CLTI Impulse Matching (6A)

•

Young Won Lim
7/30/15

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

- Impulse Matching Method

Impulse Matching Method Summary

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$\begin{aligned} h(0^-) &= 0 \\ h^{(1)}(0^-) &= 0 \\ &\vdots \\ h^{(N-2)}(0^-) &= 0 \\ h^{(N-1)}(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} h(0^+) &= k_0 \\ h^{(1)}(0^+) &= k_1 \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} \\ h^{(N-1)}(0^+) &= k_{N-1} \end{aligned}$$

assumed finite jumps

$$\begin{aligned} h^{(1)}(t) &= k_0 \delta(t) \\ h^{(2)}(t) &= k_1 \delta(t) + \textstyle\sum_i k_i \delta^{(j)}(t) \\ &\vdots \\ h^{(N-1)}(t) &= k_{N-2} \delta(t) + \textstyle\sum_i k_i \delta^{(j)}(t) \\ h^{(N)}(t) &= k_{N-1} \delta(t) + \textstyle\sum_i k_i \delta^{(j)}(t) \end{aligned}$$

$$h^{(N)}(0) + a_1 h^{(N-1)}(0) + \cdots + a_{N-1} h^{(1)}(0) + a_N h(0) = b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \cdots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$$

determine k_i — *Impulse matching*

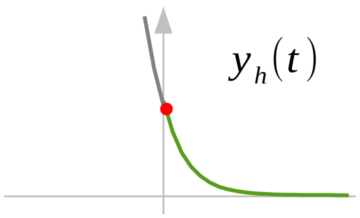
determine c_i

$$h(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}\right) \cdot u(t)$$

Impulse Response & Particular Solution

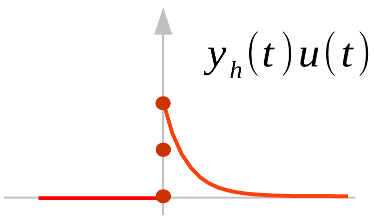
$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1}\frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1}\frac{dx(t)}{dt} + b_M x(t)$$

$y(t) = h(t)$: impulse response   when $x(t) = \delta(t)$: forcing function



forcing function

	causal system		valid interval
	$t < 0$	$t = 0$	$t > 0$
$x(t) =$	0	$\delta(t)$	0



		$t < 0$	$t = 0$	$t > 0$
particular solution	$y_p(t)$	0	$b_0\delta(t)$	0
homogeneous solution	$y_h(t)$	0	$y_h(t)$	$y_h(t)$

impulse response

$h(t) = y_h(t)u(t)$ $(N>M)$

$h(t) = y_h(t)u(t) + b_0\delta(t)$ $(N=M)$

Derivatives of $y_h(t) \cdot u(t)$

$$h(t) = y_h(t)u(t)$$

Pascal's triangle

0:									1								
1:								1		1							
2:							1		2		1						
3:						1		3		3		1					
4:					1		4		6		4		1				
5:				1		5		10		10		5		1			
6:			1		6		15		20		15		6		1		
7:		1		7		21		35		35		21		7		1	
8:	1		8		28		56		70		56		28		8		1

$$h^{(0)}(t) = y_h(t)u(t)$$

$$h^{(1)}(t) = y_h^{(1)}(t)u(t) + y_h(t)u^{(1)}(t)$$

$$h^{(2)}(t) = y_h^{(2)}(t)u(t) + 2\,y_h^{(1)}(t)u^{(1)}(t) + y_h(t)u^{(2)}(t)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$h^{(N)}(t) = \binom{N}{0} y_h^{(N)}(t)u(t) + \binom{N}{1} y_h^{(N-1)}(t)u^{(1)}(t) + \cdots \binom{N}{N-1} y_h^{(1)}(t)u^{(N-1)}(t) + \binom{N}{N} y_h^{(0)}(t)u^{(N)}(t)$$

$$= \binom{N}{0} y_h^{(N)}(t)u(t) + \binom{N}{1} y_h^{(N-1)}(t)\delta(t) + \cdots \binom{N}{N-1} y_h^{(1)}(t)\delta^{(N-2)}(t) + \binom{N}{N} y_h^{(0)}(t)\delta^{(N-1)}(t)$$

Applying delta function properties, later

$$h^{(0)}(t) = y_h(t)u(t)$$
$$h^{(1)}(t) = y_h^{(1)}(t)u(t) + y_h(t)\delta^{(0)}(t)$$
$$h^{(2)}(t) = y_h^{(2)}(t)u(t) + 2\,y_h^{(1)}(t)\delta^{(0)}(t) + y_h(t)\delta^{(1)}(t)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$h^{(N)}(t) = \binom{N}{0} y_h^{(N)}(t)u(t) + \binom{N}{1} y_h^{(N-1)}(t)\delta(t) + \cdots \binom{N}{N-1} y_h^{(1)}(t)\delta^{(N-2)}(t) + \binom{N}{N} y_h^{(0)}(t)\delta^{(N-1)}(t)$$

$$f(t)\delta(t) = f(0)\delta(t)$$

$$h^{(0)}(t) = y_h(t)u(t)$$
$$h^{(1)}(t) = y_h^{(1)}(t)u(t) + y_h(0)\delta^{(0)}(t)$$
$$h^{(2)}(t) = y_h^{(2)}(t)u(t) + 2\,y_h^{(1)}(0)\delta^{(0)}(t) + y_h(0)\delta^{(1)}(t)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$h^{(N)}(t) = y_h^{(N)}(t)u(t) + \binom{N}{1} y_h^{(N-1)}(0)\delta^{(0)}(t) + \cdots \binom{N}{N-1} y_h^{(1)}(0)\delta^{(N-2)}(t) + y_h^{(0)}(0)\delta^{(N-1)}(t)$$

Applying delta function properties, first

$$h(t) = y_h(t)u(t)$$

$$f(t)\delta(t) = f(0)\delta(t)$$

$$h^{(0)}(t) = y_h(t)u(t)$$

$$h^{(1)}(t) = y_h^{(1)}(t)u(t) + y_h(t)\delta^{(0)}(t) \Rightarrow y_h^{(1)}(t)u(t) + y_h(0)\delta^{(0)}(t)$$

$$h^{(2)}(t) = y_h^{(2)}(t)u(t) + y_h^{(1)}(t)\delta^{(0)}(t) + y_h(0)\delta^{(1)}(t) \Rightarrow y_h^{(2)}(t)u(t) + y_h^{(1)}(0)\delta^{(0)}(t) + y_h(0)\delta^{(1)}(t)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$h^{(N)}(t) = y_h^{(N)}(t)u(t) + y_h^{(N-1)}(0)\delta^{(0)}(t) + \cdots \; y_h^{(1)}(0)\delta^{(N-2)}(t) + y_h^{(0)}(0)\delta^{(N-1)}(t)$$

Values of $h^{(i)}(0+)$

$$h(t) = y_h(t)u(t)$$

$$f(t)\delta(t) = f(0)\delta(t)$$

$$h^{(0)}(t) = y_h(t)u(t)$$

$$h^{(1)}(t) = y_h^{(1)}(t)u(t) + y_h(t)\delta^{(0)}(t) \Rightarrow y_h^{(1)}(t)u(t) + y_h(0)\delta^{(0)}(t)$$

$$h^{(2)}(t) = y_h^{(2)}(t)u(t) + y_h^{(1)}(t)\delta^{(0)}(t) + y_h(0)\delta^{(1)}(t) \Rightarrow y_h^{(2)}(t)u(t) + y_h^{(1)}(0)\delta^{(0)}(t) + y_h(0)\delta^{(1)}(t)$$

$$h^{(0)}(0^+) = y_h(0^+)$$

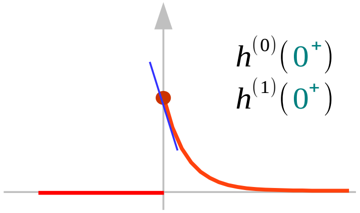
$$h^{(1)}(0^+) = y_h^{(1)}(0^+)$$

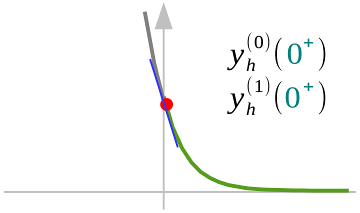
$$h^{(2)}(0^+) = y_h^{(2)}(0^+)$$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$h^{(N)}(t) = y_h^{(N)}(t)u(t) + y_h^{(N-1)}(0)\delta^{(0)}(t) + \cdots y_h^{(1)}(0)\delta^{(N-2)}(t) + y_h^{(0)}(0)\delta^{(N-1)}(t)$$

$$h^{(N)}(0^+) = y_h^{(N)}(0^+)$$





- Determining the Coefficients of an Impulse Response

Case 1) N > M : N-1 = M

Differentiate each initial condition
Substitute the differentiated $h(t)$
Find the finite jumps k_i
Find c_i in $y_h(t)$

Case 2) N > M : N-2 = M

Case 3) N = M

Differentiate each initial condition

(N > M : N-1 = M)

$$\begin{aligned} h(0^-) &= 0 \\ h^{(1)}(0^-) &= 0 \\ &\vdots \\ h^{(N-2)}(0^-) &= 0 \\ h^{(N-1)}(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} h(0^+) &= k_0 \\ h^{(1)}(0^+) &= k_1 \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} \\ h^{(N-1)}(0^+) &= k_{N-1} \end{aligned}$$

assumed finite jumps

$$\begin{aligned} h^{(1)}(t) &= k_0\delta(t) \\ h^{(2)}(t) &= k_1\delta(t) + k_0\delta^{(1)}(t) \\ &\vdots \\ h^{(N-1)}(t) &= k_{N-2}\delta(t) + k_{N-3}\delta^{(1)}(t) + \cdots + k_0\delta^{(N-2)}(t) \\ h^{(N)}(t) &= k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1\delta^{(N-2)}(t) + k_0\delta^{(N-1)}(t) \end{aligned}$$

$h^{(i)}(t)$

substitute

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-2} h^{(2)}(t) + a_{N-1} h^{(1)}(t) + a_N h(t)$$

$$= b_0\delta^{(M)}(t) + b_1\delta^{(M-1)}(t) + \cdots + b_{M-1}\delta^{(1)}(t) + b_M\delta(t)$$

Impulse matching

N variables / N equations

determine

$k_0,\ k_1,\ \cdots\ k_{N-2},\ k_{N-1}$

determine

$c_0,\ c_1,\ \cdots\ c_{N-2},\ c_{N-1}$

$$h(t) = (c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}) \cdot u(t)$$

Substitute the differentiated *h*(*t*)

($N > M$: $N-1 = M$)

substitute

Impulse matching

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) \leftrightarrow b_0 \delta^{(M)}(t) + b_1 \delta^{(M-1)}(t) + \cdots + b_{M-1} \delta^{(1)}(t) + b_M \delta(t)$$

$a_{N-1} \times$ $\quad h^{(1)}(t) = k_0 \delta(t)$

$a_{N-2} \times$ $\quad h^{(2)}(t) = k_1 \delta(t) + k_0 \delta^{(1)}(t)$

$\vdots$

$a_1 \times$ $\quad h^{(N-1)}(t) = k_{N-2} \delta(t) + k_{N-3} \delta^{(1)}(t) + \cdots + k_0 \delta^{(N-2)}(t)$

$$h^{(N)}(t) = k_{N-1} \delta(t) + k_{N-2} \delta^{(1)}(t) + \cdots + k_1 \delta^{(N-2)}(t) + k_0 \delta^{(N-1)}(t)$$

$b_M \delta(t)$ $\quad b_{M-1} \delta^{(1)}(t)$ $\quad b_1 \delta^{(M-1)}(t)$ $\quad b_0 \delta^{(M)}(t)$

Find the finite jumps k_i

(N > M :   N-1 = M)

$$
\begin{array}{ll}
a_{N-1} \times & h^{(1)}(t) = k_0\,\delta(t) \\
a_{N-2} \times & h^{(2)}(t) = k_1\,\delta(t) + k_0\,\delta^{(1)}(t) \\
 & \vdots \\
a_1 \times & h^{(N-1)}(t) = k_{N-2}\,\delta(t) + k_{N-3}\,\delta^{(1)}(t) + \cdots + k_0\,\delta^{(N-2)}(t) \\
 & h^{(N)}(t) = k_{N-1}\,\delta(t) + k_{N-2}\,\delta^{(1)}(t) + \cdots + k_1\,\delta^{(N-2)}(t) + k_0\,\delta^{(N-1)}(t)
\end{array}
$$

$$
b_M\,\delta(t) \qquad b_{M-1}\,\delta^{(1)}(t) \qquad b_1\,\delta^{(M-1)}(t) \qquad b_0\,\delta^{(M)}(t)
$$

N variables / N equations

determine

$k_0,\ k_1,\ \cdots\ k_{N-2},\ k_{N-1}$

$$
\left\{
\begin{array}{l}
k_{N-1} + a_1 k_{N-2} \cdots + a_{N-2} k_1 + a_{N-1} k_0 = b_M \\
k_{N-2} \cdots + a_{N-3} k_1 + a_{N-2} k_0 = b_{M-1} \\
\cdots \quad \cdots \quad \cdots \\
k_2 + a_1 k_1 + a_2 k_0 = b_2 \\
k_1 + a_1 k_0 = b_1 \\
k_0 = b_0
\end{array}
\right.
$$

Find c_i in $y_h(t)$

($N > M$: $N-1 = M$)

$$\begin{aligned} h(0^+) &= k_0 \\ h^{(1)}(0^+) &= k_1 \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} \\ h^{(N-1)}(0^+) &= k_{N-1} \end{aligned} \quad \Rightarrow \quad \begin{aligned} &y_n(0^+) \\ &y_n^{(1)}(0^+) \\ &\vdots \\ &y_n^{(N-2)}(0^+) \\ &y_n^{(N-1)}(0^+) \end{aligned}$$

found finite jumps

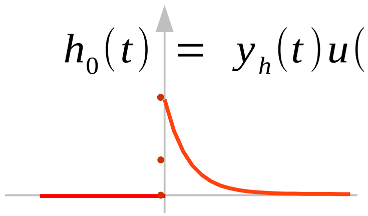
$$\begin{aligned} y_n(t) &= c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-1} e^{\lambda_{N-1} t} \\ y_n^{(1)}(t) &= c_0 \lambda_0 e^{\lambda_0 t} + c_1 \lambda_1 e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1} e^{\lambda_{N-1} t} \\ y_n^{(N-2)}(t) &= c_0 \lambda_0^{(N-2)} e^{\lambda_0 t} + c_1 \lambda_1^{(N-2)} e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1}^{(N-2)} e^{\lambda_{N-1} t} \\ y_n^{(N-1)}(t) &= c_0 \lambda_0^{(N-1)} e^{\lambda_0 t} + c_1 \lambda_1^{(N-1)} e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1}^{(N-1)} e^{\lambda_{N-1} t} \end{aligned}$$

substitute

$$\begin{aligned} h(0^+) &= k_0 = y_n(0^+) = c_0 + c_1 + \cdots + c_{N-1} \\ h^{(1)}(0^+) &= k_1 = y_n^{(1)}(0^+) = c_0 \lambda_0 + c_1 \lambda_1 + \cdots + c_{N-1} \lambda_{N-1} \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} = y_n^{(N-2)}(0^+) = c_0 \lambda_0^{(N-2)} + c_1 \lambda_1^{(N-2)} + \cdots + c_{N-1} \lambda_{N-1}^{(N-2)} \\ h^{(N-1)}(0^+) &= k_{N-1} = y_n^{(N-1)}(0^+) = c_0 \lambda_0^{(N-1)} + c_1 \lambda_1^{(N-1)} + \cdots + c_{N-1} \lambda_{N-1}^{(N-1)} \end{aligned}$$

assume distinct roots

$$h_0(t) = y_h(t) u(t)$$



determine
$c_0, \; c_1, \; \cdots \; c_{N-2}, \; c_{N-1}$

$$h(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N-1} t}\right) \cdot u(t)$$

- Determining the Coefficients of an Impulse Response

Case 1) N > M : N-1 = M

Case 2) N > M : N-2 = M

Differentiate each initial condition
Substitute the differentiated $h(t)$
Find the finite jumps k_i
Find c_i in $y_h(t)$

Case 3) N = M

Simplified Impulse Matching Method (2)

$$P(D) = (b_0 D^M + \cdots + b_{N-1} D + b_N)$$

$$h(t) = P(D)[y_n(t)u(t)] \Rightarrow [P(D)y_n(t)]u(t)$$

$$\begin{aligned}
b_M &\times & h^{(0)}(t) &= y_h(t)u(t) \\
b_{M-1} &\times & h^{(1)}(t) &= y_h^{(1)}(t)u(t) \\
b_{M-2} &\times & h^{(2)}(t) &= y_h^{(2)}(t)u(t) \\
b_0 &\times & h^{(N-1)}(t) &= y_h^{(N-1)}(t)u(t) \\
& & h^{(N)}(t) &= y_h^{(N)}(t)u(t) + \delta(t)
\end{aligned}$$

$$\begin{aligned}
b_M &\times & h^{(0)}(t) &= y_h(t)u(t) \\
b_{M-1} &\times & h^{(1)}(t) &= y_h^{(1)}(t)u(t) \\
b_{M-2} &\times & h^{(2)}(t) &= y_h^{(2)}(t)u(t) \\
b_1 &\times & h^{(N-1)}(t) &= y_h^{(N-1)}(t)u(t) \\
b_0 &\times & h^{(N)}(t) &= y_h^{(N)}(t)u(t) + \delta(t)
\end{aligned}$$

$$\begin{aligned}
b_M &\times y_h(t) \\
b_{M-1} &\times y_h^{(1)}(t) \\
b_{M-2} &\times y_h^{(2)}(t) \\
b_0 &\times y_h^{(N-1)}(t)
\end{aligned}$$

$$\begin{aligned}
b_M &\times y_h(t) \\
b_{M-1} &\times y_h^{(1)}(t) \\
b_{M-2} &\times y_h^{(2)}(t) \\
b_1 &\times y_h^{(N-1)}(t) \\
b_0 &\times y_h^{(N)}(t) + \delta(t)
\end{aligned}$$

Differentiate each initial condition

(N > M : N-2 = M)

$$
\begin{aligned}
h(0^-) &= 0 \\
h^{(1)}(0^-) &= 0 \\
&\vdots \\
h^{(N-2)}(0^-) &= 0 \\
h^{(N-1)}(0^-) &= 0
\end{aligned}
$$

initially at rest

$$
\begin{aligned}
h(0^+) &= k_0 \\
h^{(1)}(0^+) &= k_1 \\
&\vdots \\
h^{(N-2)}(0^+) &= k_{N-2} \\
h^{(N-1)}(0^+) &= k_{N-1}
\end{aligned}
$$

assumed finite jumps

$$
\begin{aligned}
h^{(1)}(t) &= k_0 = 0 \\
h^{(2)}(t) &= k_1\delta(t) + 0 \\
&\vdots \\
h^{(N-1)}(t) &= k_{N-2}\delta(t) + k_{N-3}\delta^{(1)}(t) + \cdots + 0 \\
h^{(N)}(t) &= k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1\delta^{(N-2)}(t) + 0
\end{aligned}
$$

→ continuous (no jump)

$h^{(i)}(t)$

substitute

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-3} h^{(3)}(t) + a_{N-2} h^{(2)}(t) + a_{N-1} h^{(1)}(t) + a_N h(t)$$

$$= b_0 \delta^{(M)}(t) + b_1 \delta^{(M-1)}(t) + \cdots + b_{M-1}\delta^{(1)}(t) + b_M \delta(t)$$

Impulse matching

N-1 variables / N-1 equations

determine
$k_0, \ k_1, \ \cdots \ k_{N-2}, \ k_{N-1}$

determine
$c_0, \ c_1, \ \cdots \ c_{N-2}, \ c_{N-1}$

$$h(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}\right)\cdot u(t)$$

Substitute the differentiated *h*(*t*)

(N > M : N-2 = M)

substitute

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-3} h^{(3)}(t) + a_{N-2} h^{(2)}(t) + a_{N-1} h^{(1)}(t) + a_N h(t)$$

Impulse matching

$$= b_0 \delta^{(M)}(t) + b_1 \delta^{(M-1)}(t) + \cdots + b_{M-1} \delta^{(1)}(t) + b_M \delta(t)$$

$a_{N-1} \times$ $h^{(1)}(t) = k_0 = 0$

$a_{N-2} \times$ $h^{(2)}(t) = k_1 \delta(t)$ $+ k_0 = 0$

$\vdots$

$a_1 \times$ $h^{(N-1)}(t) = k_{N-2}\delta(t) + k_{N-3}\delta^{(1)}(t) + \cdots + k_0 = 0$

$h^{(N)}(t) = k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1 \delta^{(N-2)}(t) + k_0 = 0$

$b_M \delta(t)$ $\quad b_{M-1}\delta^{(1)}(t)$ $\quad b_0 \delta^{(M)}(t)$

Find the finite jumps k_i

($N > M$: $N-2 = M$)

$$\begin{aligned}
a_{N-1} \times \quad & h^{(1)}(t) = k_0 = 0 \\
a_{N-2} \times \quad & h^{(2)}(t) = k_1\delta(t) + k_0 = 0 \\
& \vdots \\
a_1 \times \quad & h^{(N-1)}(t) = k_{N-2}\delta(t) + k_{N-3}\delta^{(1)}(t) + \cdots + k_0 = 0 \\
& h^{(N)}(t) = k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1\delta^{(N-2)}(t) + k_0 = 0
\end{aligned}$$

$$b_M\delta(t) \qquad b_{M-1}\delta^{(1)}(t) \qquad b_0\delta^{(M)}(t)$$

N-1 variables / N-1 equations

determine
$k_0=0,\ k_1,\ \cdots\ k_{N-2},\ k_{N-1}$

$$\begin{cases}
k_{N-1} + a_1 k_{N-2} \cdots + a_{N-2}k_1 + a_{N-1}k_0 = b_M \\
k_{N-2} \cdots + a_{N-3}k_1 + a_{N-2}k_0 = b_{M-1} \\
\cdots \quad \cdots \quad \cdots \\
k_2 + a_1 k_1 + a_2 k_0 = b_2 \\
k_1 + a_1 k_0 = b_1
\end{cases}$$

$$k_0 = 0$$

Find c_i in $y_h(t)$

$(N > M : \quad N-2 = M)$

$$\begin{aligned}
h(0^+) &= k_0 = 0 &\Rightarrow\ & y_n(0^+) \\
h^{(1)}(0^+) &= k_1 &\Rightarrow\ & y_n^{(1)}(0^+) \\
&\vdots & & \vdots \\
h^{(N-2)}(0^+) &= k_{N-2} &\Rightarrow\ & y_n^{(N-2)}(0^+) \\
h^{(N-1)}(0^+) &= k_{N-1} &\Rightarrow\ & y_n^{(N-1)}(0^+)
\end{aligned}$$

found finite jumps

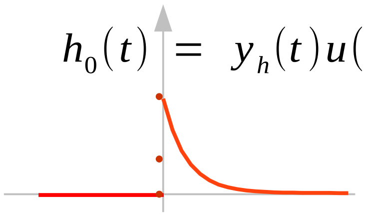
$$\begin{aligned}
y_n(t) &= c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-1} e^{\lambda_{N-1} t} \\
y_n^{(1)}(t) &= c_0 \lambda_0 e^{\lambda_0 t} + c_1 \lambda_1 e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1} e^{\lambda_{N-1} t} \\
y_n^{(N-2)}(t) &= c_0 \lambda_0^{(N-2)} e^{\lambda_0 t} + c_1 \lambda_1^{(N-2)} e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1}^{(N-2)} e^{\lambda_{N-1} t} \\
y_n^{(N-1)}(t) &= c_0 \lambda_0^{(N-1)} e^{\lambda_0 t} + c_1 \lambda_1^{(N-1)} e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1}^{(N-1)} e^{\lambda_{N-1} t}
\end{aligned}$$

substitute

$$\begin{aligned}
h(0^+) &= k_0 = 0 = y_n(0^+) &&= c_0 + c_1 + \cdots + c_{N-1} \\
h^{(1)}(0^+) &= k_1 = y_n^{(1)}(0^+) &&= c_0\lambda_0 + c_1\lambda_1 + \cdots + c_{N-1}\lambda_{N-1} \\
&\vdots \\
h^{(N-2)}(0^+) &= k_{N-2} = y_n^{(N-2)}(0^+) &&= c_0\lambda_0^{(N-2)} + c_1\lambda_1^{(N-2)} + \cdots + c_{N-1}\lambda_{N-1}^{(N-2)} \\
h^{(N-1)}(0^+) &= k_{N-1} = y_n^{(N-1)}(0^+) &&= c_0\lambda_0^{(N-1)} + c_1\lambda_1^{(N-1)} + \cdots + c_{N-1}\lambda_{N-1}^{(N-1)}
\end{aligned}$$

assume distinct roots

$$h_0(t) = y_h(t)u(t)$$



determine
$c_0,\ c_1,\ \cdots\ c_{N-2},\ c_{N-1}$

$$h(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}\right)\cdot u(t)$$

- Determining the Coefficients of an Impulse Response

Case 1) N > M : N-1 = M

Case 2) N > M : N-1 = M

Case 3) N = M

Differentiate each initial condition
A $\delta(t)$ needs to be in $h(t)$
Substitute the differentiated $h(t)$
Find the finite jumps k_i
Find c_i in $y_h(t)$

Differentiate each initial condition (N = M)

$$\begin{aligned} h(0^-) &= 0 \\ h^{(1)}(0^-) &= 0 \\ &\vdots \\ h^{(N-2)}(0^-) &= 0 \\ h^{(N-1)}(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} h(0^+) &= k_0 \\ h^{(1)}(0^+) &= k_1 \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} \\ h^{(N-1)}(0^+) &= k_{N-1} \end{aligned}$$

assumed finite jumps

$$\begin{aligned} h^{(1)}(t) &= k_0\delta(t) \\ h^{(2)}(t) &= k_1\delta(t) + k_0\delta^{(1)}(t) \\ &\vdots \\ h^{(N-1)}(t) &= k_{N-2}\delta(t) + k_{N-3}\delta^{(1)}(t) + \cdots + k_0\delta^{(N-2)}(t) \\ h^{(N)}(t) &= k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1\delta^{(N-2)}(t) + k_0\delta^{(N-1)}(t) \end{aligned}$$

$h^{(i)}(t)$

substitute

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t)$$

$$= b_0\delta^{(M)}(t) + b_1\delta^{(M-1)}(t) + \cdots + b_{M-1}\delta^{(1)}(t) + b_M\delta(t)$$

h(*t*) must include *δ*(*t*)

Impulse matching

N variables / N equations

determine $k_0, \; k_1, \; \cdots \; k_{N-2}, \; k_{N-1}$

determine $c_0, \; c_1, \; \cdots \; c_{N-2}, \; c_{N-1}$

$$h(t) = y_h(t)\cdot u(t) + b_0\delta(t)$$

$$h(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}\right)\cdot u(t) + b_0\delta(t)$$

A $\delta(t)$ needs to be in $h(t)$ (N = M)

$$h(t) = y_h(t)u(t) + b_0\delta(t)$$

$+b_0\delta(t)$	⟺	$+b_N\delta(t)$

$$\begin{aligned}
h^{(1)}(0) &= k_0\delta(t) & +b_0\delta^{(1)}(t) &\Leftrightarrow +b_{N-1}\delta^{(1)}(t)\\
h^{(2)}(0) &= k_1\delta(t) + k_0\delta^{(1)}(t) & +b_0\delta^{(2)}(t) &\Leftrightarrow +b_{M-2}\delta^{(2)}(t)\\
&\vdots\\
h^{(N-1)}(0) &= k_{N-2}\delta(t) + \cdots + k_1\delta^{(N-1)}(t) + k_0\delta^{(N-2)}(t) & +b_0\delta^{(N-1)}(t) &\Leftrightarrow +b_1\delta^{(N-1)}(t)\\
h^{(N)}(0) &= k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + k_1\delta^{(N-2)}(t) + k_0\delta^{(N-1)}(t) & +b_0\delta^{(N)}(t) &\Leftrightarrow +b_0\delta^{(N)}(t)
\end{aligned}$$

$\delta^{(N)}(t)$ $\delta^{(N-1)}(t)$ $\delta^{(1)}(t)$

$$h^{(N)}(0) + a_1 h^{(N-1)}(0) + \cdots + a_{N-1}h^{(1)}(0) + a_N h(0) = b_0\delta^{(N)}(t) + b_1\delta^{(N-1)}(t) + \cdots + b_{N-1}\delta^{(1)}(t) + b_N\delta(t)$$

Impulse matching

Substitute the differentiated $h(t)$ $(N = M)$

substitute

Impulse matching

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) \Leftrightarrow b_0 \delta^{(M)}(t) + b_1 \delta^{(M-1)}(t) + \cdots + b_{M-1} \delta^{(1)}(t) + b_M \delta(t)$$

$b_0 \delta(t)$

$$\begin{aligned}
a_{N-1} \times \quad h^{(1)}(t) &= k_0 \delta(t) + b_0 \delta^{(1)}(t) \\
a_{N-2} \times \quad h^{(2)}(t) &= k_1 \delta(t) + k_0 \delta^{(1)}(t) + b_0 \delta^{(2)}(t) \\
&\vdots \\
a_1 \times \quad h^{(N-1)}(t) &= k_{N-2} \delta(t) + k_{N-3} \delta^{(1)}(t) + \cdots + k_0 \delta^{(N-2)}(t) + b_0 \delta^{(N-1)}(t) \\
h^{(N)}(t) &= k_{N-1} \delta(t) + k_{N-2} \delta^{(1)}(t) + \cdots + k_1 \delta^{(N-2)}(t) + k_0 \delta^{(N-1)}(t) + b_0 \delta^{(N)}(t)
\end{aligned}$$

$b_M \delta(t)$ $\quad b_{M-1} \delta^{(1)}(t)$ $\quad b_1 \delta^{(M-1)}(t)$ $\quad b_0 \delta^{(M)}(t)$

Find the finite jumps k_i

$(N = M)$

$b_0\delta(t)$

$$a_{N-1} \times \quad h^{(1)}(t) = k_0\delta(t) + b_0\delta^{(1)}(t)$$
$$a_{N-2} \times \quad h^{(2)}(t) = k_1\delta(t) + k_0\delta^{(1)}(t) + b_0\delta^{(2)}(t)$$
$$\vdots$$
$$a_1 \times \quad h^{(N-1)}(t) = k_{N-2}\delta(t) + k_{N-3}\delta^{(1)}(t) + \cdots + \cdots + k_0\delta^{(N-2)}(t) + b_0\delta^{(N-1)}(t)$$
$$h^{(N)}(t) = k_{N-1}\delta(t) + k_{N-2}\delta^{(1)}(t) + \cdots + \cdots + k_1\delta^{(N-2)}(t) + k_0\delta^{(N-1)}(t) + b_0\delta^{(N)}(t)$$

$b_M\delta(t)$ $\quad b_{M-1}\delta^{(1)}(t)$ $\quad b_1\delta^{(M-1)}(t)$ $\quad b_0\delta^{(M)}(t)$

$$
\begin{cases}
k_{N-1} + a_1 k_{N-2} \cdots + a_{N-2}k_1 + a_{N-1}k_0 + b_0 = b_M \\
k_{N-2} \cdots + a_{N-3}k_1 + a_{N-2}k_0 + b_0 = b_{M-1} \\
\cdots \quad \cdots \quad \cdots \\
k_2 + a_1 k_1 + a_2 k_0 + b_0 = b_3 \\
k_1 + a_1 k_0 + b_0 = b_2 \\
k_0 + b_0 = b_1
\end{cases}
$$

$$+b_0 = b_0$$

N variables / N equations

determine
$k_0, \; k_1, \; \cdots \; k_{N-2}, \; k_{N-1}$

Find c_i in $y_h(t)$

(N = M)

$$\begin{aligned} h(0^+) &= k_0 \\ h^{(1)}(0^+) &= k_1 \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} \\ h^{(N-1)}(0^+) &= k_{N-1} \end{aligned} \quad \Rightarrow \quad \begin{aligned} & y_n(0^+) \\ & y_n^{(1)}(0^+) \\ & \vdots \\ & y_n^{(N-2)}(0^+) \\ & y_n^{(N-1)}(0^+) \end{aligned}$$

found finite jumps

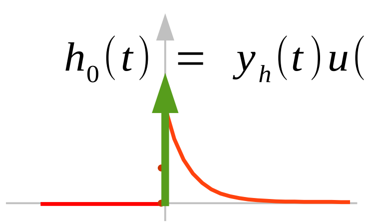
$$\begin{aligned} y_n(t) &= c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-1} e^{\lambda_{N-1} t} \\ y_n^{(1)}(t) &= c_0 \lambda_0 e^{\lambda_0 t} + c_1 \lambda_1 e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1} e^{\lambda_{N-1} t} \\ y_n^{(N-2)}(t) &= c_0 \lambda_0^{(N-2)} e^{\lambda_0 t} + c_1 \lambda_1^{(N-2)} e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1}^{(N-2)} e^{\lambda_{N-1} t} \\ y_n^{(N-1)}(t) &= c_0 \lambda_0^{(N-1)} e^{\lambda_0 t} + c_1 \lambda_1^{(N-1)} e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1}^{(N-1)} e^{\lambda_{N-1} t} \end{aligned}$$

substitute

$$\begin{aligned} h(0^+) &= k_0 = y_n(0^+) &&= c_0 + c_1 + \cdots + c_{N-1} \\ h^{(1)}(0^+) &= k_1 = y_n^{(1)}(0^+) &&= c_0 \lambda_0 + c_1 \lambda_1 + \cdots + c_{N-1} \lambda_{N-1} \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} = y_n^{(N-2)}(0^+) &&= c_0 \lambda_0^{(N-2)} + c_1 \lambda_1^{(N-2)} + \cdots + c_{N-1} \lambda_{N-1}^{(N-2)} \\ h^{(N-1)}(0^+) &= k_{N-1} = y_n^{(N-1)}(0^+) &&= c_0 \lambda_0^{(N-1)} + c_1 \lambda_1^{(N-1)} + \cdots + c_{N-1} \lambda_{N-1}^{(N-1)} \end{aligned}$$

assume distinct roots

$$h_0(t) = y_h(t) u(t) + b_0 \delta(t)$$



determine

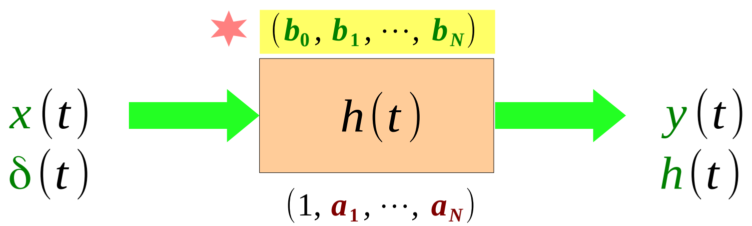
$c_0, \ c_1, \ \cdots \ c_{N-2}, \ c_{N-1}$

$$h(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}\right) \cdot u(t) + b_0 \delta(t)$$

- Superposition of an Impulse Function and its Derivatives (General System S) and (Base System S0)

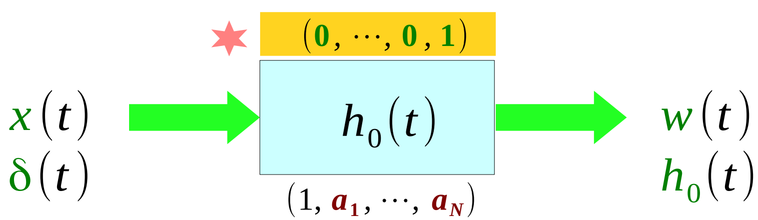
A General System S and a Base System S0

General System S



$h(t)$: the impulse response of **S**

Base System S0



$h_0(t)$: the impulse response of **S0**

ZIR of a Base System S0

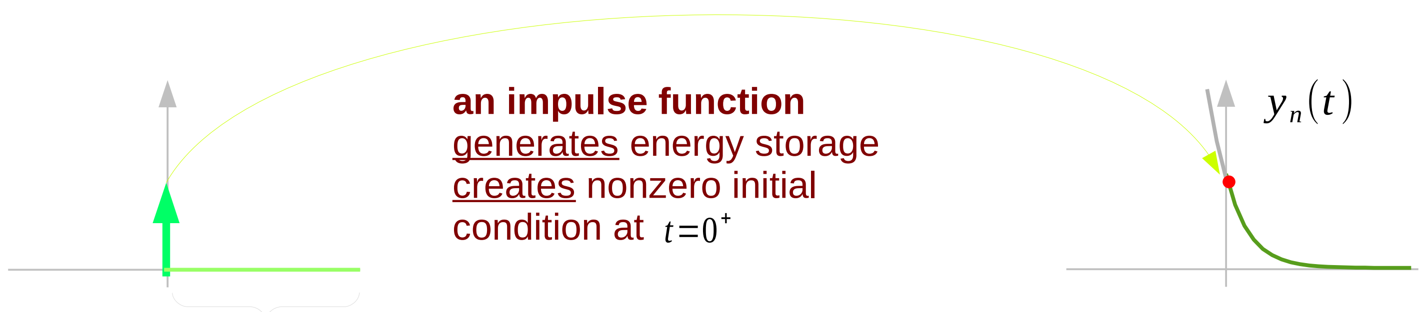
N ≥ M : $h_0(t)$ include no impulse $\delta(t)$ (t > 0)

➡ $h_0(t)$ include characteristic modes only

➡ System **S** and **S0** have the same characteristic modes

$y_n(t)$: viewed as the zero input response of **S0**

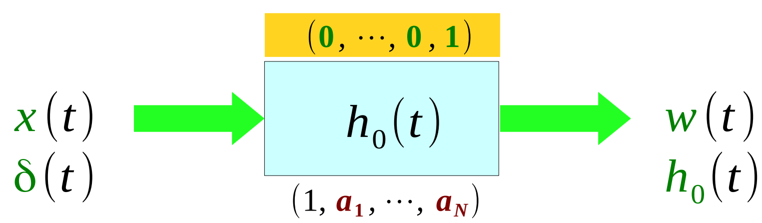
cf) natural response all the lumped char modes
homogeneous response



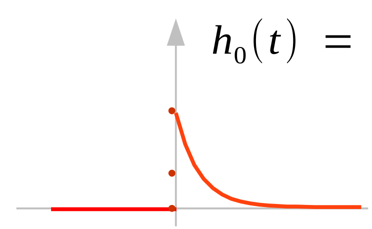
an impulse function
generates energy storage
creates nonzero initial
condition at $t = 0^+$

zero input

zero input response



$$h_0(t) = y_n(t)u(t)$$



Superposition of inputs (1)

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) = b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \cdots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$$

$h_N(t)$	$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta^{(N)}$	N	⇒	$h_0^{(N)}(t)$
$h_{N-1}(t)$	$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta^{(N-1)}$	$N-1$	⇒	$h_0^{(N-1)}(t)$
$\vdots$	$\vdots$			$\vdots$
$h_1(t)$	$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta^{(1)}$	1	⇒	$h_0^{(1)}(t)$
$h_0(t)$	$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h(t) = \delta$	0	⇒	$h_0^{(0)}(t)$

$$h(t) = b_0 h_N + b_1 h_{N-1} + \cdots + b_{N-1} h_1 + b_N h_0$$

$$= b_0 h_0^{(N)} + b_1 h_0^{(N-1)} + \cdots + b_{N-1} h_0^{(1)} + b_N h_0^{(0)}$$

Superposition of inputs (2)

Base System S0

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) = \delta(t)$$

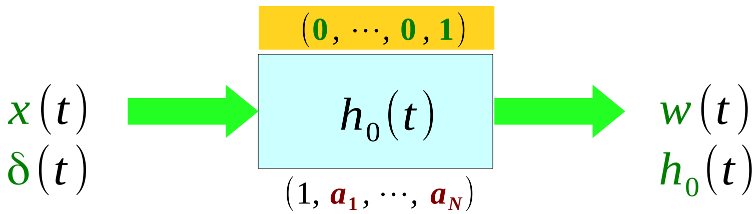
$$\Rightarrow \quad h_0(t)$$

General System S

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \cdots + a_{N-1} h^{(1)}(t) + a_N h(t) = b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \cdots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$$

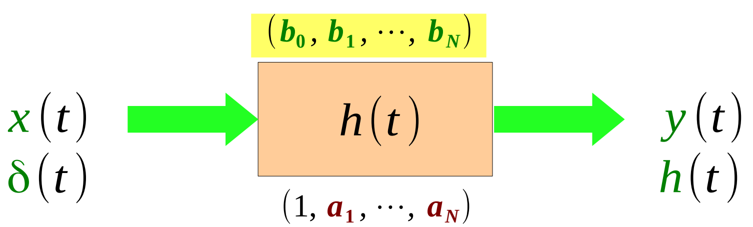
$$h(t) = b_0 h_0^{(N)} + b_1 h_0^{(N-1)} + \cdots + b_{N-1} h_0^{(1)} + b_N h_0^{(0)}$$

h(t) of a General System S



$$h_0 = \frac{1}{a_N}\left[\delta - (h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_{N-1} h_0^{(1)})\right]$$

Base System S0 causal $h_0(t) = y_n u$



$$h = b_0 h_0^{(N)} + b_1 h_0^{(N-1)} + \cdots + b_{N-1} h_0^{(1)} + b_N h_0^{(0)}$$

General System S causal $h(t) = b_0 \{y_n u\}^{(N)} + b_1 \{y_n u\}^{(N-1)} + \cdots + b_N \{y_n u\}^{(0)}$

Single Impulse Matching of $h_0(t)$

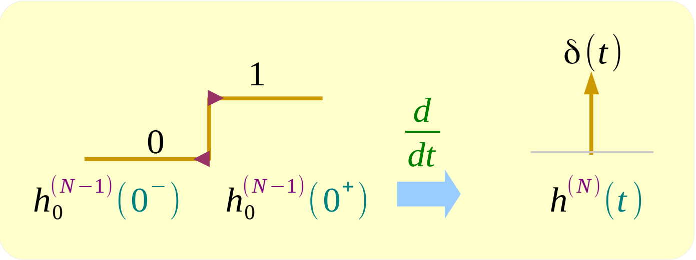
causal $h_0(t) = y_n(t)u(t)$

N > M ➡ include no impulse

$$h_0 = \frac{1}{a_N}\left(\delta - h_0^{(N)} - a_1 h_0^{(N-1)} - \cdots - a_{N-1} h_0^{(1)}\right)$$

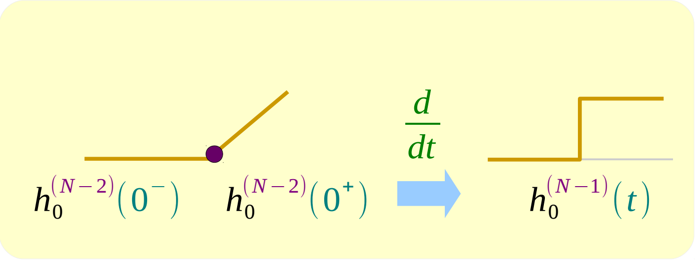
$h_0^{(N-1)}(t)$ There must be a finite jump

$\delta - h_0^{(N)} \rightarrow 0$



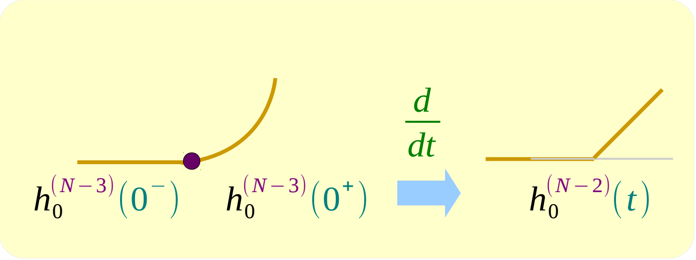
$h_0^{(N-1)}(0^+) = 1$

$h_0^{(N-2)}(t)$ must be continuous



$h_0^{(N-2)}(0^+) = 0$

$h_0^{(N-3)}(t)$ must be continuous



$h_0^{(N-3)}(0^+) = 0$

Initial conditions of h_0 at 0^+ created by $\delta(t)$

$$h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_{N-1} h_0^{(1)} + a_N h_0(t) = \delta(t)$$

Single Impulse at the right hand side creates a unique initial condition

$$\frac{d^N h_0}{dt^N} + a_1 \frac{d^{N-1} h_0}{dt^{N-1}} + a_2 \frac{d^{N-2} h_0}{dt^{N-2}} \quad \cdots \quad + a_2 \frac{d h_0}{dt} + a_1 h_0(t) = \delta(t)$$

integrate → integrate → … integrate → integrate

← differentiate ← differentiate … ← differentiate ← differentiate

$\frac{d^N h_0}{dt^N}$	$\frac{d^{N-1} h_0}{dt^{N-1}}$	$\frac{d^{N-2} h_0}{dt^{N-2}}$	$\frac{d h_0}{dt}$	$h_0(t)$
$\delta(t)$	$u(t)$	$t\,u(t)$	$t^{N-2}\,u(t)$	$t^{N-1}\,u(t)$

A unit ***impulse*** *at t = 0*

A unit jump ***discontinuity*** *at t = 0*

no jump discontinuity at t = 0

$$h_0^{(N-1)}(0^+) = \mathbf{1}$$

$$h_0^{(N-2)}(0^+) = h_0^{(N-1)}(0^+) = \cdots \; h_0^{(1)}(0^+) = h_0(0^+) = \mathbf{0}$$

Base System's IVP

The original IVP

$$h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_{N-1} h_0^{(1)} + a_N h_0(t) = \delta(t)$$ M = 0

$$h_0^{(N-1)_n}(0^+) = 1 \qquad h_0^{(N-2)}(0^+) = h_0^{(N-1)}(0^+) = \cdots \; h_0^{(1)}(0^+) = h_0(0^+) = 0$$

$$h_0(t) = y_n(t)u(t)$$

$$\delta(t) \in h^{(N)}(t)$$

discard ($t < 0$) part

Solve this IVP

$$y_n^{(N)} + a_1 y_n^{(N-1)} + \cdots + a_{N-1} y_n^{(1)} + a_N y_n(t) = 0$$ homogeneous solution

$$y_n^{(N-1)_n}(0^+) = 1 \qquad y_n^{(N-2)}(0^+) = y_n^{(N-1)}(0^+) = \cdots \; y_n^{(1)}(0^+) = y_n(0^+) = 0$$

$$y_n(t)$$

$$\delta(t) \notin y_n^{(N)}(t)$$

$$y_n(t) = (c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t})$$

homogeneous solution

$t < 0$, $t = 0$, $t > 0$

Derivatives of $y_h(t) \cdot u(t)$ with the initial conditions

$$f(t)\delta(t) = f(0)\delta(t)$$

If this is applied <u>not</u> first,

$$\begin{aligned}
a_N \times \quad & h^{(0)}(t) = y_h(t)u(t) \\
a_{N-1} \times \quad & h^{(1)}(t) = y_h^{(1)}(t)u(t) + y_h(0)\delta^{(0)}(t) \\
a_{N-2} \times \quad & h^{(2)}(t) = y_h^{(2)}(t)u(t) + 2\,y_h^{(1)}(0)\delta^{(0)}(t) + y_h(0)\delta^{(1)}(t) \\
& \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
& h^{(N)}(t) = y_h^{(N)}(t)u(t) + \binom{N}{1} y_h^{(N-1)}(0)\delta^{(0)}(t) + \cdots \binom{N}{N-1} y_h^{(1)}(0)\delta^{(N-2)}(t) + y_h^{(0)}(0)\delta^{(N-1)}(t)
\end{aligned}$$

$$y_n^{(N)} + a_1 y_n^{(N-1)} + \cdots + a_{N-1} y_n^{(1)} + a_N y_n(t) = 0$$

$$h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_{N-1} h_0^{(1)} + a_N h_0(t) = \delta(t)$$

$$N\delta(t)$$

Applying delta function properties, first

$$f(t)\delta(t) = f(0)\delta(t)$$

$$h^{(0)}(t) = y_h(t)u(t)$$

$$h^{(1)}(t) = y_h^{(1)}(t)u(t) + y_h(t)\delta^{(0)}(t) \qquad = y_h^{(1)}(t)u(t) + y_h(0)\delta^{(0)}(t)$$

$$h^{(2)}(t) = y_h^{(2)}(t)u(t) + y_h^{(1)}(t)\delta^{(0)}(t) + y_h(0)\delta^{(1)}(t) \qquad = y_h^{(2)}(t)u(t) + y_h^{(1)}(0)\delta^{(0)}(t) + y_h(0)\delta^{(1)}(t)$$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$h^{(N)}(t) = y_h^{(N)}(t)u(t) + \underbrace{y_h^{(N-1)}(0)}_{=\,1}\delta(t) + \cdots \underbrace{y_h^{(1)}(0)}_{=\,0}\delta^{(N-2)}(t) + \underbrace{y_h^{(0)}(0)}_{=\,0}\delta^{(N-1)}(t)$$

$$h^{(N)}(t) = y_h^{(N)}(t)u(t) + \delta(t)$$

$$y_h^{(N-1)}(0^+) = 1 \qquad y_h^{(N-2)}(0^+) = y_h^{(N-1)}(0^+) = \cdots\ y_h^{(1)}(0^+) = y_h(0^+) = 0$$

No δ(t) in $y_n(t)$

All continuous at $t = 0$

$$\begin{aligned}
y_n(t) &= c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-1} e^{\lambda_{N-1} t} \\
y_n^{(1)}(t) &= c_0 \lambda_0 e^{\lambda_0 t} + c_1 \lambda_1 e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1} e^{\lambda_{N-1} t} \\
y_n^{(N-2)}(t) &= c_0 \lambda_0^{(N-2)} e^{\lambda_0 t} + c_1 \lambda_1^{(N-2)} e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1}^{(N-2)} e^{\lambda_{N-1} t} \\
y_n^{(N-1)}(t) &= c_0 \lambda_0^{(N-1)} e^{\lambda_0 t} + c_1 \lambda_1^{(N-1)} e^{\lambda_1 t} + \cdots + c_{N-1} \lambda_{N-1}^{(N-1)} e^{\lambda_{N-1} t}
\end{aligned}$$

$t = 0^-$ $\quad t = 0$ $\quad t = 0^+$

$$\begin{aligned}
y_n(0^-) = y_n(0) = y_n(0^+) &= c_0 + c_1 + \cdots + c_{N-1} \\
y_n^{(1)}(0^-) = y_n^{(1)}(0) = y_n^{(1)}(0^+) &= c_0\lambda_0 + c_1\lambda_1 + \cdots + c_{N-1}\lambda_{N-1} \\
&\vdots \\
y_n^{(N-2)}(0^-) = y_n^{(N-2)}(0) = y_n^{(N-2)}(0^+) &= c_0\lambda_0^{(N-2)} + c_1\lambda_1^{(N-2)} + \cdots + c_{N-1}\lambda_{N-1}^{(N-2)} \\
y_n^{(N-1)}(0^-) = y_n^{(N-1)}(0) = y_n^{(N-1)}(0^+) &= c_0\lambda_0^{(N-1)} + c_1\lambda_1^{(N-1)} + \cdots + c_{N-1}\lambda_{N-1}^{(N-1)}
\end{aligned}$$

~~$$\begin{aligned}
y_n(0^-) &= 0 \\
y_n^{(1)}(0^-) &= 0 \\
&\vdots \\
y_n^{(N-2)}(0^-) &= 0 \\
y_n^{(N-1)}(0^-) &= 0
\end{aligned}$$~~

Not initially rest before $t = 0$

All continuous initial conditions $t = 0$

Can include no delta function

$$\delta(t) \notin y_n^{(N)}(t)$$

All continuous $y_n(t)$ and its derivatives

continuous

$$y_n^{(N-1)}(0^+) = 1$$

$$y_n^{(N-1)}(0) = 1$$

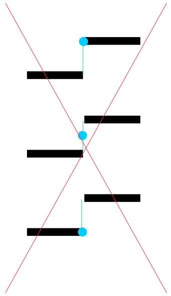
continuity

$$y_n^{(N-2)}(0^+) = y_n^{(N-3)}(0^+) = \cdots \; y_n^{(1)}(0^+) = y_n(0^+) = 0$$

$$y_n^{(N-2)}(0) = y_n^{(N-3)}(0) = \cdots \; y_n^{(1)}(0) = y_n(0) = 0$$

***t* > 0**

***t* = 0**



Solve this IVP

$$y_n^{(N)} + a_1 y_n^{(N-1)} + \cdots + a_{N-1} y_n^{(1)} + a_N y_n(t) = 0$$

homogeneous solution

***t* < 0, *t* = 0, *t* > 0**

$$y_n^{(N-1)}(0) = 1 \qquad y_n^{(N-2)}(0) = y_n^{(N-1)}(0) = \cdots \; y_n^{(1)}(0) = y_n(0) = 0$$

$$\delta(t) \notin y_n^{(N)}(t)$$

$y_n(t)$ Discard the t < 0 part $\Rightarrow$

$$h_0(t) = y_n(t)u(t)$$

Verifying the original IVP

$$h^{(N)}(t) = y_h^{(N)}(t)u(t) + \underbrace{y_h^{(N-1)}(0)}_{=\,1}\delta(t) + \cdots \underbrace{y_h^{(1)}(0)}_{=\,0}\delta^{(N-2)}(t) + \underbrace{y_h^{(0)}(0)}_{=\,0}\delta^{(N-1)}(t)$$

$$h^{(N)}(t) = y_h^{(N)}(t)u(t) + \delta(t)$$

		$h^{(N)}(t) = y_h^{(N)}(t)u(t)$
a_1	$\times$	$h^{(N-1)}(t) = y_h^{(N-1)}(t)u(t)$
		$\vdots \quad \vdots$
a_{N-1}	$\times$	$h^{(1)}(t) = y_h^{(1)}(t)u(t)$
a_N	$\times$	$h^{(0)}(t) = y_h^{(0)}(t)u(t)$

$\Rightarrow$ 0

$$h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_{N-1}h_0^{(1)} + a_N h_0(t) = \delta(t) \qquad t = 0$$

$$y_h^{(N)} + a_1 y_h^{(N-1)} + \cdots + a_{N-1}y_h^{(1)} + a_N y_h(t) = 0 \qquad t = 0$$

$$h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_{N-1}h_0^{(1)} + a_N h_0(t) = 0 \qquad t > 0$$

$$y_h^{(N)} + a_1 y_h^{(N-1)} + \cdots + a_{N-1}y_h^{(1)} + a_N y_h(t) = 0 \qquad t > 0$$

- Simplified Impulse Matching

Impulse Response $h(t)$, $h_0(t)$

Derived System S

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,x(t)}{dt} + b_M\, x(t)$$

Base System S0

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = x(t)$$

shares the same characteristic modes

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot h(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1} D + b_M) \cdot \delta(t)$$ **Derived System S**

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot h_0(t) = \delta(t)$$ **Base System S0**

$$Q(D) \cdot h(t) = P(D) \cdot \delta(t) \qquad h(t) = P(D) \cdot h_0(t)$$

$$Q(D) \cdot h_0(t) = \delta(t)$$

Getting the Impulse Response $h_0(t)$

Base System S0

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = x(t)$$

Solve this IVP $y_n(t)$ *linear combination of characteristic modes*

$$y_n^{(N)} + a_1 y_n^{(N-1)} + \cdots + a_{N-1} y_n^{(1)} + a_N y_n(t) = 0$$

homogeneous solution

$t < 0$, $t = 0$, $t > 0$

$$y_n^{(N-1)}(0) = 1 \qquad y_n^{(N-2)}(0) = y_n^{(N-1)}(0) = \cdots \; y_n^{(1)}(0) = y_n(0) = 0$$

$\Rightarrow$ $\delta(t) \notin y_n^{(N)}(t)$

$y_n(t)$ Discard the t < 0 part $\Rightarrow$ $h_0(t) = y_n(t)u(t)$

Getting the Impulse Response $h(t)$ from $h_o(t)$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) w(t) = x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + \cdots + b_{M-1} D + b_M) x(t)$$

$$y(t) = (b_0 D^M + \cdots + b_{N-1} D + b_N) w(t)$$

$$Q(D) w(t) = x(t)$$

$$Q(D) P(D) w(t) = P(D) x(t)$$

$$y(t) = P(D) w(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h_0(t) = \delta(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_0 D^M + \cdots + b_{M-1} D + b_M) \delta(t)$$

$$h(t) = (b_0 D^M + \cdots + b_{N-1} D + b_N) h_0(t)$$

$$Q(D) h_0(t) = \delta(t)$$

$$Q(D) P(D) h_0(t) = P(D) \delta(t)$$

$$h(t) = P(D) h_0(t)$$

Causality is considered *causal* $y_n(t) u(t)$

$$h(t) = P(D)[\ y_n(t) u(t)\ ] \Rightarrow [\ P(D) y_n(t)\ ] u(t)$$

$$h(t) = b_o \delta(t) + P(D) y_n(t), \quad t \geq 0$$

$$h(t) = b_o \delta(t) + [\ P(D) y_n(t)\ ] u(t)$$

Getting the Impulse Response $h(t)$ from $y_n(t)$

$$P(D) = (b_0 D^M + \cdots + b_{N-1} D + b_N)$$

$$h(t) = P(D)[\, h_0(t)\,] = P(D)[\, y_n(t) u(t)\,] \quad \Rightarrow \quad [P(D) y_n(t)\,] u(t)$$

$$h(t) = b_0 \delta(t) + P(D)[\, y_n(t) \cdot u(t)\,] \quad \Rightarrow \quad h(t) = b_0 \delta(t) + [\, P(D)\, y_n(t)\,] \cdot u(t)$$

$$h^{(0)}(t) = y_h(t) u(t) \quad \Rightarrow \quad h^{(0)}(t) = y_h(t) u(t)$$

$$h^{(1)}(t) = y_h^{(1)}(t) u(t) + y_h(0) \delta^{(0)}(t) \quad \Rightarrow \quad h^{(1)}(t) = y_h^{(1)}(t) u(t)$$

$$h^{(2)}(t) = y_h^{(2)}(t) u(t) + y_h^{(1)}(0) \delta^{(0)}(t) + y_h(0) \delta^{(1)}(t) \quad \Rightarrow \quad h^{(2)}(t) = y_h^{(2)}(t) u(t)$$

$$\Rightarrow \quad h^{(N-1)}(t) = y_h^{(N-1)}(t) u(t)$$

$$h^{(N)}(t) = y_h^{(N)}(t) u(t) + y_h^{(N-1)}(0) \delta(t) + \cdots \; y_h^{(1)}(0) \delta^{(N-2)}(t) + y_h^{(0)}(0) \delta^{(N-1)}(t) \quad \Rightarrow \quad h^{(N)}(t) = y_h^{(N)}(t) u(t) + \delta(t)$$

Verification

$P(D)[\, y_n(t)u(t) \,]$ N-1 = M

$$
\begin{array}{lclcl}
b_M & \times & h^{(0)}(t) & = & y_h(t)u(t) & \times & b_M \\
b_{M-1} & \times & h^{(1)}(t) & = & y_h^{(1)}(t)u(t) & \times & b_{M-1} \\
b_{M-2} & \times & h^{(2)}(t) & = & y_h^{(2)}(t)u(t) & \times & b_{M-2} \\
b_0 & \times & h^{(N-1)}(t) & = & y_h^{(N-1)}(t)u(t) & \times & b_0 \\
 & & h^{(N)}(t) & = & y_h^{(N)}(t)u(t) + \delta(t) & &
\end{array}
$$

$[\, P(D)y_n(t) \,]\, u(t)$

$$
\left.
\begin{array}{lcl}
b_M & \times & y_h(t) \\
b_{M-1} & \times & y_h^{(1)}(t) \\
b_{M-2} & \times & y_h^{(2)}(t) \\
b_0 & \times & y_h^{(N-1)}(t)
\end{array}
\right\} u(t)
$$

$P(D)[\, y_n(t)u(t) \,]$ N = M

$$
\begin{array}{lclcl}
b_M & \times & h^{(0)}(t) & = & y_h(t)u(t) & \times & b_M \\
b_{M-1} & \times & h^{(1)}(t) & = & y_h^{(1)}(t)u(t) & \times & b_{M-1} \\
b_{M-2} & \times & h^{(2)}(t) & = & y_h^{(2)}(t)u(t) & \times & b_{M-2} \\
b_1 & \times & h^{(N-1)}(t) & = & y_h^{(N-1)}(t)u(t) & \times & b_1 \\
b_0 & \times & h^{(N)}(t) & = & y_h^{(N)}(t)u(t) + \delta(t) & \times & b_1
\end{array}
$$

$[\, P(D)y_n(t) \,]\, u(t)$

$$
\left.
\begin{array}{lcl}
b_M & \times & y_h(t) \\
b_{M-1} & \times & y_h^{(1)}(t) \\
b_{M-2} & \times & y_h^{(2)}(t) \\
b_1 & \times & y_h^{(N-1)}(t) \\
b_0 & \times & y_h^{(N)}(t) \quad + \delta(t)
\end{array}
\right\} u(t)
$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)