Differentiation Rules (2A)

Young Won Lim
2/22/16

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Derivative Product and Quotient Rule

Product and Quotient Rule

$$fg \;\longrightarrow\; \boxed{\frac{d}{dx}} \;\longrightarrow\; f'g + fg'$$

$$\frac{d}{dx}(f\,g) = \left(\frac{df}{dx}g + f\frac{dg}{dx}\right)$$

$f(x), \quad g(x)$

$$\frac{f}{g} \;\longrightarrow\; \boxed{\frac{d}{dx}} \;\longrightarrow\; \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \left(\frac{df}{dx}g - f\frac{dg}{dx}\right)\left(\frac{1}{g^2}\right)$$

$f(x), \quad g(x)$

Product Rule

$$f\,g \;\rightarrow\; \boxed{\frac{d}{dx}} \;\rightarrow\; f'g + f\,g'$$

$$f(x),\quad g(x)$$

$$f(x)g(x) \;\rightarrow\; \boxed{\frac{d}{dx}} \;\rightarrow\; f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}(f\cdot g) = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx}$$

$$d(f\cdot g) = df\cdot g + f\cdot dg$$

$$\frac{d}{dx}(f\cdot g)\,dx = \left(\frac{df}{dx}\cdot g\right)dx + \left(f\cdot\frac{dg}{dx}\right)dx$$

$$= \left(\frac{df}{dx}dx\right)\cdot g + f\cdot\left(\frac{dg}{dx}dx\right)$$

$$= df\cdot g + f\cdot dg$$

Quotient Rule

$$f(x)g(x) \;\rightarrow\; \boxed{\frac{d}{dx}} \;\rightarrow\; f'(x)g(x) + f(x)g'(x)$$

$$f(x)\left\{\frac{1}{g(x)}\right\} \;\rightarrow\; \boxed{\frac{d}{dx}} \;\rightarrow\; f'(x)\left\{\frac{1}{g(x)}\right\} + f(x)\left\{\frac{1}{g(x)}\right\}'$$

$$= f'(x)\left\{\frac{1}{g(x)}\right\} + f(x)\left\{\frac{-g'(x)}{g^2(x)}\right\}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$\frac{d}{dx}g^{-1}(x) = -g^{-2}(x)\frac{d}{dx}g(x)$$

Young Won Lim

Example 1

$$f\,g \;\rightarrow\; \boxed{\frac{d}{dx}} \;\rightarrow\; f'g \;+\; f\,g'$$

$$x^2 \cdot x^3 \;\rightarrow\; \boxed{\frac{d}{dx}} \;\rightarrow\; (2x)\cdot x^3 \;+\; x^2(3x^2)$$

$$\frac{d}{dx}x^5 = 5x^4$$

$$e^{2x} \cdot e^{3x} \;\rightarrow\; \boxed{\frac{d}{dx}} \;\rightarrow\; (2e^{2x})\cdot e^{3x} \;+\; e^{2x}(3e^{3x})$$

$$\frac{d}{dx}e^{5x} = 5e^{5x}$$

Example 2

$$x^2 \cdot \cos(x) \Rightarrow \boxed{\frac{d}{dx}} \Rightarrow (2x) \cdot \cos(x) - x^2(-\sin(x))$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$x^2 \cdot \cos(x) = x^2\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)$$

$$\frac{d}{dx} x^2 \cdot \cos(x) = 2x\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)$$
$$+ \; x^2\left(-x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \frac{1}{7!}x^7 - \cdots\right)$$

Example 3

$$(x+p)(x+q) \rightarrow \frac{d}{dx} \rightarrow (x+q)+(x+p)$$

$$x^2+(p+q)x+pq \qquad 2x+(p+q)$$

$$\frac{1}{(x+p)(x+q)} \rightarrow \frac{d}{dx} \rightarrow -\frac{(x+q)+(x+p)}{(x+p)^2(x+q)^2}$$

$$\frac{(x+r)}{(x+p)(x+q)} \rightarrow \frac{d}{dx} \rightarrow \frac{(x+p)(x+q)-(x+r)((x+q)+(x+p))}{(x+p)^2(x+q)^2}$$

$$(x+r)\frac{1}{(x+p)(x+q)} \qquad \frac{1}{(x+p)(x+q)}-(x+r)\frac{(x+q)+(x+p)}{(x+p)^2(x+q)^2}$$

$$\frac{d}{dx} x^{n+1} = \frac{d}{dx}\left(x^n \cdot x\right)$$

$$= \left(\frac{d}{dx} x^n\right) \cdot x + x^n \cdot \left(\frac{d}{dx} x\right)$$

$$= \left(n\, x^{n-1}\right) \cdot x + x^n \cdot 1$$

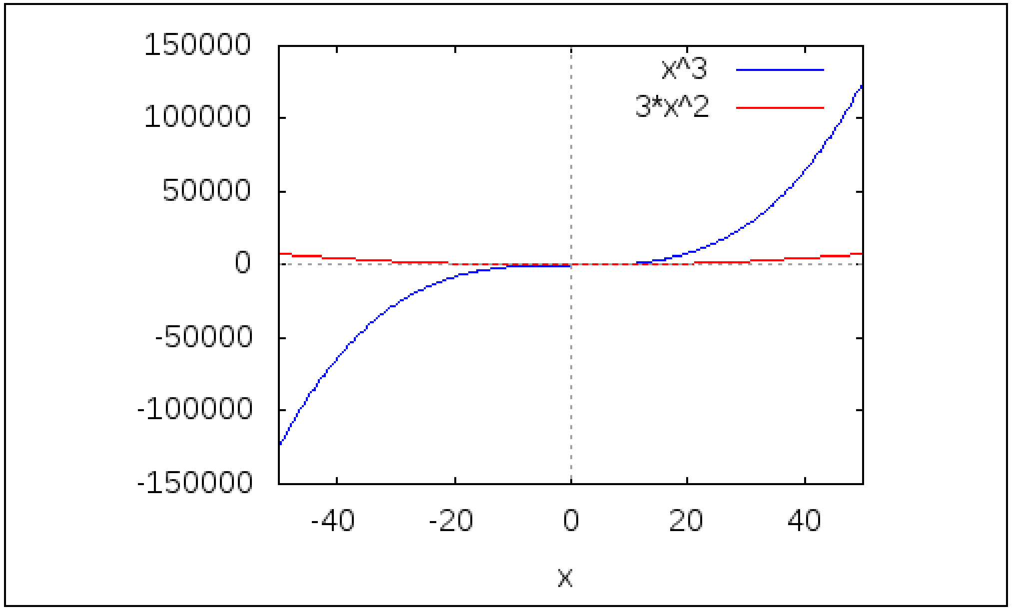
$$= (n+1)\, x^n$$

$$\frac{d}{dx} x^{n+1} = (n+1)\, x^{n+1-1}$$

$$\frac{d}{dx} x^{n} = n\, x^{n-1}$$

```
(%i18) f(x) := x; g(x) := x^2;
(%o18) f(x):=x
(%o19) g(x):=x^2
(%i23) wxplot2d([f(x)*g(x), h(x)], [x,-50,50])$
```

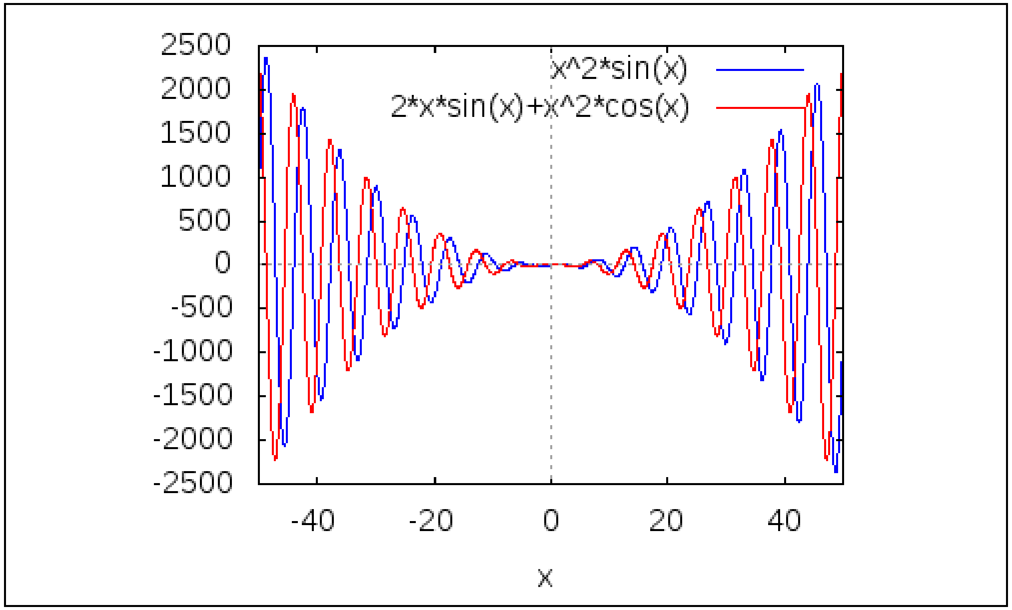
(%t23)




```
(%i21) h(x) := diff(f(x)*g(x), x, 1);
(%o21) h(x):=diff(f(x) g(x), x, 1)
```

```
(%i24) f(x) := sin(x); g(x) := x^2;
(%o24) f(x):=sin(x)
(%o25) g(x):=x^2
(%i27) wxplot2d([f(x)*g(x), h(x)], [x,-50,50])$
```

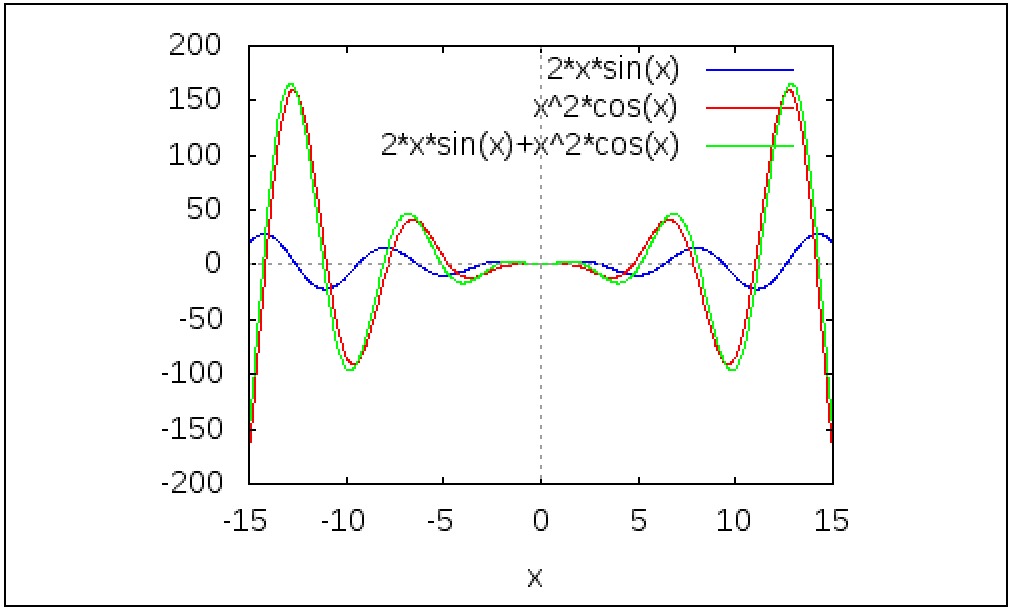
(%t27)




```
(%i26) h(x) := diff(f(x)*g(x), x, 1);
(%o26) h(x):=diff(f(x) g(x),x,1)
```

(%i34) wxplot2d([2*x*sin(x), x^2*cos(x), h(x)], [x,-15,15])$

(%t34)



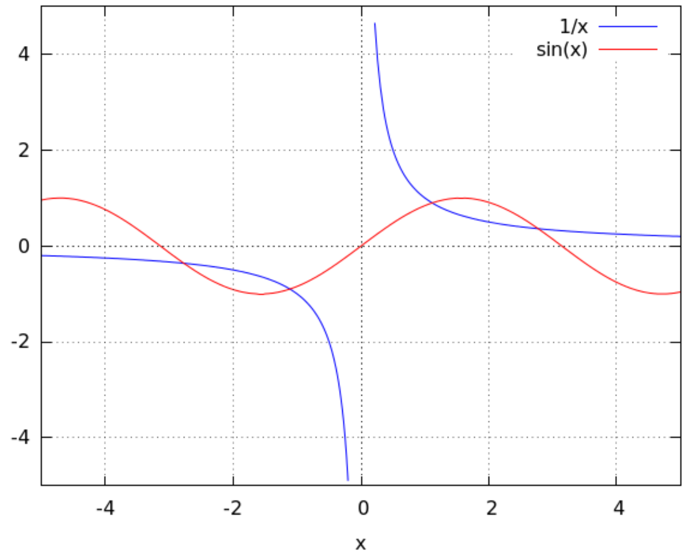
```
(%i22) f(x) := 1/x;
(%o22) f(x):=1/x
(%i23) g(x) := sin(x);
(%o23) g(x):=sin(x)
(%i24) h1(x) := diff(f(x),x) * g(x);
(%o24) h1(x):=diff(f(x),x) g(x)
(%i25) h2(x) := f(x) * diff(g(x), x);
(%o25) h2(x):=f(x) diff(g(x),x)
(%i26) h(x) := f(x) * g(x);
(%o26) h(x):=f(x) g(x)
```

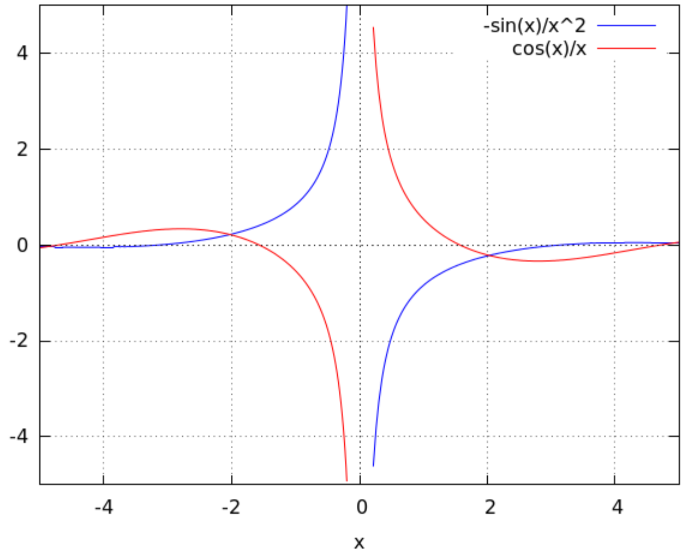
$$f(x) = \frac{1}{x} \qquad g(x) = \sin(x) \qquad f(x)\cdot g(x) = \frac{\sin(x)}{x}$$

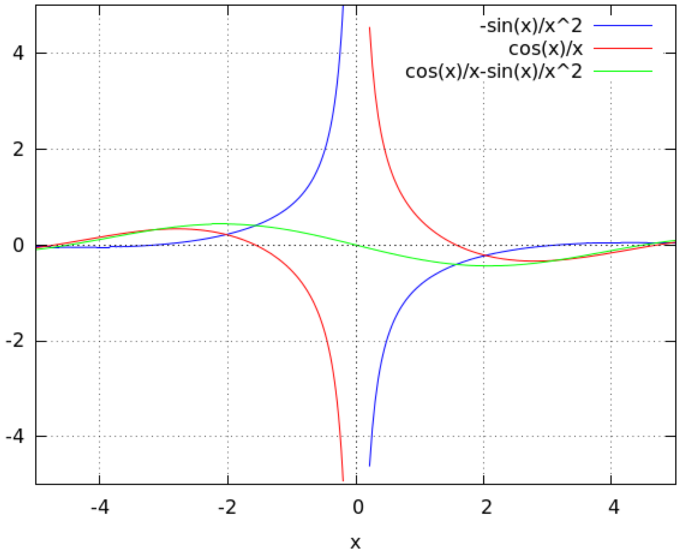
$$h(x) = h1(x) + h2(x)$$

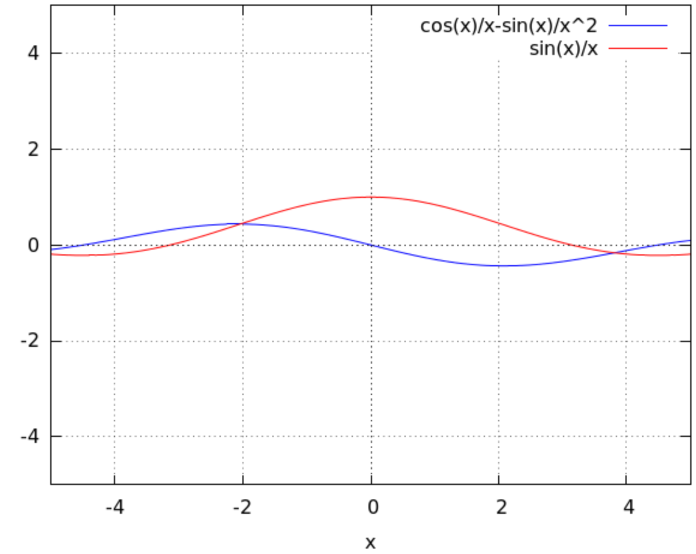
$$h1(x) = \frac{d}{dx}\frac{1}{x}\cdot\sin(x) = -\frac{1}{x^2}\sin(x)$$

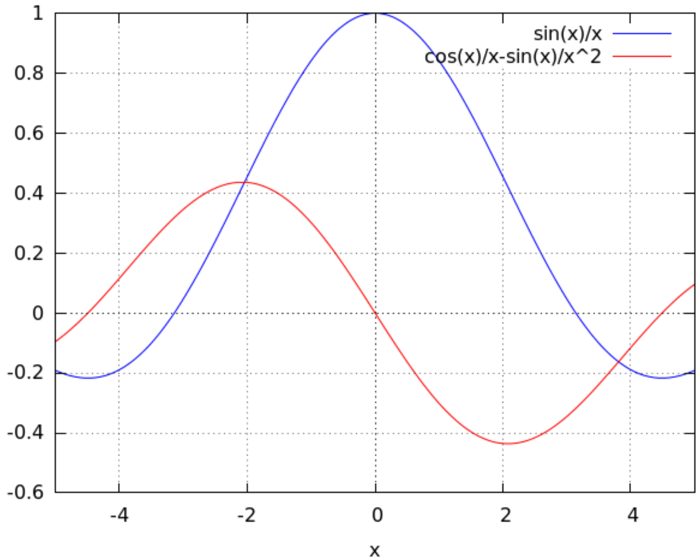
$$h2(x) = \frac{1}{x}\cdot\frac{d}{dx}\sin(x) = \frac{1}{x}\cos(x)$$

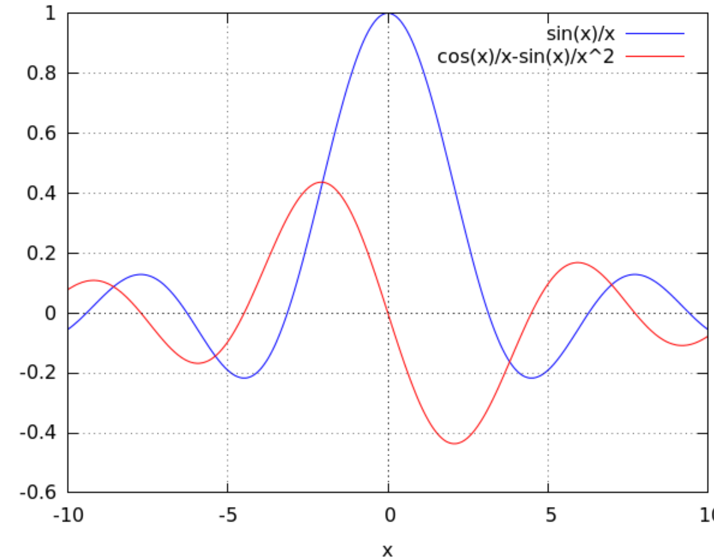





-sin(x)/x^2
cos(x)/x
cos(x)/x-sin(x)/x^2
x


cos(x)/x-sin(x)/x^2
sin(x)/x
x


sin(x)/x
cos(x)/x-sin(x)/x^2
x


sin(x)/x
cos(x)/x-sin(x)/x^2
x

```
(%i22) f(x) := 1/x;
```

(%o22) $f(x) := \frac{1}{x}$

```
(%i23) g(x) := sin(x);
```

(%o23) $g(x) := \sin(x)$

```
(%i24) h1(x) := diff(f(x),x) * g(x);
```

(%o24) $h1(x) := diff(f(x), x)\, g(x)$

```
(%i25) h2(x) := f(x) * diff(g(x), x);
```

(%o25) $h2(x) := f(x)\, diff(g(x), x)$

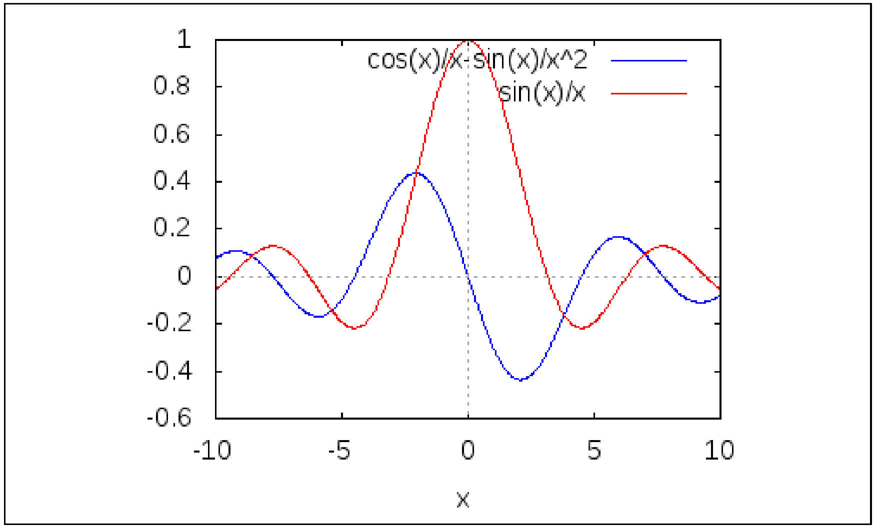
```
(%i26) h(x) := f(x) * g(x);
```

(%o26) $h(x) := f(x)\, g(x)$

```
(%i32) wxplot2d([h1(x)+h2
       (x), h(x)], [x,-10,10])$
```

plot2d: expression evaluates to non-numeric value somewhere in plotting range.
plot2d: expression evaluates to non-numeric value somewhere in plotting range

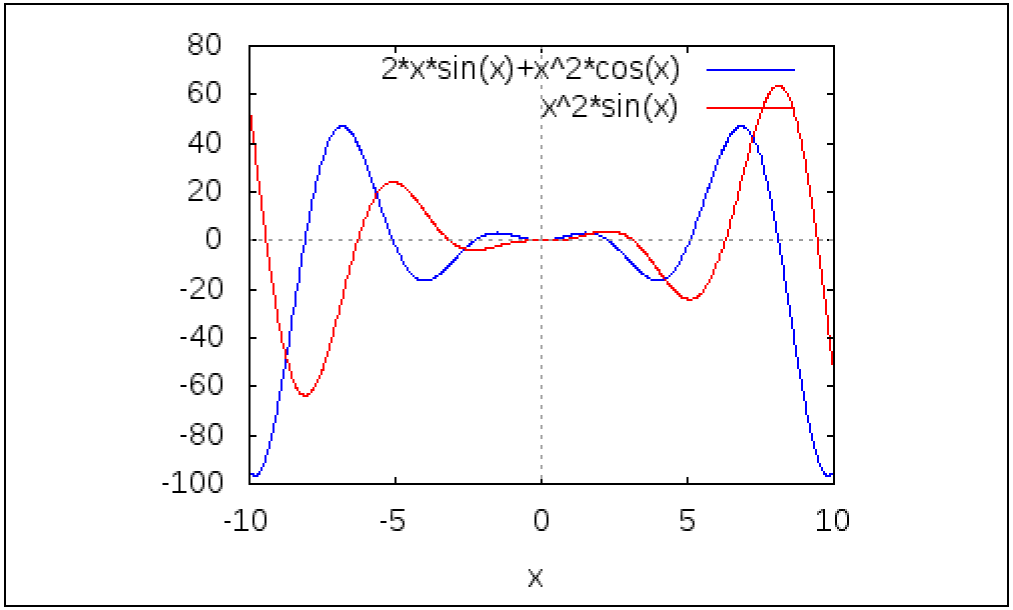
(%t32)



```
(%i33) f(x) := x^2;
(%o33) f(x):=x^2
(%i34) g(x) := sin(x);
(%o34) g(x):=sin(x)
(%i35) h1(x) := diff(f(x),x) * g(x);
(%o35) h1(x):=diff(f(x),x) g(x)
(%i36) h2(x) := f(x) * diff(g(x), x);
(%o36) h2(x):=f(x) diff(g(x),x)
(%i37) h(x) := f(x) * g(x);
(%o37) h(x):=f(x) g(x)
```

```
(%i38) wxplot2d([h1(x)+h2(x), h(x)], [x,-10,10])$
```

(%t38)



```
(%i54) f(x) := x^3;
```
(%o54) $f(x):=x^3$

```
(%i53) g(x) := sin(x);
```
(%o53) $g(x):=\sin(x)$

```
(%i55) h1(x) := f(g(x));
```
(%o55) $h1(x):=f(g(x))$

```
(%i56) h2(x) := g(f(x));
```
(%o56) $h2(x):=g(f(x))$

```
(%i61) h1(x);
```
(%o61) $\sin(x)^3$

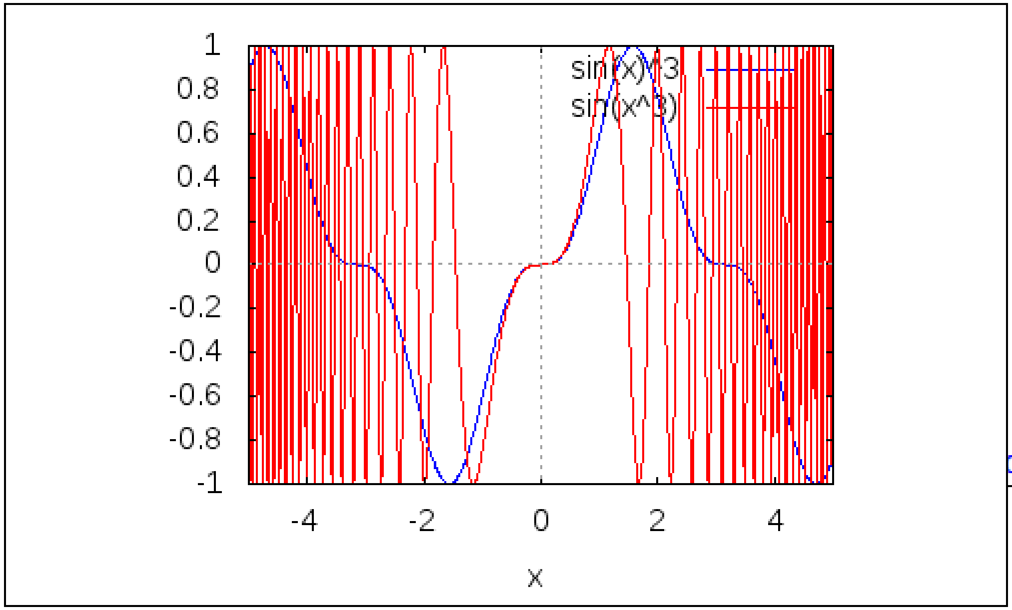
```
(%i62) diff(h1(x), x);
```
(%o62) $3\cos(x)\sin(x)^2$

```
(%i63) h2(x);
```
(%o63) $\sin(x^3)$

```
(%i64) diff(h2(x), x);
```
(%o64) $3x^2\cos(x^3)$

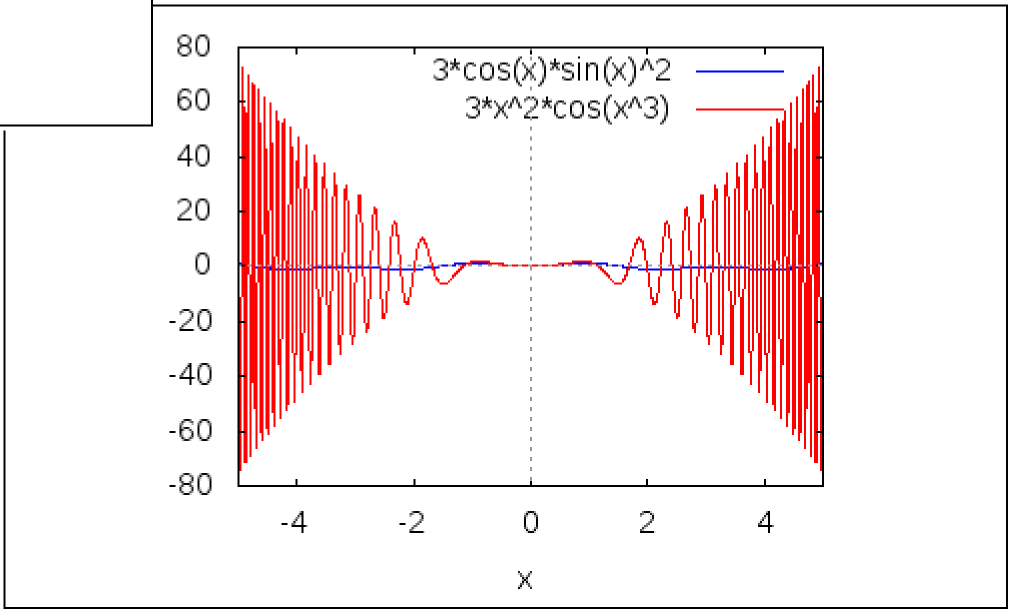
(%i65) wxplot2d([h1(x), h2(x)], [x,-5,5])$

(%t65)



d([diff(h1(x),x), diff(h2(x),x)], [x,-5,5])$

(%t59)



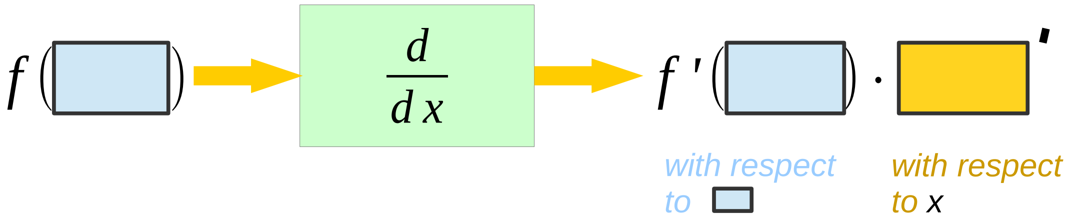
Chain Rule

Chain Rule

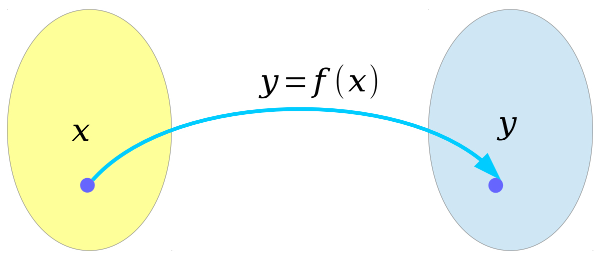
$$f(g(x)) \rightarrow \frac{d}{dx} \rightarrow f'(g) \cdot g'(x)$$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

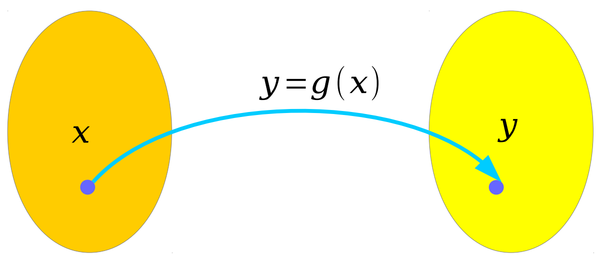
$$f(g) = f(g(x))$$



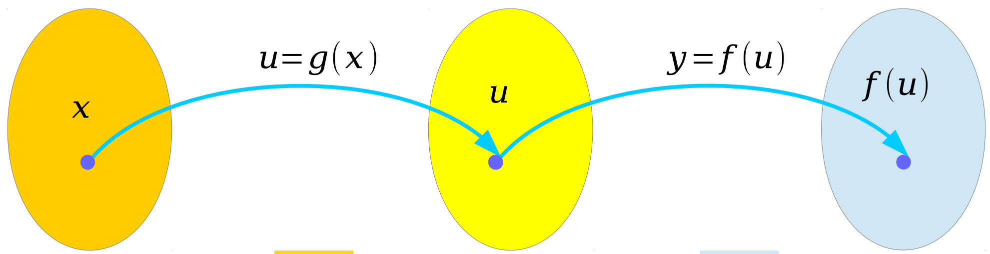
Chain Rule



$$f(x)$$



$$g(x)$$

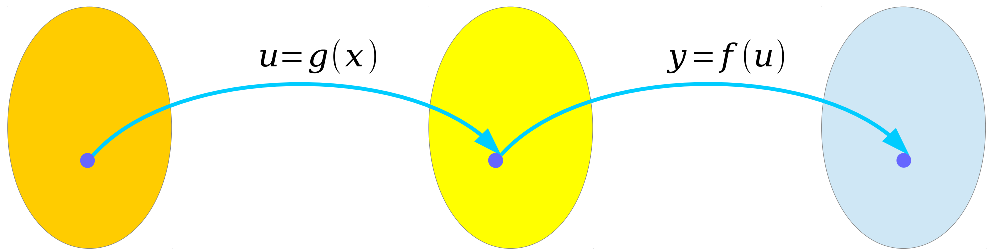


$$f(g(x))$$

$\frac{dg}{dx}$ $\frac{df}{dg}$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Example 1



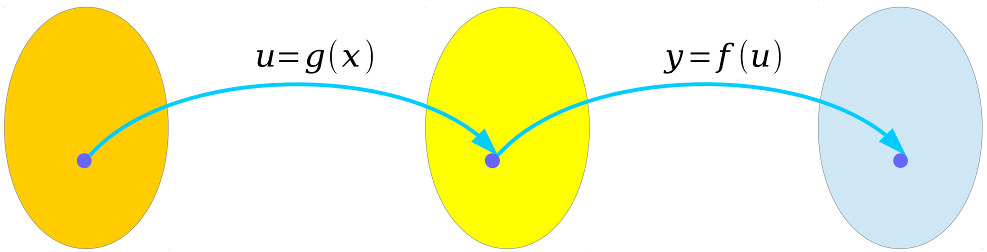
x $\quad$ x^3 $\quad$ $a(x^3)^2 + b = a x^6 + b$

$g(x) = x^3$ $\qquad$ $f(x) = a x^2 + b$

$g'(x) = 3x^2$ $\qquad$ $f'(g) = 2ag$

$$f'(g)\,g'(x) = (2ag)\cdot(3x^2)$$
$$= (2ax^3)\cdot(3x^2)$$
$$= 6ax^5$$

Example 2



$$\frac{d}{dx} f(g(x)) =$$

$$f'(x) g'(x)$$

$$x \leftarrow g(x)$$

$$f'(g(x)) g'(x)$$

$$f(x) = a x^2 + b \qquad g(x) = x^3$$

$$f'(x) = 2 a x \qquad g'(x) = 3 x^2$$

$$2a(x^3) \cdot 3 x^2$$

Chain Rule and Substitution Rule Examples

Chain Rule and Substitution Rule

$$f(g(x)) \quad \Rightarrow \boxed{\frac{d}{dx}} \Rightarrow \quad f'(g(x)) \cdot g'(x)$$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

$$f(g(x)) + C \quad \Leftarrow \boxed{\int \cdot \, dx} \Leftarrow \quad f'(g(x)) \cdot g'(x)$$

$$f(g(x)) + C = \int f'(g(x)) \cdot g'(x) \, dx$$

Chain Rule and Substitution Rule Examples

$e^{(x^2+2)}$ → $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx} = e^g(2x) = e^{x^2+2}(2x)$

$\begin{cases} f(x)=e^x \\ g(x)=x^2+2 \end{cases}$ → $f(g)=e^g$ $\quad \begin{cases} \frac{df}{dg}=e^g \\ \frac{dg}{dx}=2x \end{cases}$

$\int e^{x^2+2}(2x)\,dx$ → $\int \frac{df}{dg}dg = e^g = e^{x^2+2} + c$ or $\int \frac{df}{dg}dg = e^g = e^{x^2+2} + c$

view (I) $\begin{cases} f'(x)=e^x \\ g(x)=x^2+2 \end{cases}$ → $f'(g)=e^g$ $\quad \begin{cases} \int \frac{df}{dg}dg \\ \frac{dg}{dx}dx = 2x\,dx \end{cases}$ → $f'(g)=e^g+c$

view (II) $\begin{cases} f(x)=e^x \\ g(x)=x^2+2 \end{cases}$ → $f(g)=e^g$ $\quad \begin{cases} \int f(g)\,dg \\ \frac{dg}{dx}dx = 2x\,dx \end{cases}$ → $F(g)=e^g+c$

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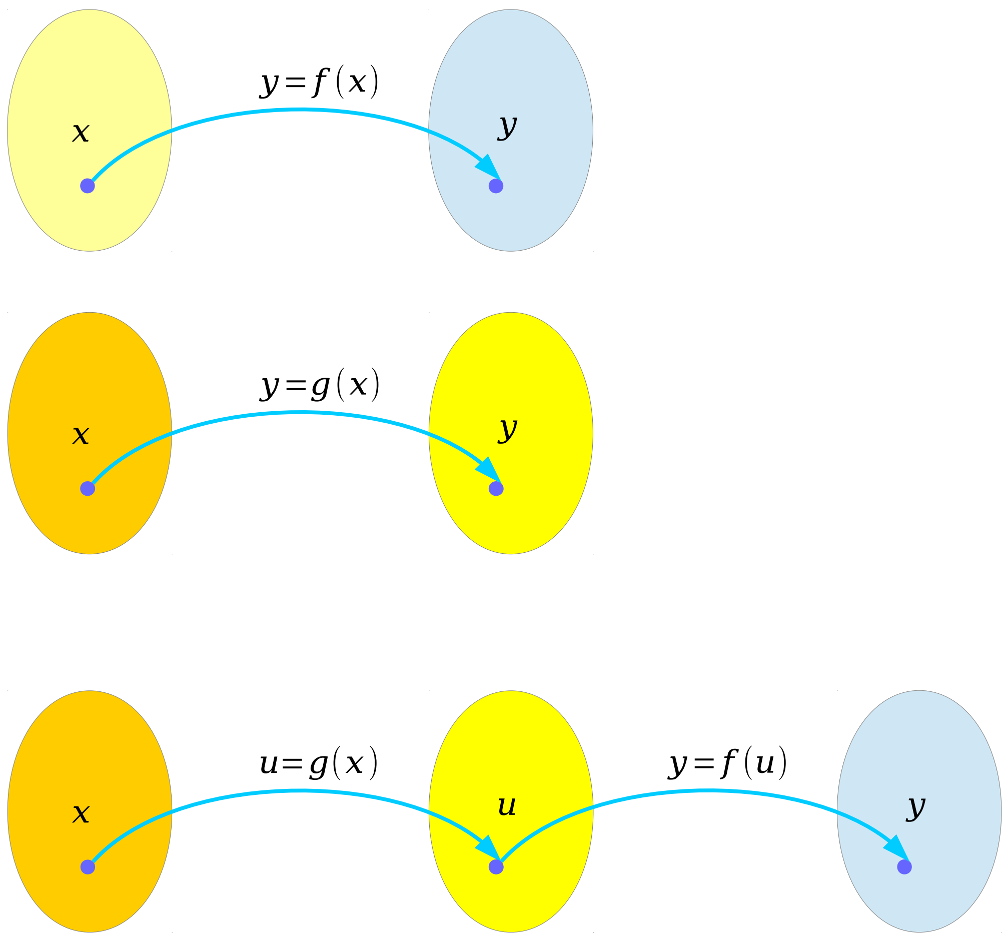
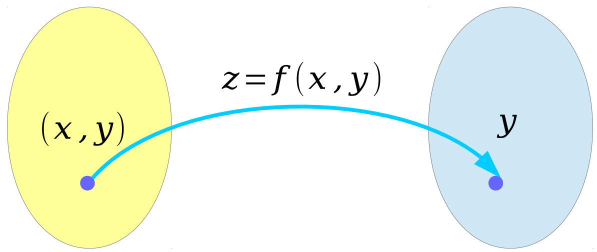
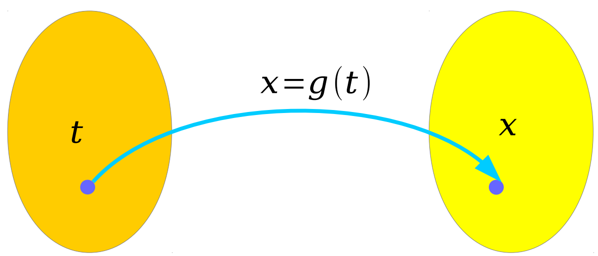
Chain Rule

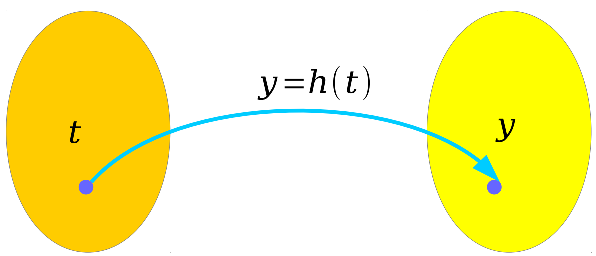


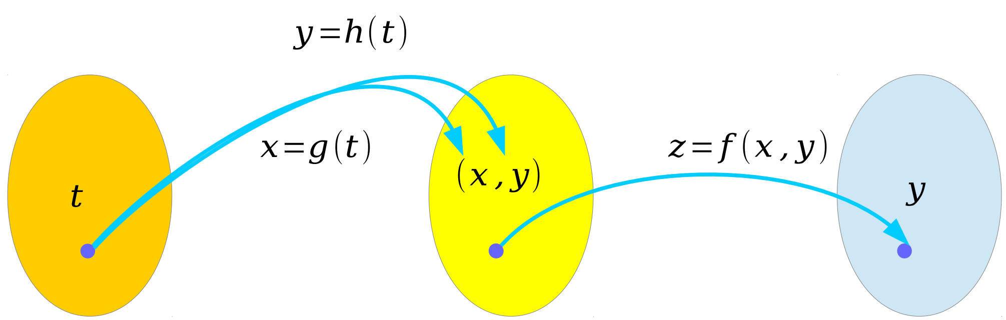
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Chain Rule and Partial Differentiation









$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Chain Rule and Total Differentials

$$z = f(x, y)$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$x(t)$$

$$dx = \frac{dx}{dt} dt$$

$$y(t)$$

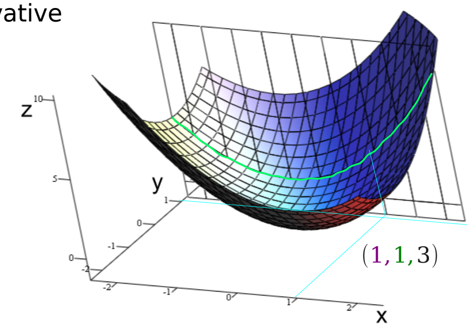
$$dy = \frac{dy}{dt} dt$$

$$\frac{dz}{dt} dt = \frac{\partial z}{\partial x} \frac{dx}{dt} dt + \frac{\partial z}{\partial y} \frac{dy}{dt} dt$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

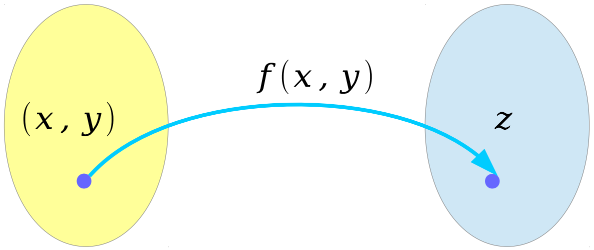
Parameterized Function of Two Variables

http://en.wikipedia.org/wiki/Partial_derivative



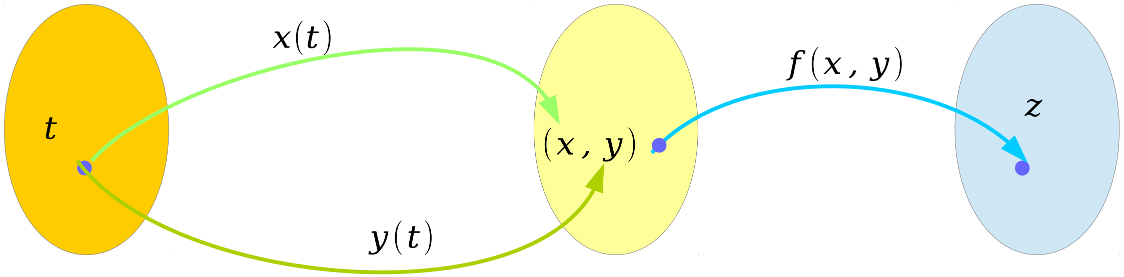
$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial x} = 2x + y$$

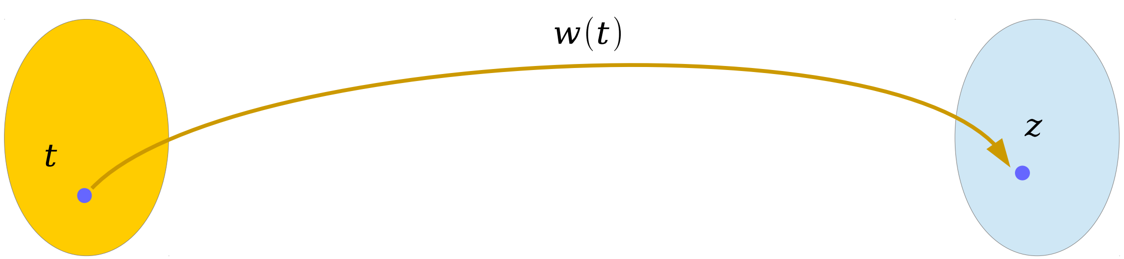
$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial y} = x + 2y$$



$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$





$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\frac{dw}{dt}(t) = \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”