Time Domain Analysis (1A)

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2nd Order Systems

$$\frac{9}{s^2+9s+9} \qquad s^2+9s+9=0 \qquad m_1, m_2 = \frac{-9\pm 3\sqrt{5}}{2}$$

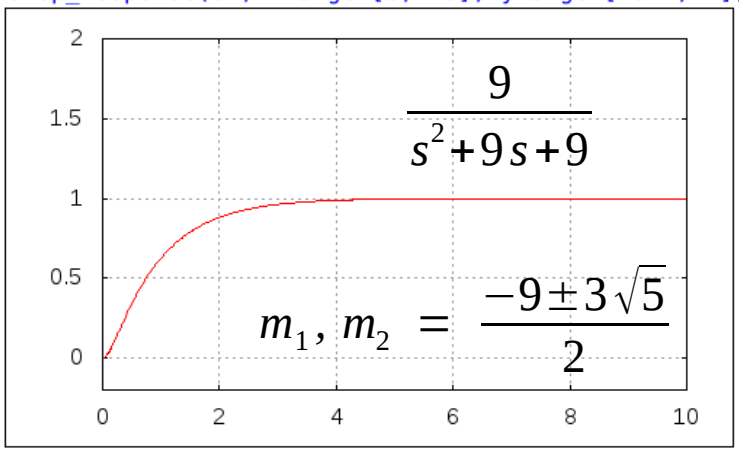
$$\frac{9}{s^2+2s+9} \qquad s^2+2s+9=0 \qquad m_1, m_2 = -1\pm i\,2\sqrt{2}$$

$$\frac{9}{s^2+9} \qquad s^2+9=0 \qquad m_1, m_2 = \pm i\,3$$

$$\frac{9}{s^2+6s+9} \qquad s^2+6s+9=0 \qquad m_1, m_2 = -3$$

Step Responses

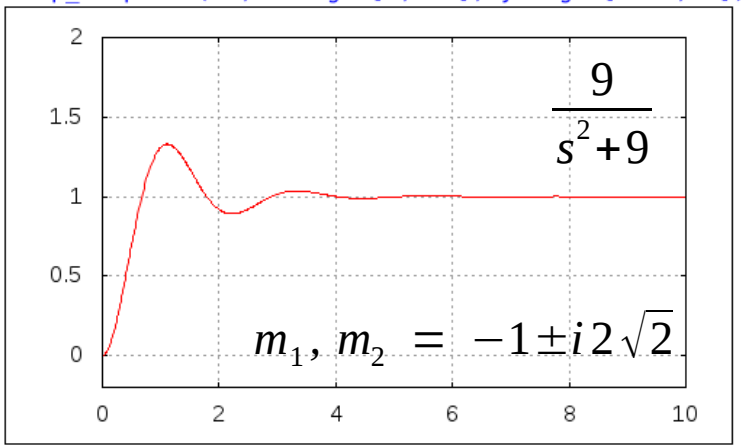
```
(%i39) step_response(G1, xrange=[0, 10], yrange=[-0.2, 2]);
```



(%t39)

(%o39)

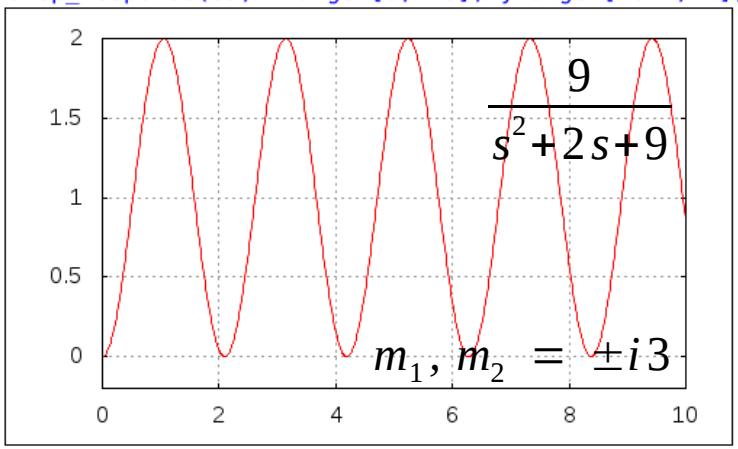
```
(%i40) step_response(G2, xrange=[0, 10], yrange=[-0.2, 2]);
```



(%t40)

(%o40)

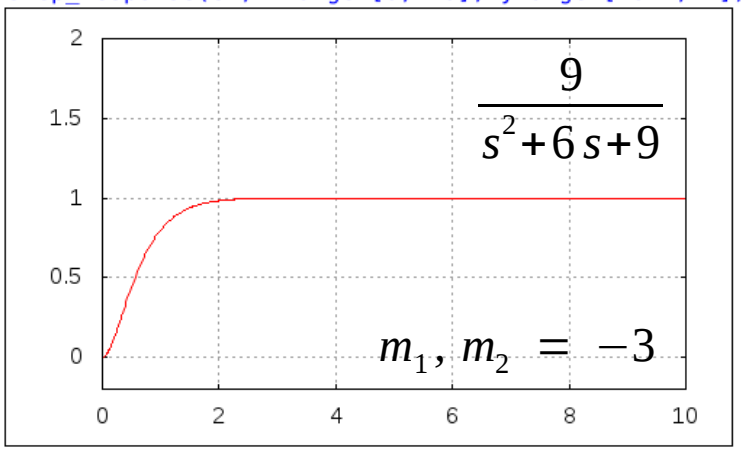
```
(%i41) step_response(G3, xrange=[0, 10], yrange=[-0.2, 2]);
```



(%t41)

(%o41)

```
(%i42) step_response(G4, xrange=[0, 10], yrange=[-0.2, 2]);
```



(%t42)

(%o42)

2nd Order Transfer Function: Standard Form

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

$$= -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

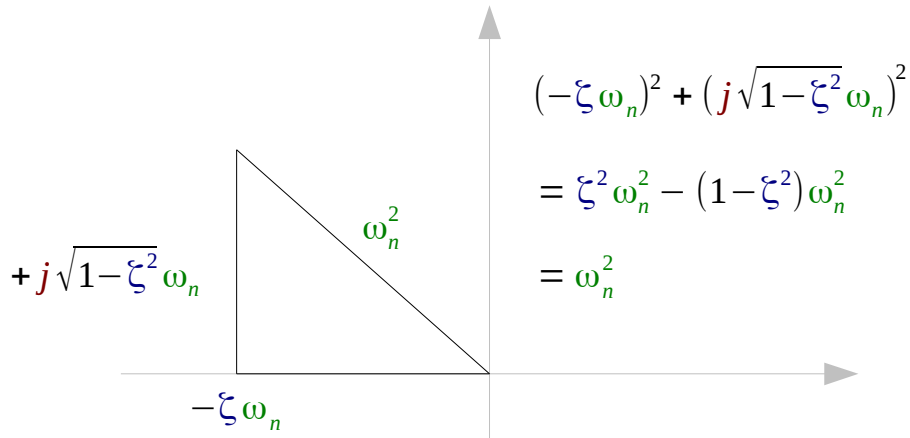
$$= -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n$$

$s = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$	$\zeta > 1$
$s = -\omega_n$	$\zeta = 1$
$s = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n$	$0 < \zeta < 1$
$s = \pm j\omega_n$	$\zeta = 0$

2nd Order Transfer Function: Standard Form

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$



$$s = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n \qquad \zeta > 1$$

$$s = -\omega_n \qquad \zeta = 1$$

$$s = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n \qquad 0 < \zeta < 1$$

$$s = \pm j\omega_n \qquad \zeta = 0$$

2nd Order Transfer Function: Standard Form

$$s = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n \qquad 0 < \zeta < 1$$

$$\zeta = 0.1, \quad \omega_n = 200 \qquad s^2 + 4s + 20\sqrt{0.99}$$

$$\zeta = 0.2, \quad \omega_n = 100 \qquad s^2 + 4s + 10\sqrt{0.96}$$

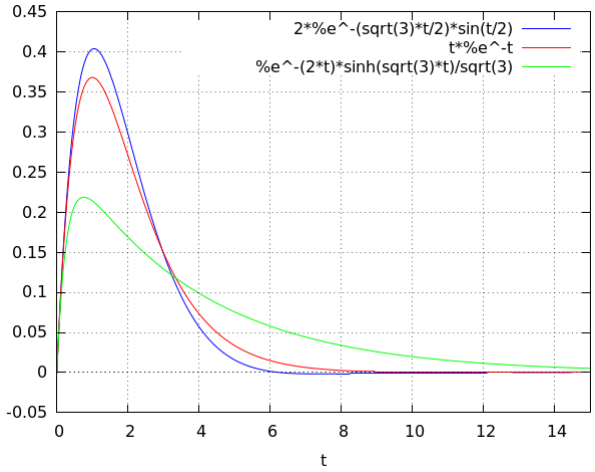
$$\zeta = 0.4, \quad \omega_n = 50 \qquad s^2 + 4s + 5\sqrt{0.84}$$

2nd Order Systems

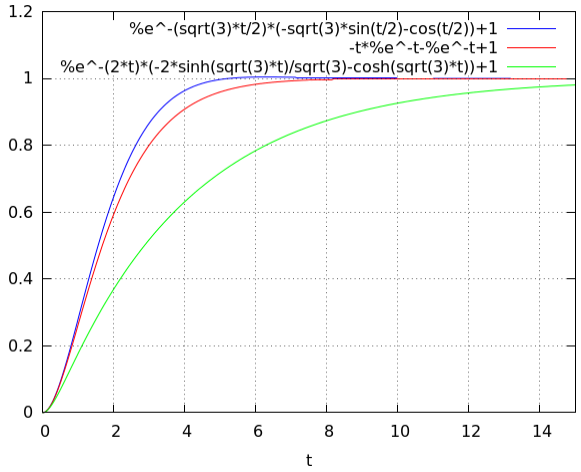
	$\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)}$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$	$\frac{\omega_n^2}{s^2(s^2+2\zeta\omega_n s+\omega_n^2)}$
$\zeta=\sqrt{0.75}$ $\omega_n=1$	$2e^{\frac{-\sqrt{3}}{2}t}\sin\left(\frac{t}{2}\right)$	$e^{\frac{-\sqrt{3}}{2}t}\left(-\cos\left(\frac{t}{2}\right)-\sqrt{3}\sin\left(\frac{t}{2}\right)\right)+1$	$e^{\frac{-\sqrt{3}}{2}t}\left(\sqrt{3}\cos\left(\frac{t}{2}\right)+\sin\left(\frac{t}{2}\right)\right)+t-\sqrt{3}$
$\zeta=1$ $\omega_n=1$	$e^{-t}t$	$-e^{-t}t-e^{-t}+1$	$e^{-t}t+2e^{-t}+t-2$
$\zeta=2$ $\omega_n=1$	$\frac{1}{\sqrt{3}}e^{-2t}\sinh(\sqrt{3}t)$	$e^{-2t}\left(-\cosh(\sqrt{3}t)-\frac{2}{\sqrt{3}}\sinh\sqrt{3}t\right)+1$	$e^{-2t}\left(4\cosh(\sqrt{3}t)+\frac{7}{\sqrt{3}}\sinh\sqrt{3}t\right)+t-4$

2nd Order Systems

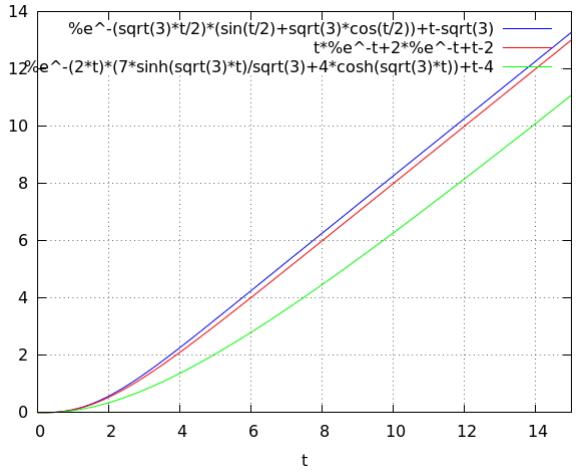
Impulse Response

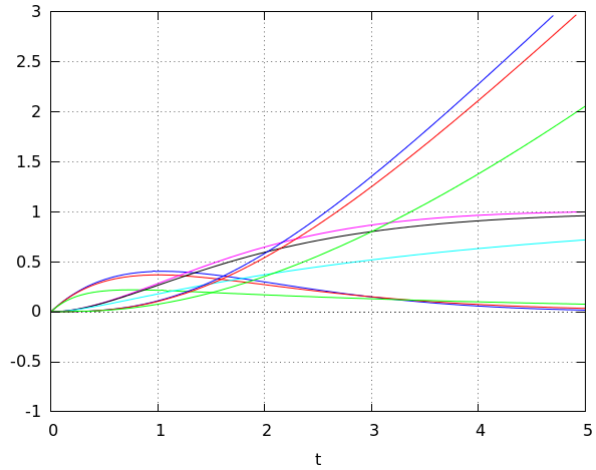


Step Response



Ramp Response





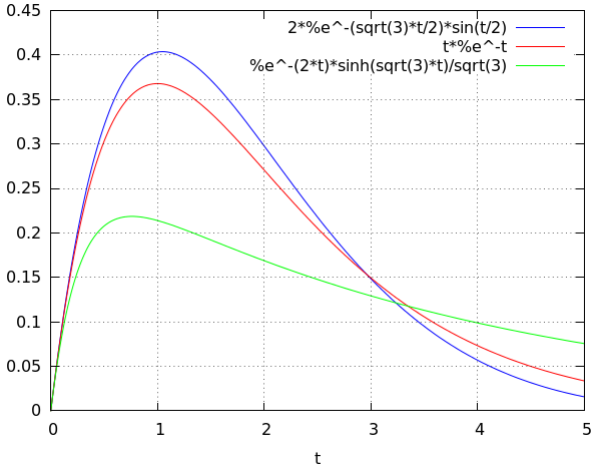
$\zeta = \sqrt{0.75}$ $\omega_n = 1$

$\zeta = 1$ $\omega_n = 1$

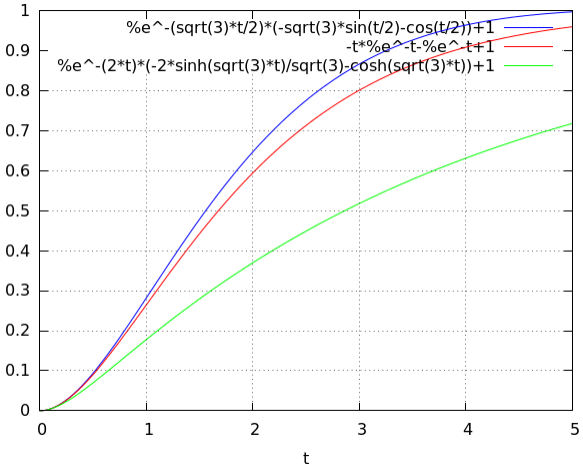
$\zeta = 2$ $\omega_n = 1$

2nd Order Systems

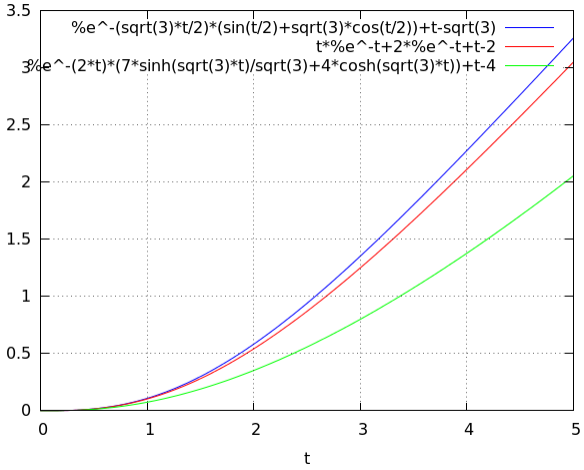
Impulse Response

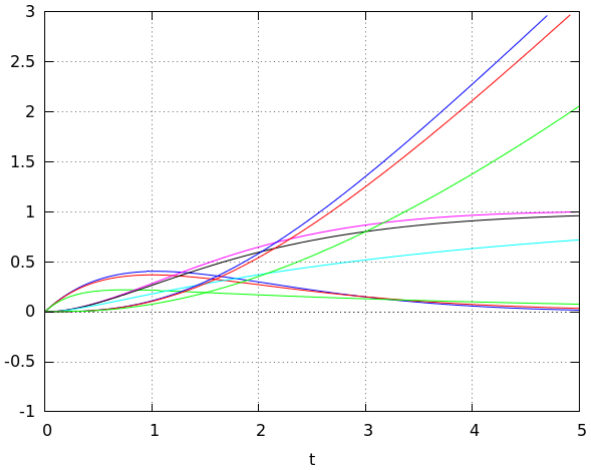


Step Response



Ramp Response





2nd Order Systems – Underdamping

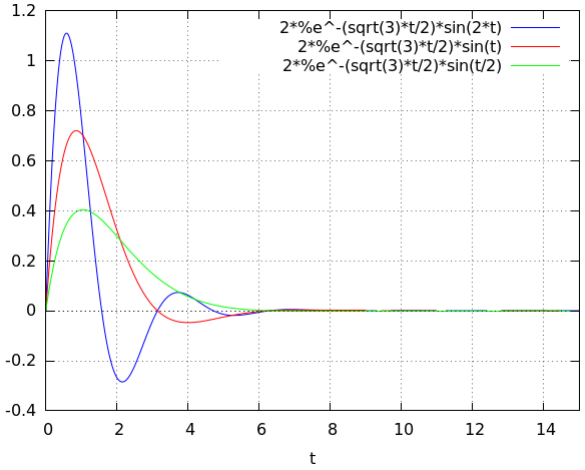
$$2e^{\frac{-\sqrt{3}}{2}t}\sin(2t) \qquad e^{\frac{-\sqrt{3}}{2}t}\left(-\cos(2t)-\sqrt{3}\sin(2t)\right)+1 \qquad e^{\frac{-\sqrt{3}}{2}t}\left(\sqrt{3}\cos(2t)+\sin(2t)\right)+t-\sqrt{3}$$

$$2e^{\frac{-\sqrt{3}}{2}t}\sin(t) \qquad e^{\frac{-\sqrt{3}}{2}t}\left(-\cos(t)-\sqrt{3}\sin(t)\right)+1 \qquad e^{\frac{-\sqrt{3}}{2}t}\left(\sqrt{3}\cos(t)+\sin(t)\right)+t-\sqrt{3}$$

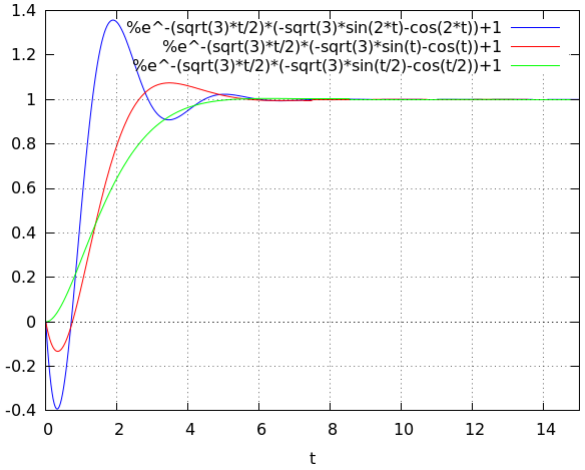
$$2e^{\frac{-\sqrt{3}}{2}t}\sin\left(\frac{t}{2}\right) \qquad e^{\frac{-\sqrt{3}}{2}t}\left(-\cos\left(\frac{t}{2}\right)-\sqrt{3}\sin\left(\frac{t}{2}\right)\right)+1 \qquad e^{\frac{-\sqrt{3}}{2}t}\left(\sqrt{3}\cos\left(\frac{t}{2}\right)+\sin\left(\frac{t}{2}\right)\right)+t-\sqrt{3}$$

2nd Order Systems

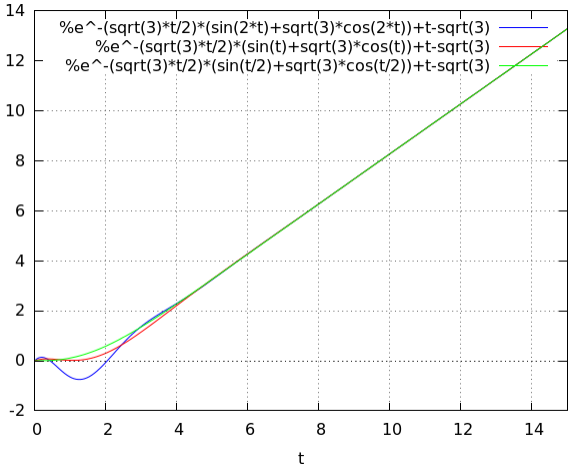
Impulse Response



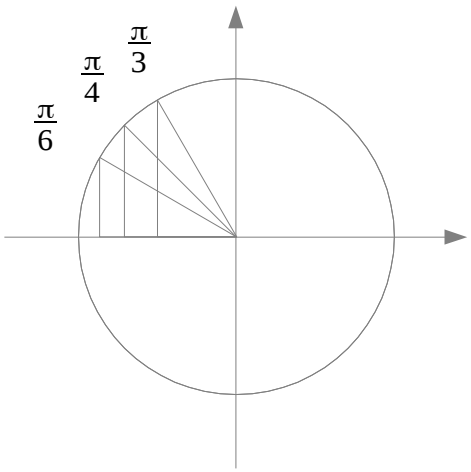
Step Response

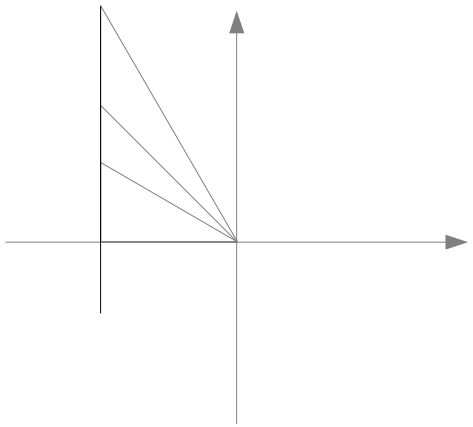


Ramp Response



2nd Order Systems – Underdamping



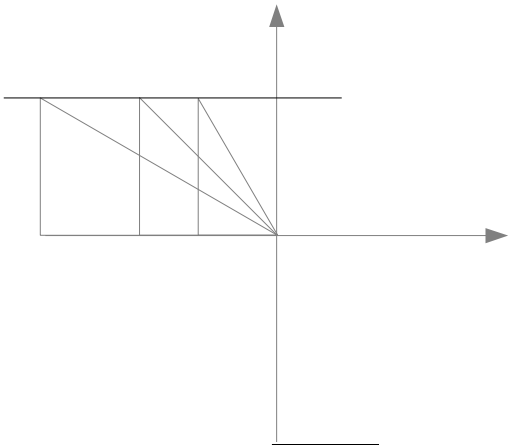


$$\sigma_d = \zeta\omega_n = 2$$

$$\zeta = \frac{1}{2} \qquad \omega_n = 2\left(\frac{1}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{2}}{2} \qquad \omega_n = 2\left(\frac{\sqrt{2}}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{3}}{2} \qquad \omega_n = 2\left(\frac{\sqrt{3}}{2}\right)^{-1}$$



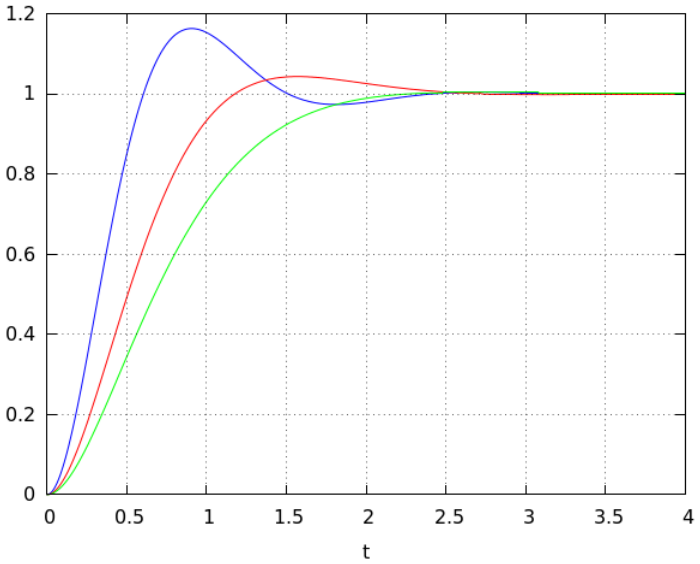
$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 2$$

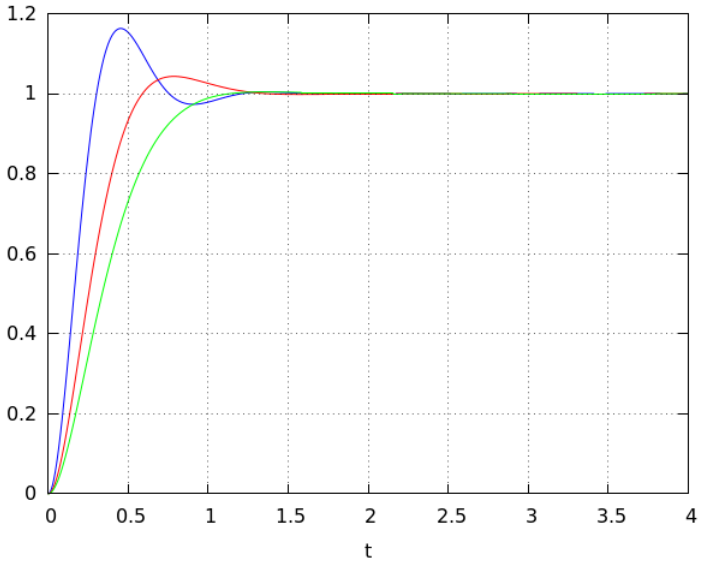
$$\zeta = \frac{1}{2} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{3}}{2} \qquad \omega_n = 2\left(\frac{\sqrt{3}}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{2}}{2} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{2}}{2} \qquad \omega_n = 2\left(\frac{\sqrt{2}}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{3}}{2} \qquad \sqrt{1-\zeta^2} = \frac{1}{2} \qquad \omega_n = 2\left(\frac{1}{2}\right)^{-1}$$

2nd Order Systems – Underdamping





$$\zeta = \frac{1}{2} \qquad \omega_n = 2\left(\frac{1}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{2}}{2} \qquad \omega_n = 2\left(\frac{\sqrt{2}}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{3}}{2} \qquad \omega_n = 2\left(\frac{\sqrt{3}}{2}\right)^{-1}$$

$$\zeta = \frac{1}{2} \qquad \omega_n = 4\left(\frac{1}{2}\right)^{-1}$$

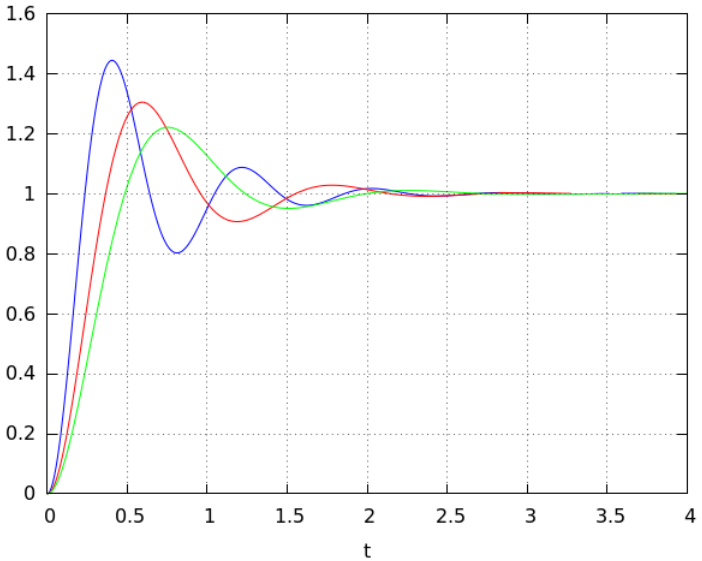
$$\zeta = \frac{\sqrt{2}}{2} \qquad \omega_n = 4\left(\frac{\sqrt{2}}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{3}}{2} \qquad \omega_n = 4\left(\frac{\sqrt{3}}{2}\right)^{-1}$$

$$\sigma_d = \zeta\omega_n = 2$$

$$\sigma_d = \zeta\omega_n = 4$$

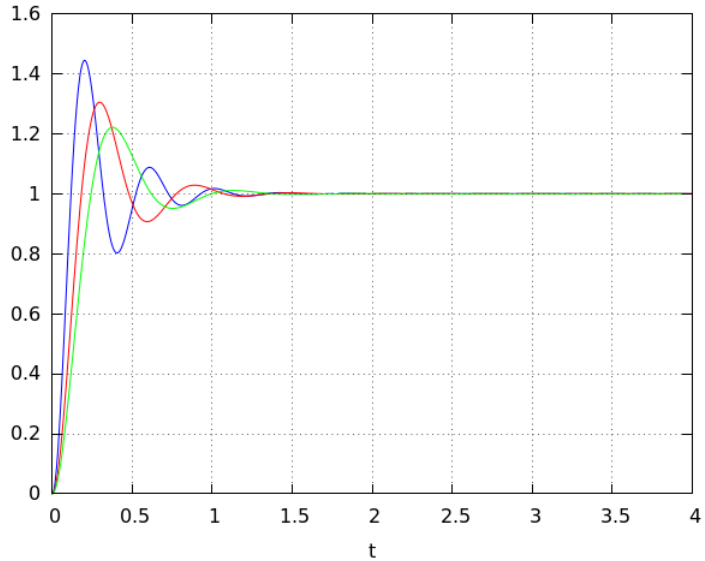
2nd Order Systems – Underdamping



$$\zeta = \frac{1}{4} \qquad \zeta = \frac{\sqrt{2}}{4} \qquad \zeta = \frac{\sqrt{3}}{4}$$

$$\omega_n = 2\left(\frac{1}{4}\right)^{-1} \qquad \omega_n = 2\left(\frac{\sqrt{2}}{4}\right)^{-1} \qquad \omega_n = 2\left(\frac{\sqrt{3}}{4}\right)^{-1}$$

$$\sigma_d = \zeta\omega_n = 2$$

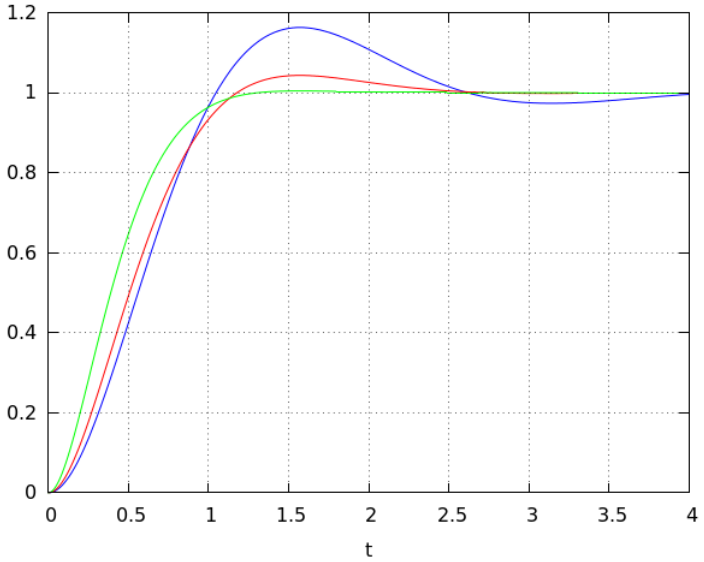


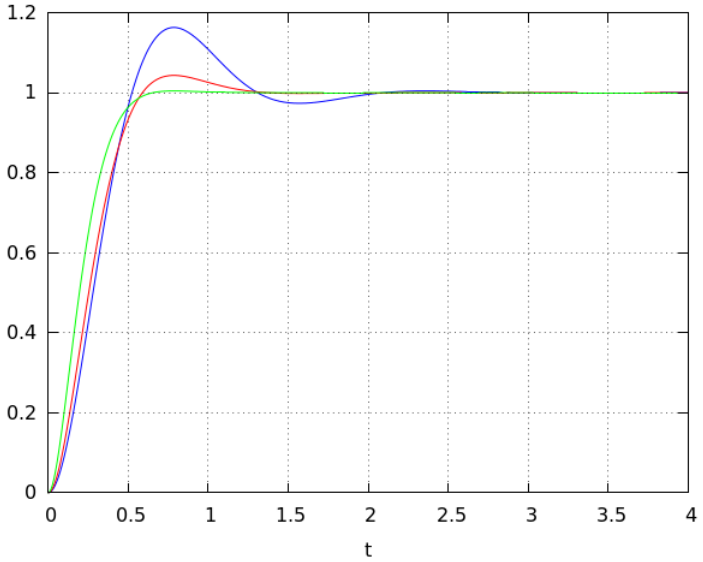
$$\zeta = \frac{1}{4} \qquad \zeta = \frac{\sqrt{2}}{4} \qquad \zeta = \frac{\sqrt{3}}{4}$$

$$\omega_n = 4\left(\frac{1}{4}\right)^{-1} \qquad \omega_n = 4\left(\frac{\sqrt{2}}{4}\right)^{-1} \qquad \omega_n = 4\left(\frac{\sqrt{3}}{4}\right)^{-1}$$

$$\sigma_d = \zeta\omega_n = 4$$

2nd Order Systems – Underdamping





$$\zeta = \frac{1}{2} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{3}}{2} \qquad \omega_n = 2\left(\frac{\sqrt{3}}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{2}}{2} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{2}}{2} \qquad \omega_n = 2\left(\frac{\sqrt{2}}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{3}}{2} \qquad \sqrt{1-\zeta^2} = \frac{1}{2} \qquad \omega_n = 2\left(\frac{1}{2}\right)^{-1}$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 2$$

$$\zeta = \frac{1}{2} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{3}}{2} \qquad \omega_n = 4\left(\frac{\sqrt{3}}{2}\right)^{-1}$$

$$\zeta = \frac{\sqrt{2}}{2} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{2}}{2} \qquad \omega_n = 4\left(\frac{\sqrt{2}}{2}\right)^{-1}$$

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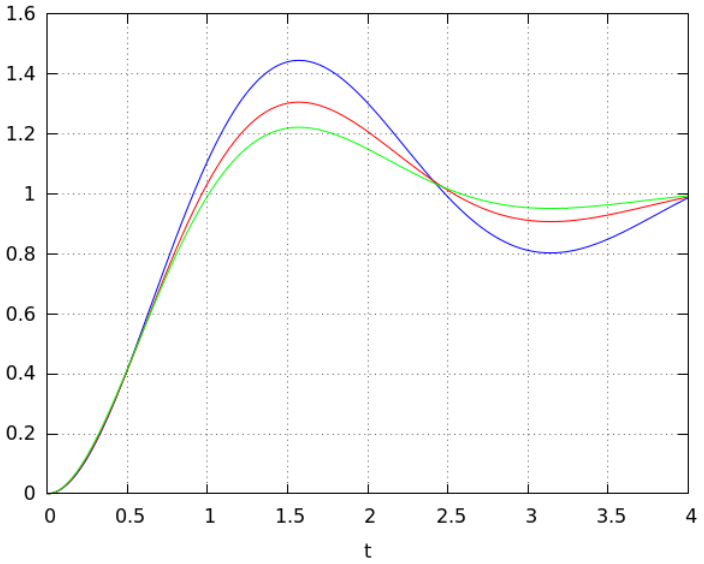
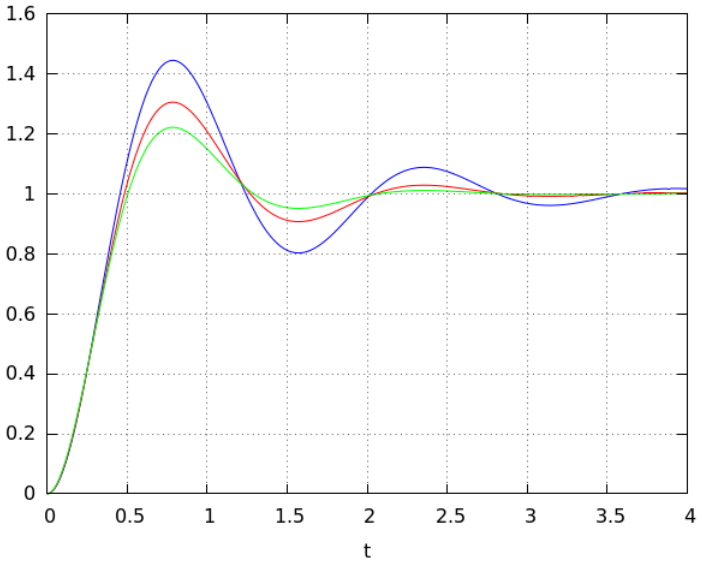
$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 4$$

2nd Order Systems – Underdamping





$$\zeta = \frac{1}{4} \qquad \zeta = \frac{\sqrt{2}}{4} \qquad \zeta = \frac{\sqrt{3}}{4}$$

$$\sqrt{1-\zeta^2} = \frac{\sqrt{15}}{4} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{14}}{4} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{13}}{4}$$

$$\omega_n = 2\left(\frac{\sqrt{15}}{4}\right)^{-1} \qquad \omega_n = 2\left(\frac{\sqrt{14}}{4}\right)^{-1} \qquad \omega_n = 2\left(\frac{\sqrt{13}}{4}\right)^{-1}$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 2$$

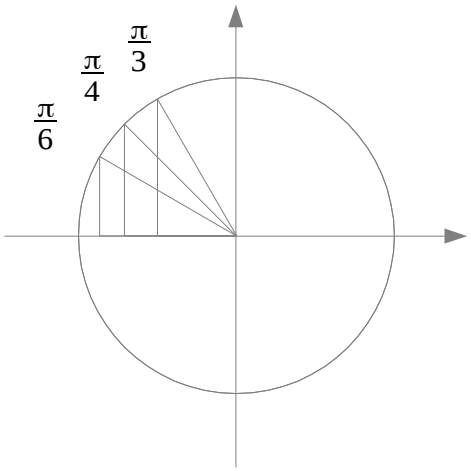
$$\zeta = \frac{1}{4} \qquad \zeta = \frac{\sqrt{2}}{4} \qquad \zeta = \frac{\sqrt{3}}{4}$$

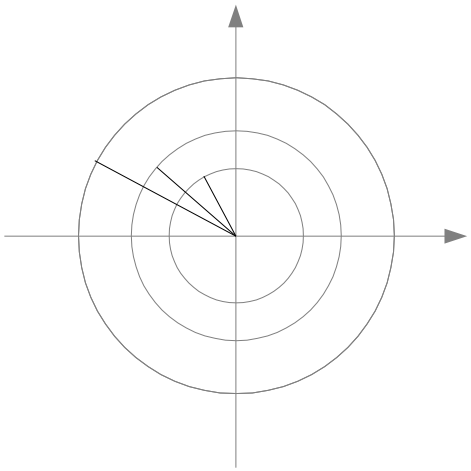
$$\sqrt{1-\zeta^2} = \frac{\sqrt{15}}{4} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{14}}{4} \qquad \sqrt{1-\zeta^2} = \frac{\sqrt{13}}{4}$$

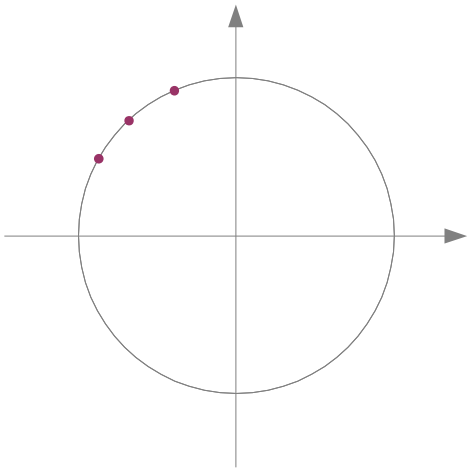
$$\omega_n = 4\left(\frac{\sqrt{15}}{4}\right)^{-1} \qquad \omega_n = 4\left(\frac{\sqrt{14}}{4}\right)^{-1} \qquad \omega_n = 4\left(\frac{\sqrt{13}}{4}\right)^{-1}$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 4$$

2nd Order Systems – Underdamping







References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”