CTFD – Frequency Response (7A)

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9/23/15

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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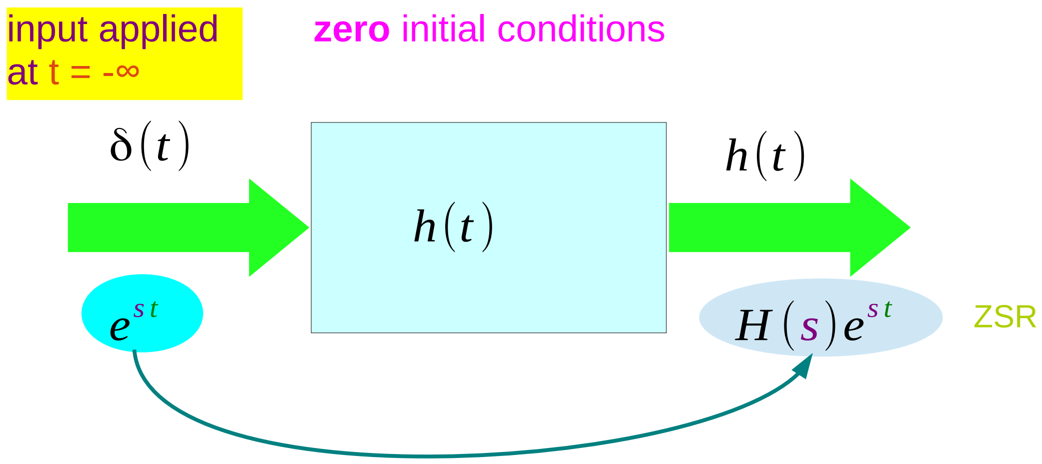
Computing a Transfer Function

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

ZSR

$$Q(s) \cdot Y(s) = P(s) \cdot X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)}$$



Transfer function (s-domain)

$$H(s) = \frac{Y(s)}{X(s)}$$

Transfer function (t-domain)

$$H(s) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{st}}$$

Polynomials of Differential Equation

$$H(s) = \frac{P(s)}{Q(s)}$$

$$(b_0, b_1, \cdots, b_{N-1}, b_N)$$

$$(1, a_1, \cdots, a_{N-1}, a_N)$$

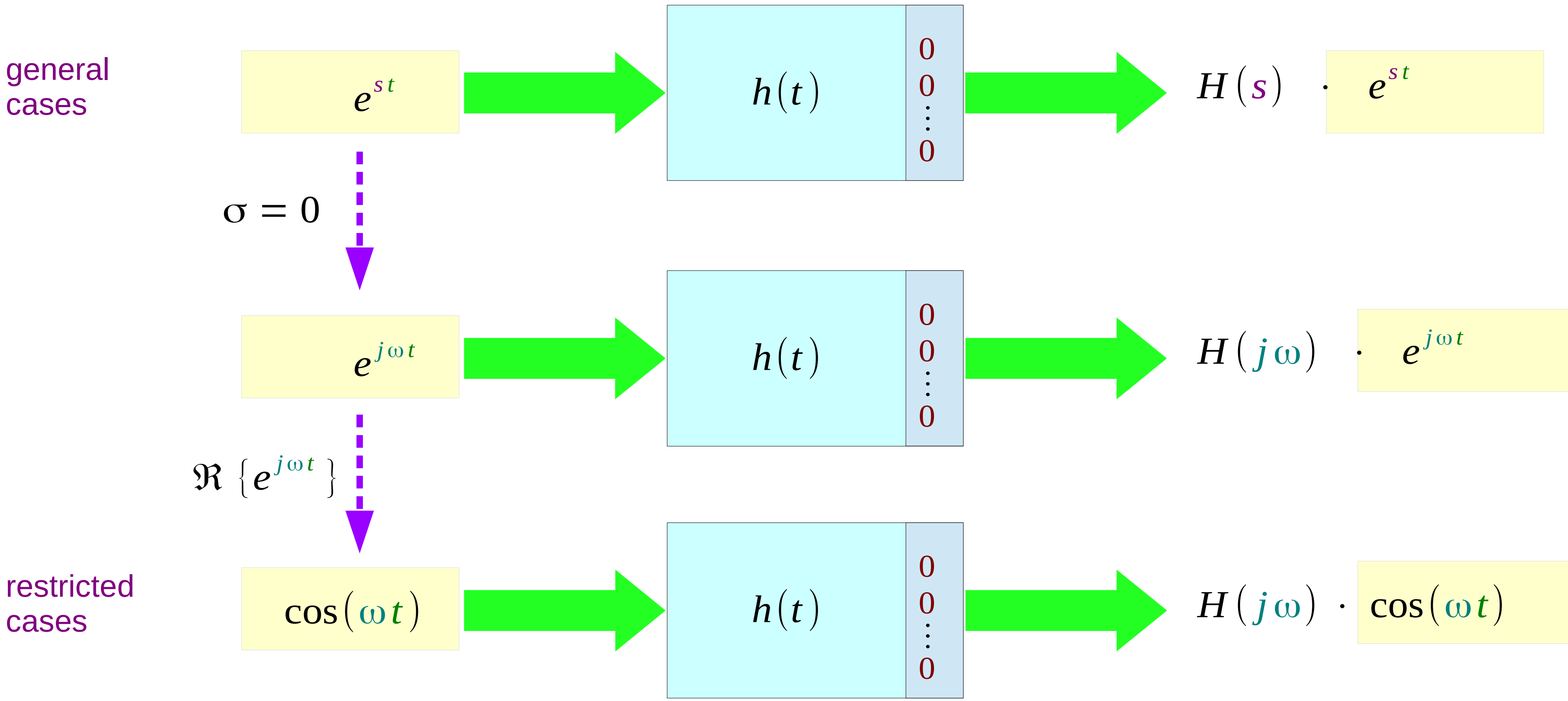
Laplace Transform of h(t)

$$H(s) = \int_0^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Special cases of H(s)

$s = \sigma + j\omega$

input applied at $t = -\infty$



Frequency Response

Laplace Transform of h(t)

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \qquad s = \sigma + j\omega$$

Fourier Transform of h(t)

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau \qquad s = j\omega$$

Polynomials of Differential Equation

$$H(s) = \frac{P(s)}{Q(s)} \qquad s = \sigma + j\omega$$

Polynomials of Differential Equation

$$H(j\omega) = \frac{P(j\omega)}{Q(j\omega)} \qquad s = j\omega$$

Transfer function (t-domain)

$$H(s) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{st}} \qquad s = \sigma + j\omega$$

Frequency response (t-domain)

$$H(j\omega) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{j\omega t}} \qquad s = j\omega$$

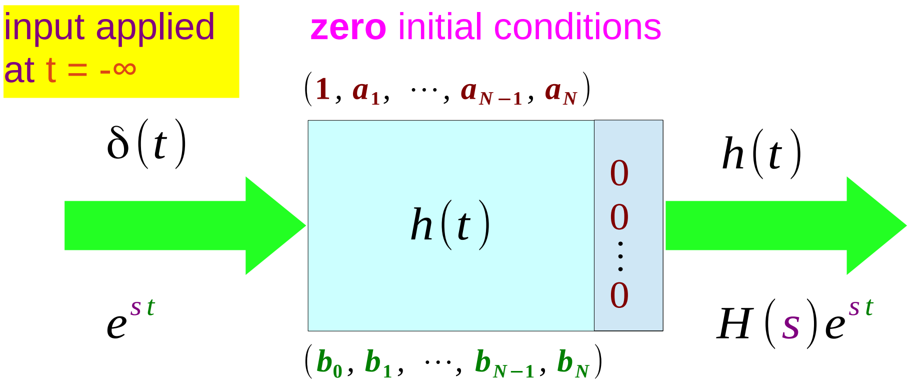
Transfer function (s-domain)

$$H(s) = \frac{Y(s)}{X(s)} \qquad s = \sigma + j\omega$$

Frequency response (ω-domain)

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \qquad s = j\omega$$

Transfer Function & Frequency Response

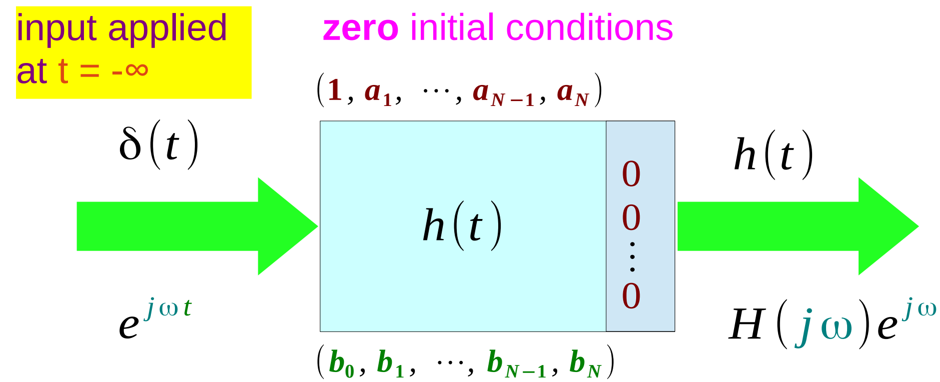


$$
\begin{aligned}
y(t) &= h(t) * e^{st} \qquad s = \sigma + j\omega \\
&= \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= e^{st} \cdot \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \\
&= e^{st} \cdot H(s)
\end{aligned}
$$

$h(t) = 0 \quad (t < 0)$

$$H(s) = \int_{0}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Transfer function



$$
\begin{aligned}
y(t) &= h(t) * e^{j\omega t} \qquad s = j\omega \\
&= \int_{-\infty}^{+\infty} h(\tau) e^{j\omega(t-\tau)} d\tau \\
&= e^{j\omega t} \cdot \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau \\
&= e^{j\omega t} \cdot H(j\omega)
\end{aligned}
$$

$h(t) = 0 \quad (t < 0)$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$

Frequency response

Total Response to everlasting sinusoidal inputs

$e^{+j\omega t}$ → $h(t)$ → $H(+j\omega)e^{+j\omega t} = |H(+j\omega)|e^{+j\,arg\{H(+j\omega)\}}\cdot e^{+j\omega t}$

$e^{-j\omega t}$ → $h(t)$ → $H(-j\omega)e^{-j\omega t} = |H(-j\omega)|e^{+j\,arg\{H(-j\omega)\}}\cdot e^{-j\omega t}$

$e^{+j\omega t} + e^{-j\omega t}$ → $h(t)$ → $|H(+j\omega)|e^{[+j\omega t+arg\{H(+j\omega)\}]} + |H(-j\omega)|e^{[-j\omega t+arg\{H(-j\omega)\}]}$

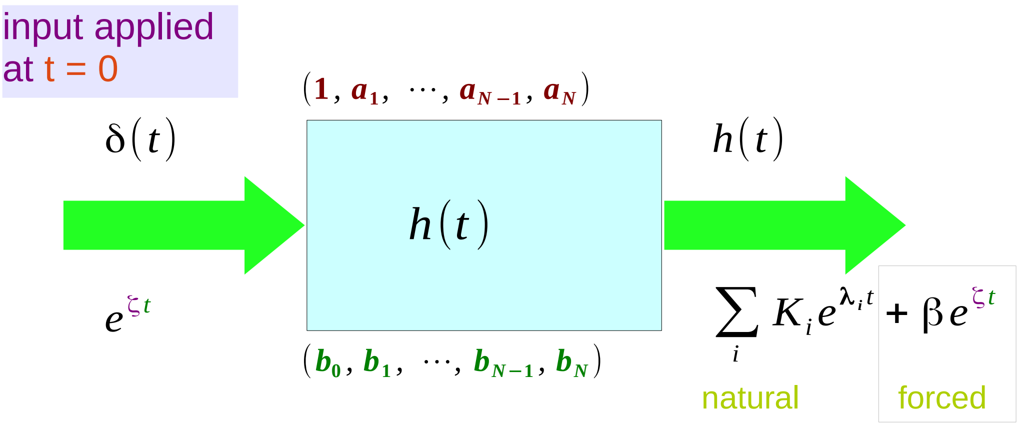
$$= |H(+j\omega)|\{e^{[+j\omega t+arg\{H(+j\omega)\}]} + e^{[-j\omega t-arg\{H(+j\omega)\}]}\}$$

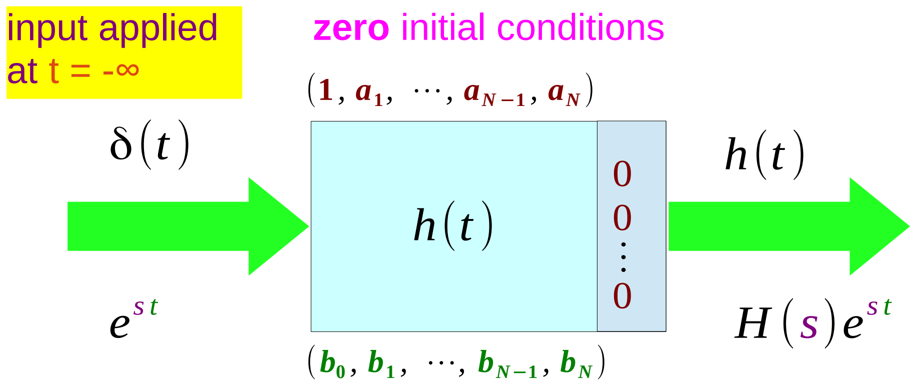
$2\cos(\omega t)$ → $h(t)$ → $|H(j\omega)|\,2\cos(\omega t+arg\{H(j\omega)\})$

$A\cos(\omega t+\alpha)$ → $h(t)$ → $A|H(j\omega)|\cos(\omega t+\alpha+arg\{H(j\omega)\})$

$A\sin(\omega t+\alpha)$ → $h(t)$ → $A|H(j\omega)|\sin(\omega t+\alpha+arg\{H(j\omega)\})$

Sinusoidal Steady State Response (1)





total response

natural forced

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) e^{\zeta t} \qquad t \geq 0$$

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) x(t) \qquad x(t) = e^{\zeta t}$$

$$Y(s) = \sum_i \frac{K_i}{(s - \lambda_i)} + H(s) X(s)$$

steady state response

forced

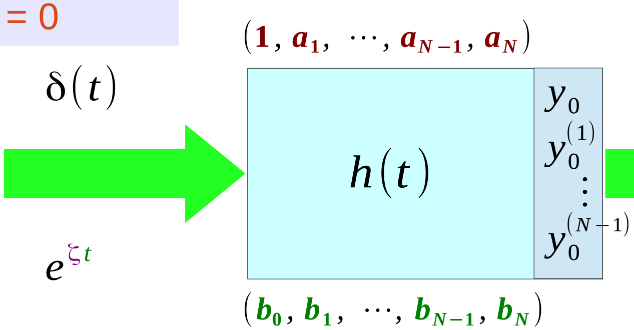
$$y_{ss}(t) = H(\zeta) e^{\zeta t} \qquad t \to \infty$$

$$y_{ss}(t) = H(\zeta) x(t) \qquad x(t) = e^{\zeta t}$$

$$Y_{ss}(s) = H(s) X(s)$$

Sinusoidal Steady State Response (2)

input applied at t = 0



ZIR ZSR

$$\sum_i c_i e^{\lambda_i t} + h(t)$$

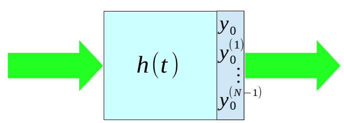
$$\sum_i K_i e^{\lambda_i t} + \beta e^{\xi t}$$

natural forced

steady state response

$x(t) = A$

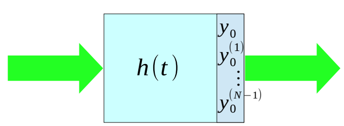
$\xi = 0$



$$y_{ss}(t) = H(0) \cdot A e^{0t} = A \cdot H(0)$$

$x(t) = A e^{j\omega t}$

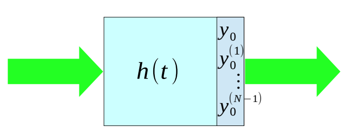
$\xi = j\omega$



$$y_{ss}(t) = H(j\omega) \cdot A e^{j\omega t} = A \cdot H(j\omega) e^{j\omega}$$

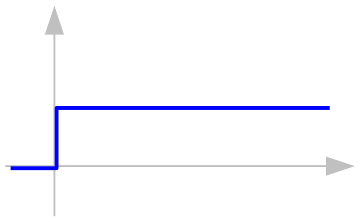
$x(t) = A\cos(\omega t)$

$\xi = j\omega$



$$y_{ss}(t) = H(j\omega) \cdot A\cos(\omega t) = A \cdot H(j\omega) \cos(\omega t)$$

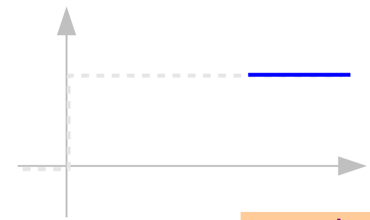
Sinusoidal Steady State Response (3)



$$y_{ss}(t) = H(0) \cdot A e^{0t} = A \cdot H(0)$$

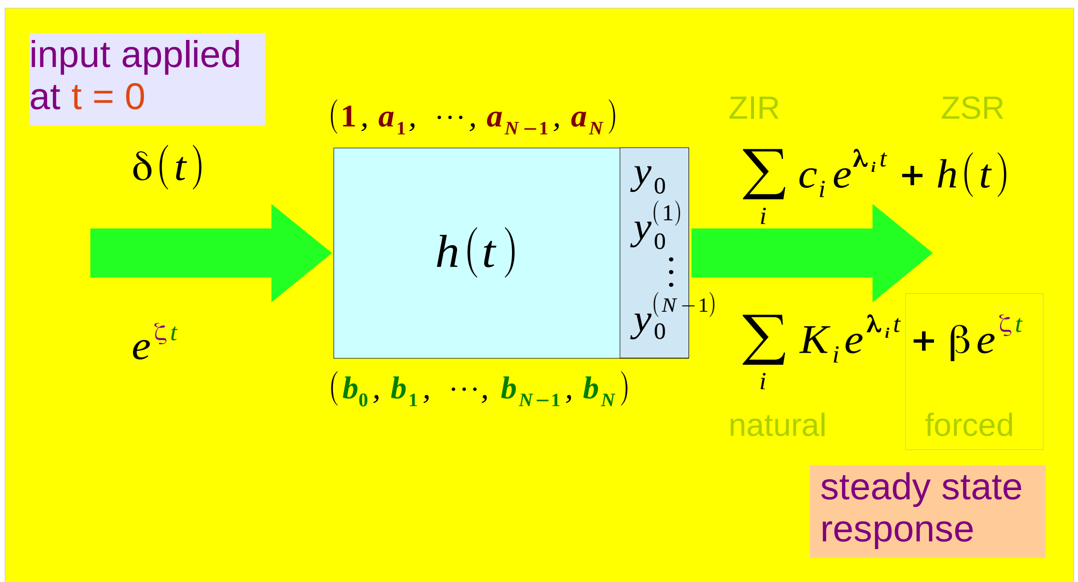
$x(t) = A$

$\xi = 0$



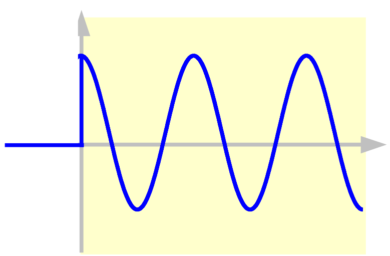
steady state response

Forced response



Forced response

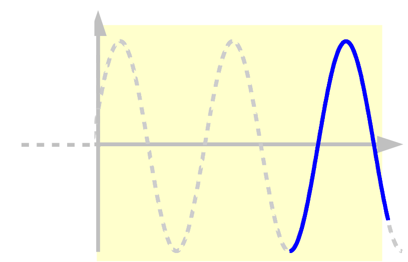
steady state response



$$y_{ss}(t) = H(j\omega) \cdot A\cos(\omega t) = A \cdot H(j\omega)\cos(\omega t)$$

$x(t) = A\cos(\omega t)$

$\xi = j\omega$



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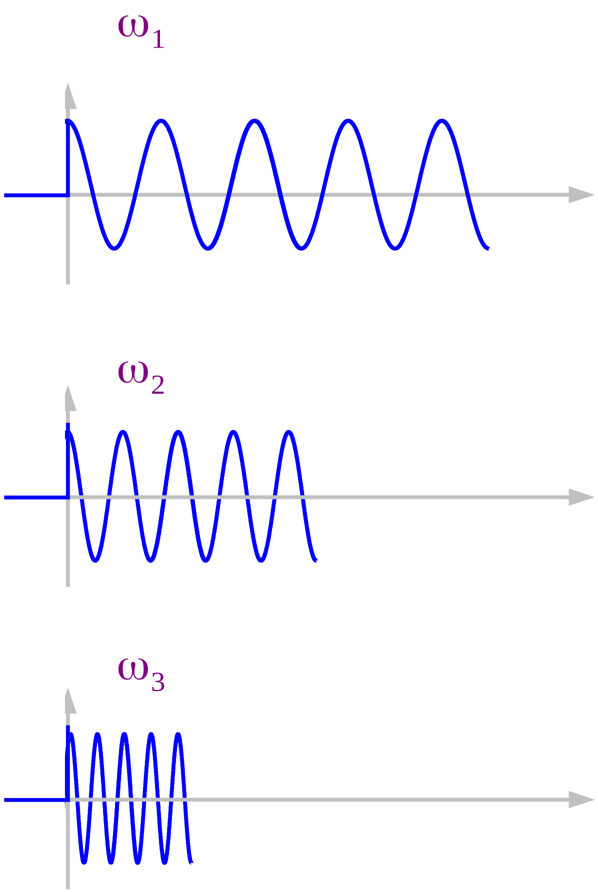
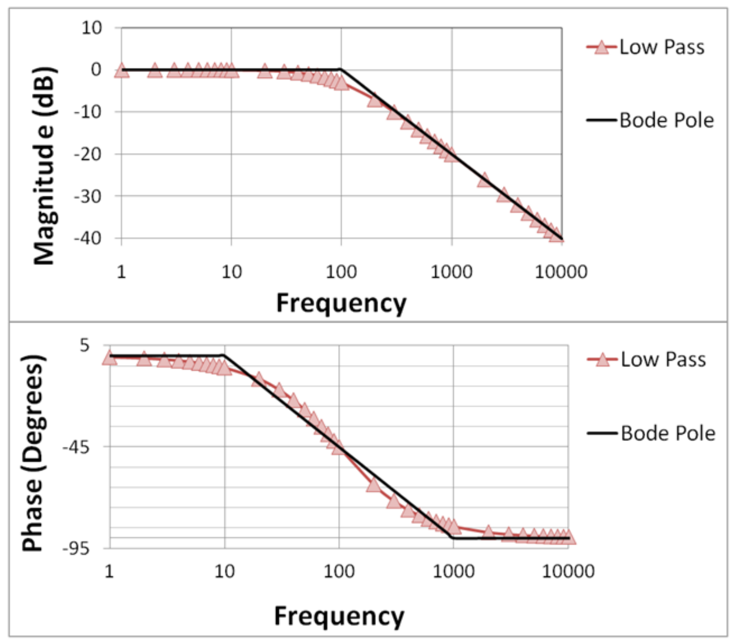
Frequency Response

$$y_{ss}(t) = H(j\omega)\cdot A\cos(\omega t) = A\cdot H(j\omega)\cos(\omega t)$$

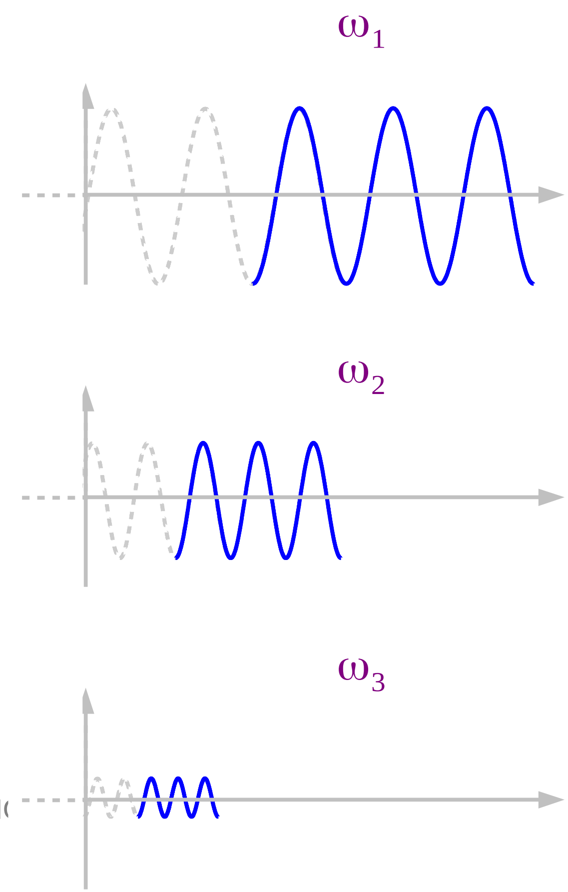
$$x(t) = A\cos(\omega t)$$
$$\xi = j\omega$$



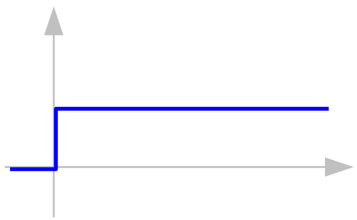
ω_1 ω_2 ω_3



http://en.wikipedia.org/wiki/File:Bode_Low-Pass.PN(



Transient Response

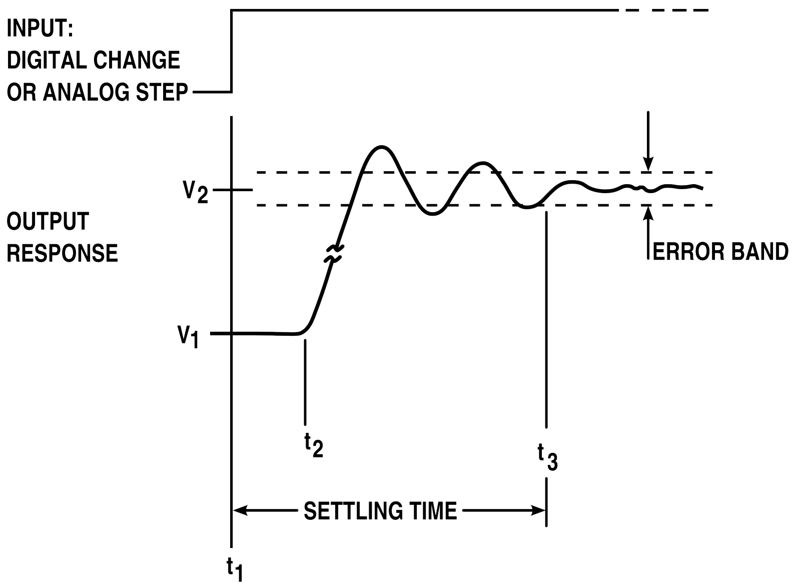


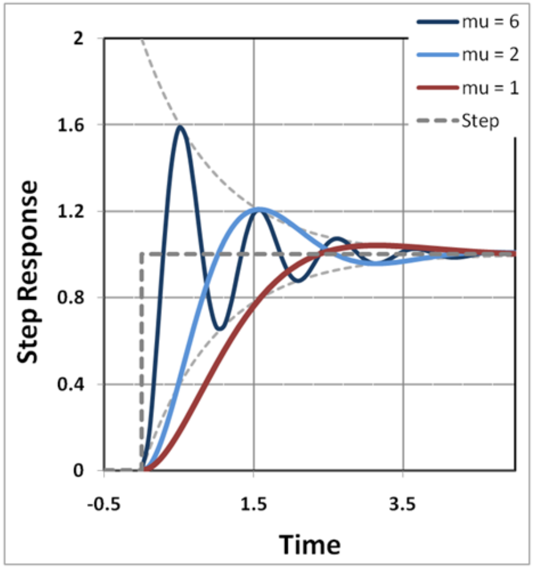
$$y_{ss}(t) = H(0) \cdot A e^{0t} = A \cdot H(0)$$

$$x(t) = A$$
$$\xi = 0$$

Natural + Forced response

transient response





http://en.wikipedia.org/wiki/File:High_accuracy_settling_time_measurements_figure_1.png
http://en.wikipedia.org/wiki/File:Step_response_for_two-pole_feedback_amplifier.PNG

Frequency Response in Control Theory (1)

$$\sqrt{A^2+B^2}\cos(\omega t-\theta) \rightarrow \boxed{G(s)} \rightarrow |G(j\omega)|\sqrt{A^2+B^2}\cos\left(\omega t-\theta+arg\{G(j\omega)\}\right)$$

$$A\cos(\omega t)+B\sin(\omega t)$$

$$=\sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\cos(\omega t)+\frac{B}{\sqrt{A^2+B^2}}\sin(\omega t)\right]$$

$$=\sqrt{A^2+B^2}\left[\cos(\theta)\cos(\omega t)+\sin(\theta)\sin(\omega t)\right]$$

$$=\sqrt{A^2+B^2}\cos(\theta-\omega t)$$

$$=\sqrt{A^2+B^2}\cos(\omega t-\theta)$$

$$A\cos(\omega t)+B\sin(\omega t)$$

$$=\sqrt{A^2+B^2}\cos(\omega t-\theta)$$

$$\cos(\theta)=\frac{A}{\sqrt{A^2+B^2}}$$

$$\sin(\theta)=\frac{B}{\sqrt{A^2+B^2}}$$

Frequency Response in Control Theory (2)

$$\sqrt{A^2+B^2}\cos(\omega t-\theta) \rightarrow G(s) \rightarrow |G(j\omega)|\sqrt{A^2+B^2}\cos(\omega t-\theta+arg\{G(j\omega)\})$$

$$A\cos(\omega t)+B\sin(\omega t) = \sqrt{A^2+B^2}\cos(\omega t-\theta) \quad \Longleftrightarrow \quad \frac{A s}{s^2+\omega^2}+\frac{B\omega}{s^2+\omega^2} = \frac{A s+B\omega}{s^2+\omega^2}$$

$$Y(s) = \frac{A s+B\omega}{s^2+\omega^2}G(s) = \frac{A s+B\omega}{(s+j\omega)(s-j\omega)}G(s) = \frac{K_1}{s+j\omega}+\frac{K_2}{s-j\omega}+F(s)$$

Partial Fraction

$$K_1 = \left[\frac{A s+B\omega}{s^2+\omega^2}(s+j\omega)G(s)\right]_{s=-j\omega} = \left[\frac{A s+B\omega}{(s-j\omega)}G(s)\right]_{s=-j\omega} = \frac{-A j\omega+B\omega}{-2j\omega}G(-j\omega) = \frac{1}{2}(A+jB)G(-j\omega)$$

$$K_2 = \left[\frac{A s+B\omega}{s^2+\omega^2}(s-j\omega)G(s)\right]_{s=+j\omega} = \left[\frac{A s+B\omega}{(s+j\omega)}G(s)\right]_{s=+j\omega} = \frac{A j\omega+B\omega}{+2j\omega}G(+j\omega) = \frac{1}{2}(A-jB)G(+j\omega)$$

Frequency Response in Control Theory (3)

$$\sqrt{A^2+B^2}\cos(\omega t-\theta) \rightarrow G(s) \rightarrow |G(j\omega)|\sqrt{A^2+B^2}\cos(\omega t-\theta+arg\{G(j\omega)\})$$

$$Y(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} + F(s) \qquad K_1 = \frac{1}{2}(A+jB)G(-j\omega) \qquad K_2 = \frac{1}{2}(A-jB)G(+j\omega)$$

$$A \pm jB = \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}} \pm j\frac{B}{\sqrt{A^2+B^2}}\right]$$

$$= \sqrt{A^2+B^2}\,[\cos\theta \pm j\sin\theta]$$

$$= \sqrt{A^2+B^2}\,e^{\pm j\theta}$$

Stable System

system modes

$$e^{p_i} \rightarrow 0 \qquad e^{\sigma_i} \rightarrow 0 \qquad p_i<0,\ \sigma_i<0$$

$$\lim_{t\rightarrow\infty} f(t) = 0 \qquad \lim_{s\rightarrow\infty} sF(s) = 0$$

Ignore $F(s)$

$$Y_{ss}(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} \qquad K_1 = \frac{1}{2}\sqrt{A^2+B^2}\,e^{+j\theta}G(-j\omega) \qquad K_2 = \frac{1}{2}\sqrt{A^2+B^2}\,e^{-j\theta}G(-j\omega)$$

Frequency Response in Control Theory (4)

$$\sqrt{A^2+B^2}\cos(\omega t-\theta) \rightarrow G(s) \rightarrow |G(j\omega)|\sqrt{A^2+B^2}\cos(\omega t-\theta+arg\{G(j\omega)\})$$

$$Y_{ss}(s)=\frac{K_1}{s+j\omega}+\frac{K_2}{s-j\omega} = \frac{1}{2}(A+jB)G(-j\omega)\frac{1}{s+j\omega}+\frac{1}{2}(A-jB)G(+j\omega)\frac{1}{s-j\omega}$$

$$= \frac{1}{2}\sqrt{A^2+B^2}e^{+j\theta}G(-j\omega)\frac{1}{s+j\omega}+\frac{1}{2}\sqrt{A^2+B^2}e^{-j\theta}G(+j\omega)\frac{1}{s-j\omega}$$

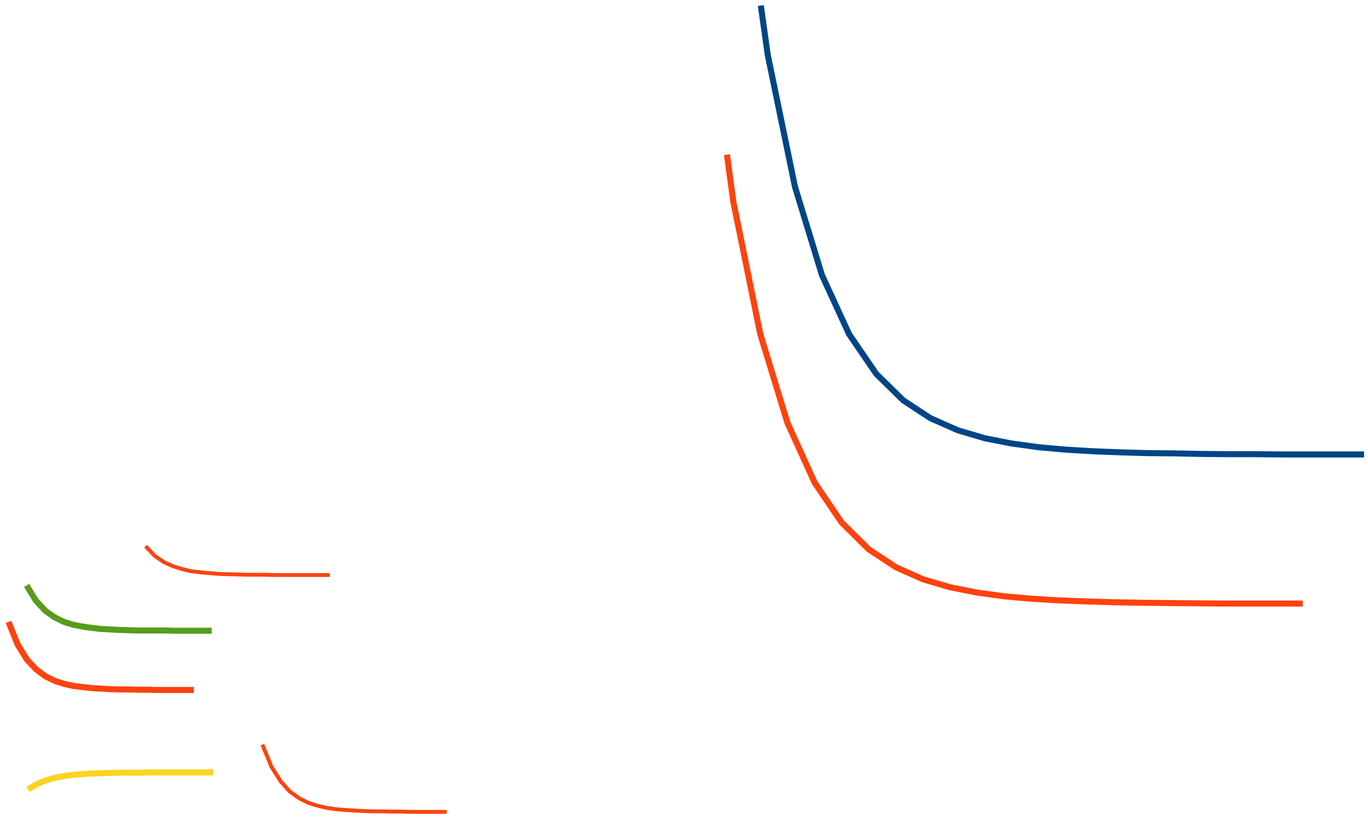
$$y_{ss}(t) = \frac{\sqrt{A^2+B^2}}{2}\left[G(-j\omega)e^{-j\omega t}e^{+j\theta} + G(+j\omega)e^{+j\omega t}e^{-j\theta}\right]$$

$$= \frac{\sqrt{A^2+B^2}}{2}\left[G(+j\omega)e^{-j(\omega t-\theta)} + G(+j\omega)e^{+j(\omega t-\theta)}\right]$$

$$= \sqrt{A^2+B^2}\ \Re\{G(+j\omega)e^{+j(\omega t-\theta)}\}$$

$$= \sqrt{A^2+B^2}\ |G(+j\omega)|\cos(\omega t-\theta+arg\{G(+j\omega)\})$$

Impulse Response h(t)



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] M. J. Roberts, Fundamentals of Signals and Systems
[4] S. J. Orfanidis, Introduction to Signal Processing
[5] B. P. Lathi, Signals and Systems