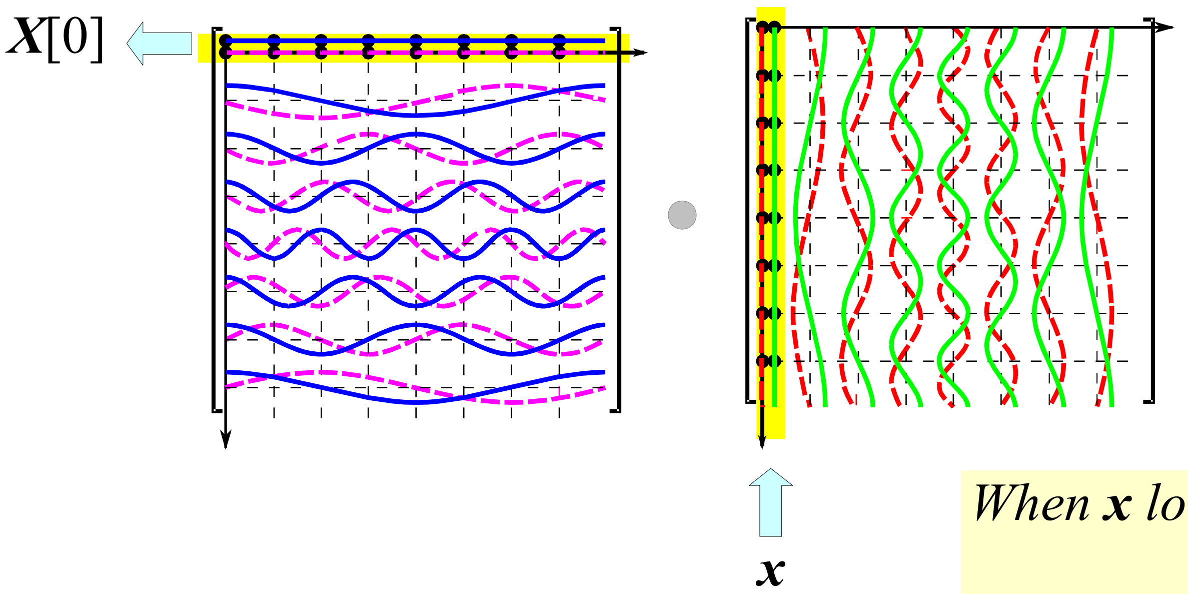
0 cycle
-1 cycles
-2 cycles
-3 cycles
-4 cycles
+3 cycles
+2 cycles
+1 cycles

—— $Re\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

- - - - $Im\left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : X[0] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$A \cdot B = A \cdot A^H = N\, I$$

$$U \cdot U^H = I$$

Unitary Matrix

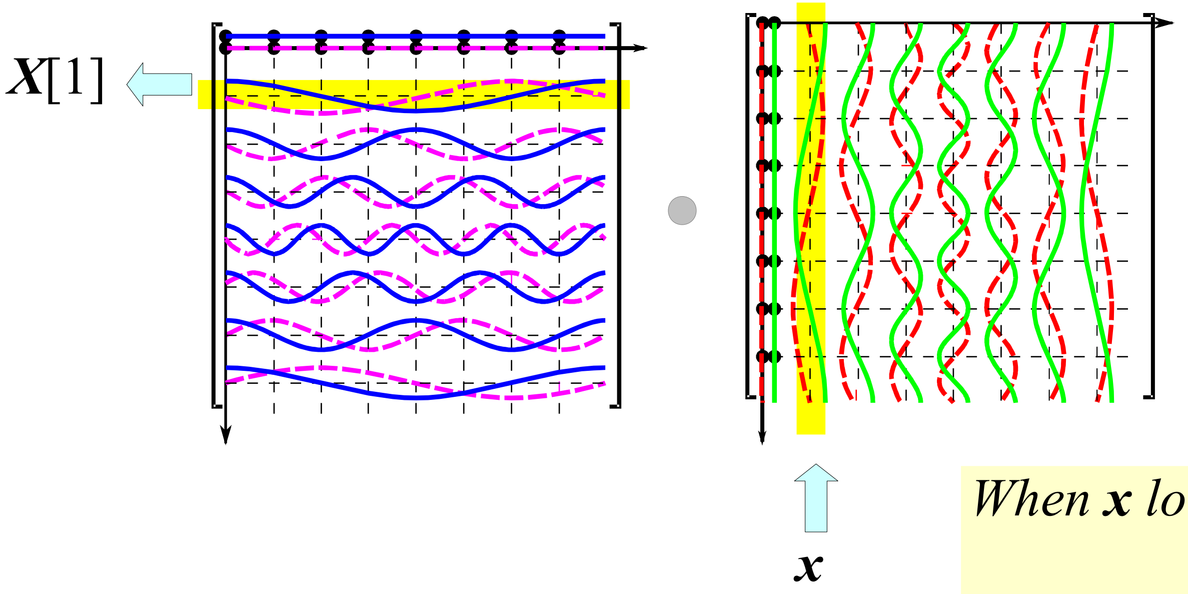
When x looks like this, $X[0]$ is max. (=N)

$X[k] = 0$ for $k \neq 0$

$$X[0] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 0} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[1] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad\qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U} \cdot \boldsymbol{U}^H = \boldsymbol{I}$$

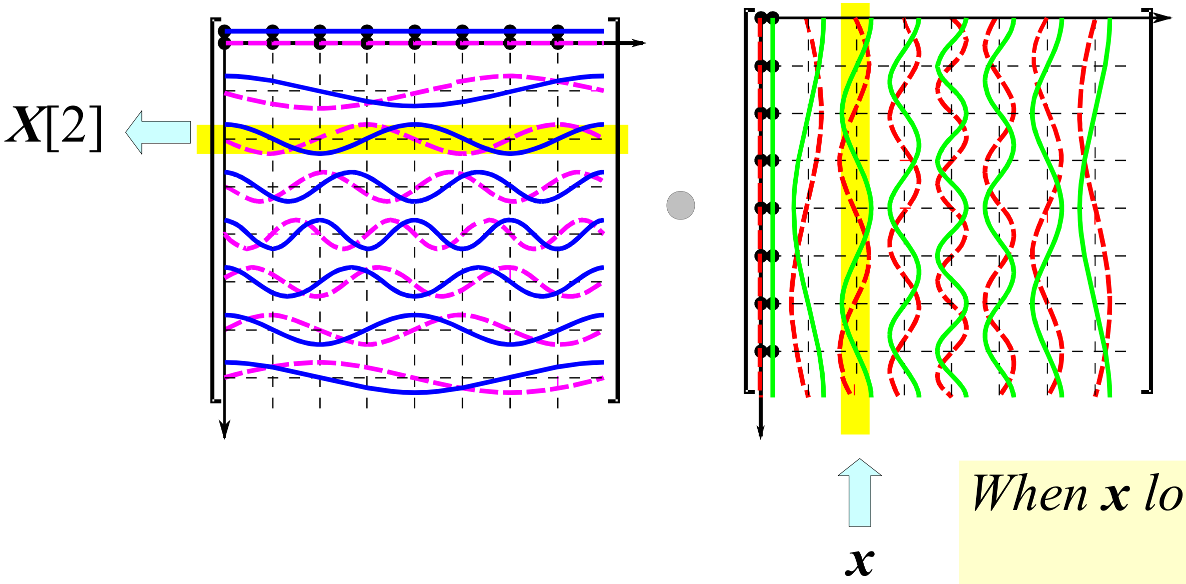
Unitary Matrix

*When **x** looks like this, **X**[1] is max. (=**N**)*
$\boldsymbol{X}[k] = 0$ for $k \neq 1$

$$\boldsymbol{X}[1] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 7} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[2] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad\qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U} \cdot \boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

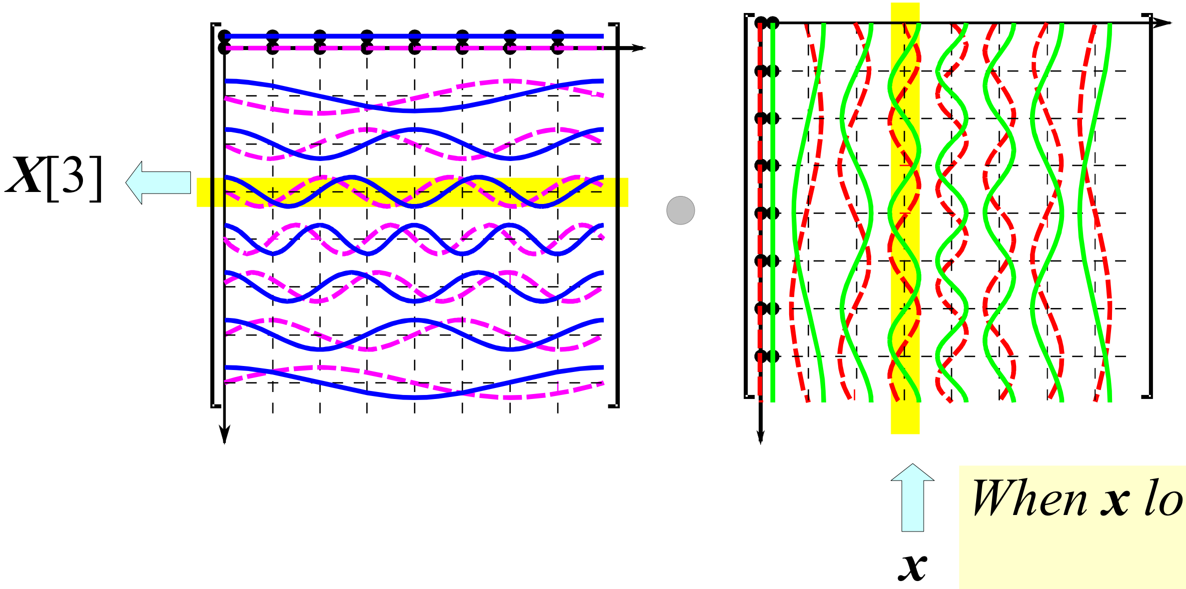
*When **x** looks like this, **X**[2] is max. (=**N**)*

$\boldsymbol{X}[k] = 0$ for $k \neq 2$

$$\boldsymbol{X}[2] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 6} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[3] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

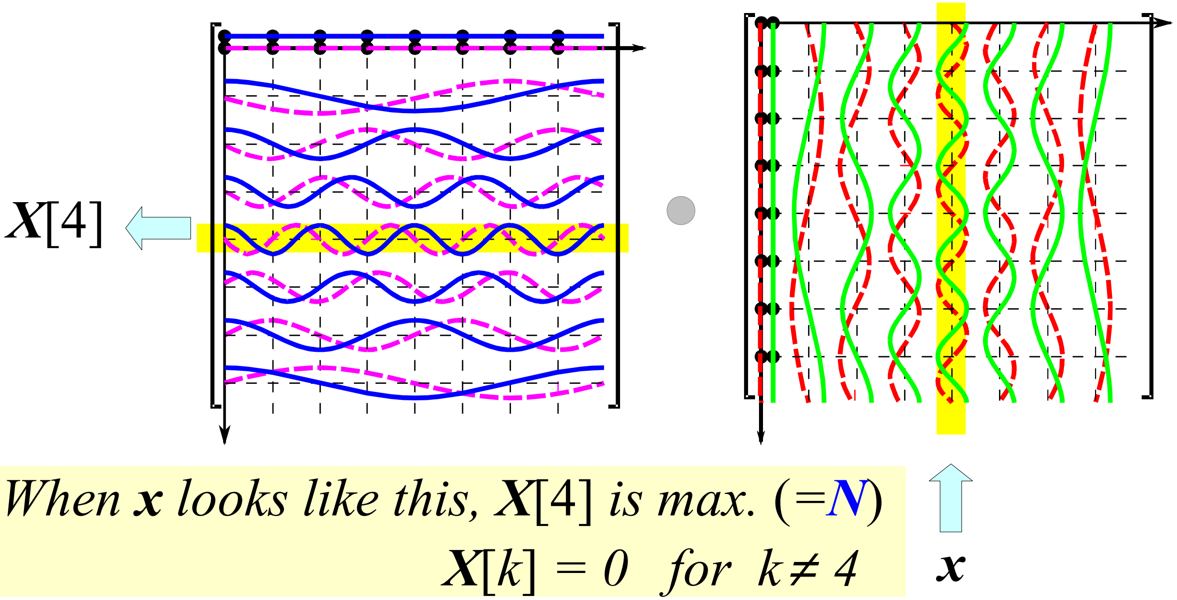
*When **x** looks like this, **X**[3] is max. (=**N**)*

$\boldsymbol{X}[k] = 0$ for $k \neq 3$

$$\color{red}{X[3]} = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 5} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[4] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

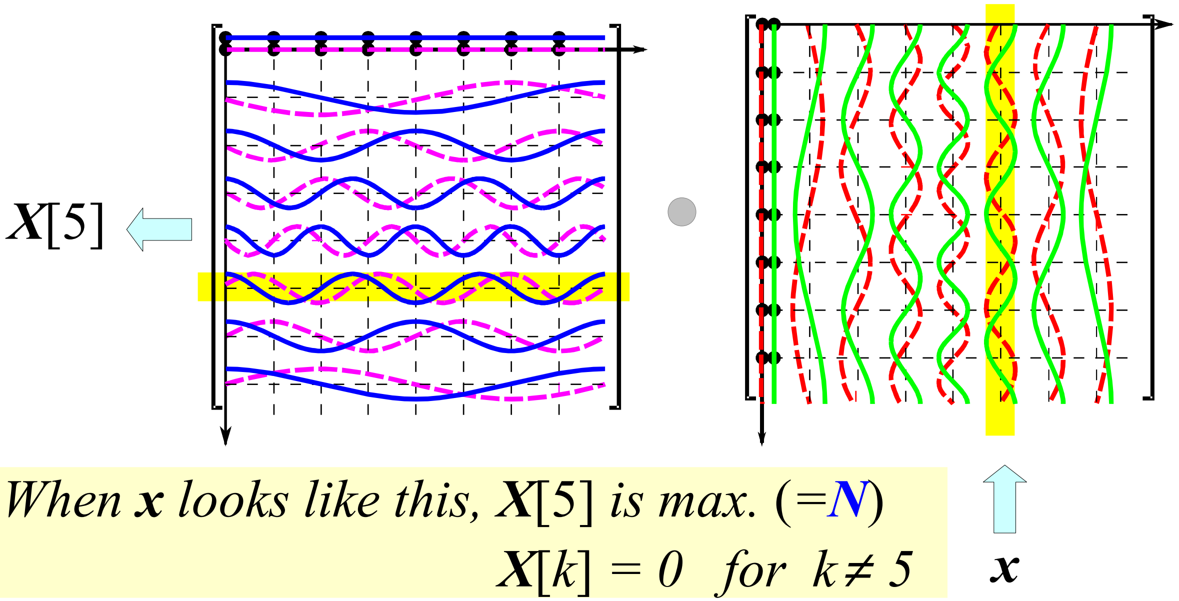
*When **x** looks like this, **X**[4] is max. (=**N**)*

$\boldsymbol{X}[k] = 0$ for $k \neq 4$

$$\boldsymbol{X}[4] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 4} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[5] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad\qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$A\cdot B = A\cdot A^H = N\,I$$

$$U\cdot U^H = I$$

Unitary Matrix

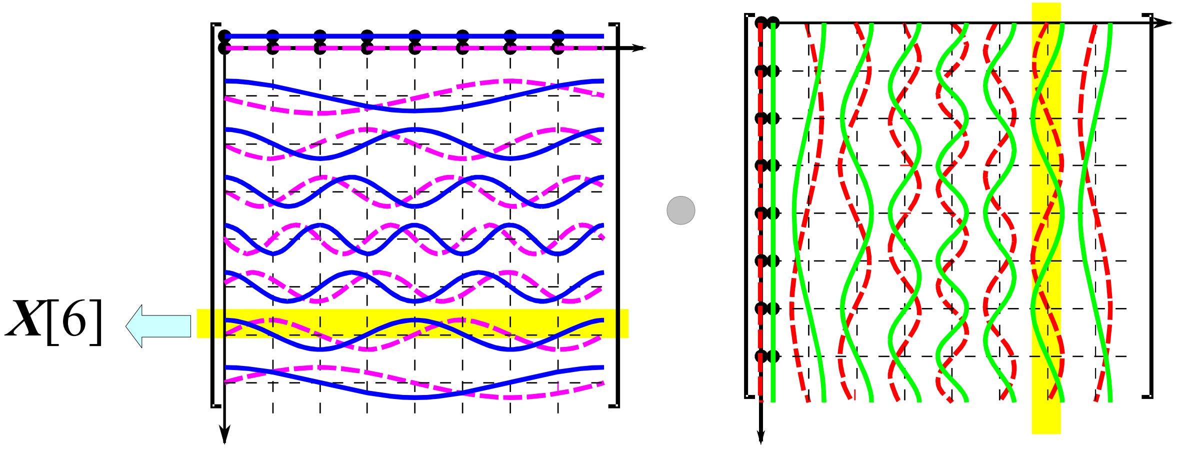
*When **x** looks like this, **X**[5] is max. (=**N**)*

$\mathbf{X}[k] = 0$ for $k \neq 5$

$$X[5] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 1} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 3} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[6] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad\qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{A}^H = N\,\boldsymbol{I}$$

$$\boldsymbol{U}\cdot\boldsymbol{U}^H = \boldsymbol{I}$$

Unitary Matrix

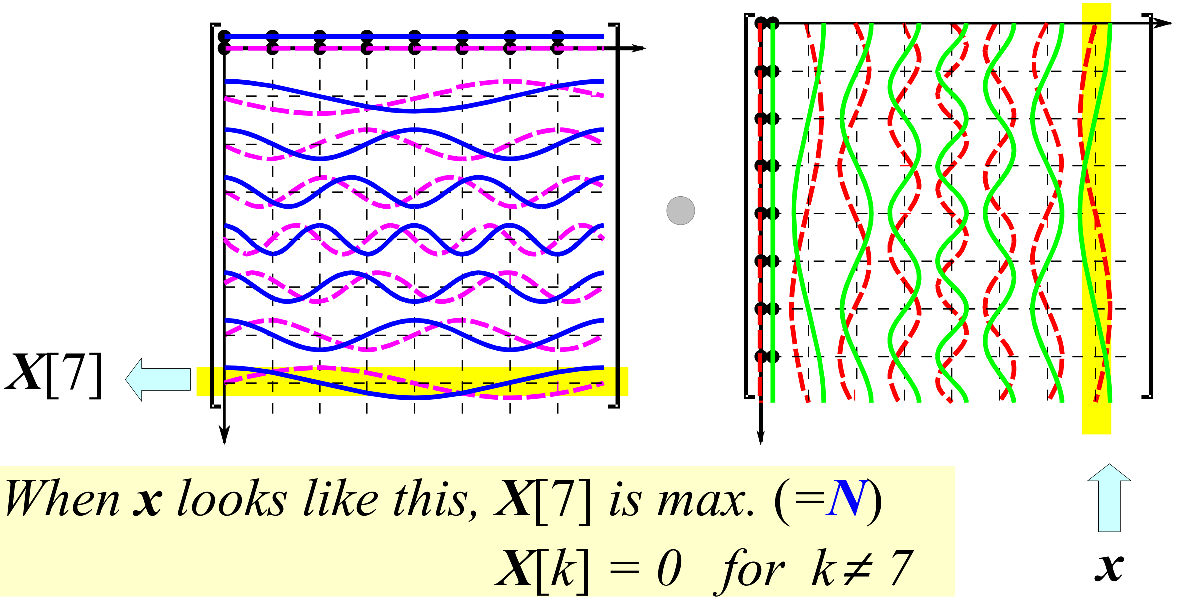
*When **x** looks like this, **X**[6] is max. (=**N**)*
$\boldsymbol{X}[k] = 0$ for $k \neq 6$

$\boldsymbol{x}$

$$X[6] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 2} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[7] in A·B

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$



$$A \cdot B = A \cdot A^H = N\, I$$

$$U \cdot U^H = I$$

Unitary Matrix

When ***x*** *looks like this,* ***X***[7] *is max. (=**N**)*

X[*k*] *= 0 for k≠ 7*

$$X[7] = \begin{pmatrix} e^{-j\cdot\frac{\pi}{4}\cdot 0} & e^{-j\cdot\frac{\pi}{4}\cdot 7} & e^{-j\cdot\frac{\pi}{4}\cdot 6} & e^{-j\cdot\frac{\pi}{4}\cdot 5} & e^{-j\cdot\frac{\pi}{4}\cdot 4} & e^{-j\cdot\frac{\pi}{4}\cdot 3} & e^{-j\cdot\frac{\pi}{4}\cdot 2} & e^{-j\cdot\frac{\pi}{4}\cdot 1} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann