CLTI Impulse Response (5B)

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Solutions of Differential Equations : h(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,x(t)}{dt} + b_M\, x(t)$$

requirement at time $t = 0$

All the derivatives of h(t) up to ***N*** must match a corresponding derivatives of the impulse up to ***M*** at time t=0

requirement at time $t \neq 0$

The linear combination of all the derivatives of h(t) must add to zero for any time $t \neq 0$

$y_h(t)u(t)$ is such a function

$y_h(t)$ is the homogeneous solution

Case 1 N > M

The derivatives of the $y_h(t)u(t)$ provide all the singularity functions necessary to match the impulse and derivatives of the impulse on the right side and no other terms need to be added

Case 2 N = M

Need to add an impulse term $K_0\delta(t)$.. and solve for K_0 by matching coefficients of impulses on both sides

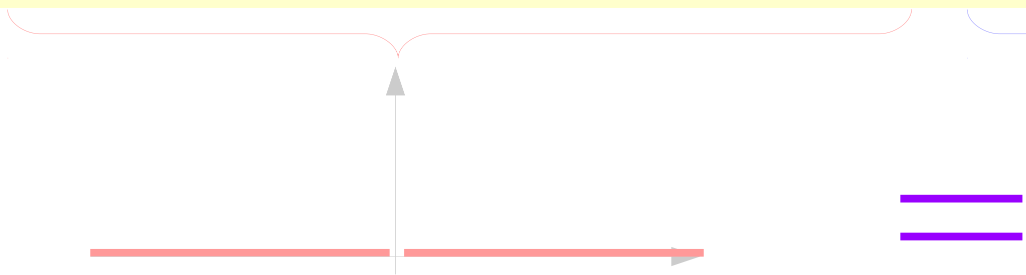
Case 3 N < M

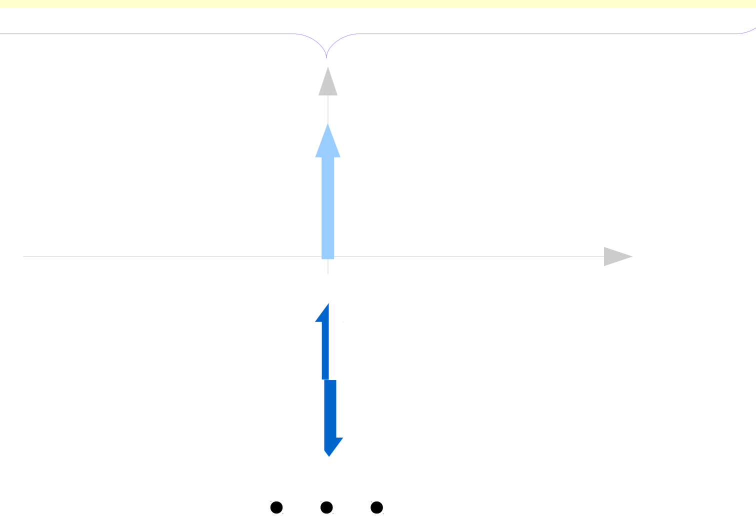
The ***N**-th* derivative of the function we add to $y_h(t)u(t)$ must have a term that matches the ***M**-th* derivative of the unit impulse. We have to add these terms

$$K_{m-n} u_{m-n}(t) + \cdots + K_1 u_1(t) + K_0 u_0(t)$$
$$= K_{m-n} \delta^{(m-n)}(t) + \cdots + K_1 \delta^{(1)}(t) + K_0 \delta^{(0)}(t)$$

Requirements at t≠0 (1)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$





requirements at *t ≠ 0*

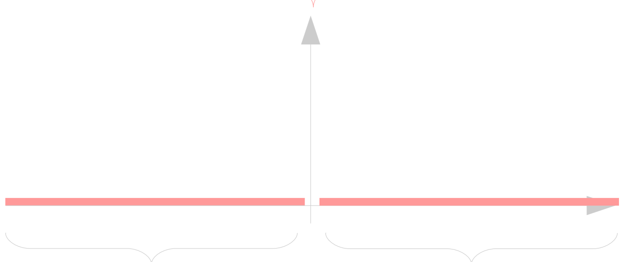
$$h^{(N)}(t) + a_1 h^{(N-1)}(t) \cdots + a_N h(t) = 0 \qquad (t \neq 0)$$

The linear combination
of all the derivatives of *h(t)*
must add to zero for any time t≠0

all the derivatives of δ(t)
exists only t=0.
It is zero for any time t≠0

Requirements at t≠0 (2)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$



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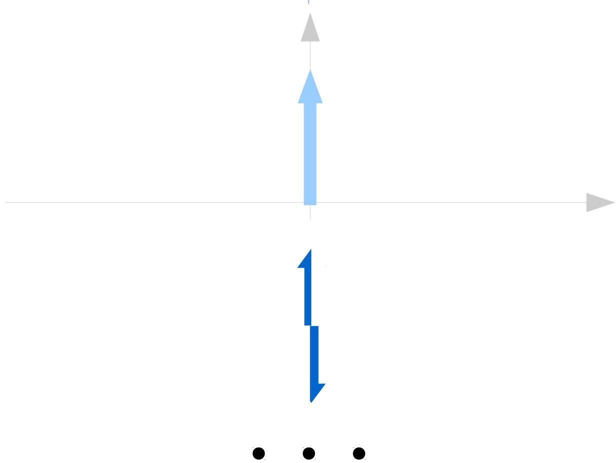


u(t)=0
: step function

the linear combination of all the derivatives of $y_p(t)$ results to zero
: homogeneous solution

$u(t) = 0$

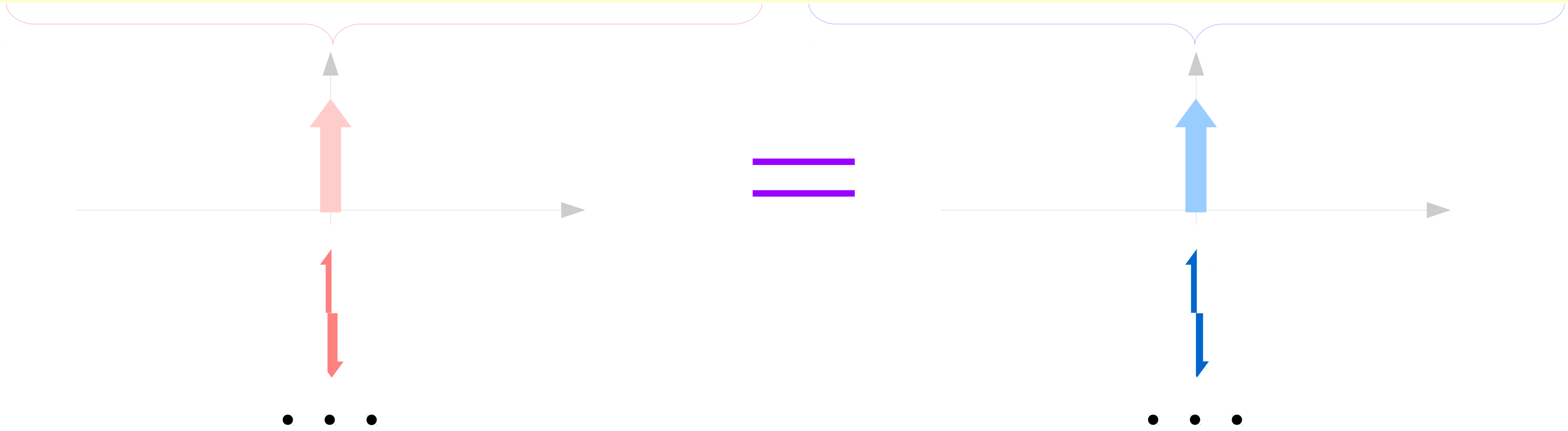
$y_h^{(N)} + a_1 y_h^{(N-1)} + \cdots + a_N y_h = 0$

for $t < 0$, $u(t) = 0$
for $t > 0$, $y_h^{(N)} + a_1 y_h^{(N-1)} + \cdots + a_N y_h = 0$
derivatives of $\{y_h \cdot u\}$ produce
derivatives of δ when $t = 0$

$y_h(t) u(t)$ when $t \neq 0$

➡ *A possible candidate of* $h(t)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$



requirements at *t = 0*

All the derivatives of *h(t)* up to *N* must **match** the corresponding derivatives of an impulse *δ(t)* up to *M* at time t=0

Need to add a δ(t) and its derivatives in case that (N ≤ M)

Need to integrate $y_h(t) \cdot u(t)$ several times in case that (N > M)

Requirements at t=0 (2)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1}\delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$\frac{d^N}{dt^N}\{y_h(t)u(t)\} = \frac{d^{N-1}}{dt^{N-1}}\{c_0 \delta(t)\} + \cdots$$

(N≤M) $\Leftrightarrow$ $b_0 \frac{d^M \delta(t)}{dt^M}$ the highest order derivatives of δ(t)

$y_h(t) \cdot u(t)$ gives the highest order (N-1)

Need to have the following terms

(N≤M)

$$m_{M-N} \frac{d^M}{dt^M}\delta(t) + \cdots + m_0 \frac{d^N}{dt^N}\delta(t) \leftarrow$$

(N≤M) $\Leftrightarrow$ $b_0 \frac{d^M \delta(t)}{dt^M}$ the highest order derivatives of δ(t)

$$m_0 \frac{d^N}{dt^N}\delta(t) + \cdots + m_{M-N} \frac{d^M}{dt^M}\delta(t) \rightarrow$$

Requirements at t=0 (3)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$m_0 \frac{d^N}{dt^N}\delta(t) + \cdots + m_{M-N} \frac{d^M}{dt^M}\delta(t)$$

all the derivatives of $y_h(t) \cdot u(t)$ may not include all the required the derivatives of $\delta(t)$ at the time $t=0$ in case that $N \le M$.

Need to add a $\delta(t)$ and its derivatives in case that ($N \le M$)

$$h(t) = y_h(t)u(t) + m_0\delta(t) \qquad (N = M)$$

$$h(t) = y_h(t)u(t) + m_0\delta(t) + m_1\dot{\delta}(t) + \cdots + m_{M-N}\,\delta^{(M-N)}(t) \qquad (N < M)$$

Requirements at t=0 (4)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, \delta(t)}{dt} + b_M \delta(t)$$

$$\frac{d^N}{dt^N}\{\quad\} = \frac{d^M}{dt^M}\{c_0 \delta(t)\} + \cdots$$

(N>M) $\Longleftrightarrow$ $b_0 \frac{d^M \delta(t)}{dt^M}$

the highest order derivatives of δ(t)

h(t) gives the highest order (M)
h(t) must have the following terms

$$h(t) = \overbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}^{N-M-1} y_h(t) u(t)\, dt \cdots dt$$

(N≤M)

$$\frac{d^N}{dt^N}\{h(t)\} = \frac{d^{M+1}}{dt^{M+1}}\left\{\frac{d^{N-M-1}}{dt^{N-M-1}}\{h(t)\}\right\} = \frac{d^{M+1}}{dt^{M+1}}\{y_h(t) u(t)\}$$

$\Longleftrightarrow$ $b_0 \frac{d^M \delta(t)}{dt^M}$

the highest order derivatives of δ(t)

Requirements at t=0 (5)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, \delta(t)}{dt} + b_M \delta(t)$$

All the derivatives of *h(t)* up to *N* must generate the derivatives of an impulse *δ(t)* only up to *M* in case that ***N > M***

Need to integrate $y_h(t) \cdot u(t)$ several times in case that (N>M)

$$h(t) = \overbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}^{N-M-1} y_h(t) u(t)\, dt \cdots dt$$

$(N > M)$

- Impulse Response Representations in terms of $y_h(t) \cdot u(t)$

Derivatives of $y_h(t) \cdot u(t)$

$$h(t) = y_h(t)u(t)$$

$$u^{(i)}(t) = \delta^{(i-1)}(t)$$
$$f(t)\delta(t) = f(0)\delta(t)$$

$$h = y_h u$$
$$h^{(1)} = y_h^{(1)} u + y_h u^{(1)} \longrightarrow y_h(0)\delta(t)$$
$$h^{(2)} = y_h^{(2)} u + 2 y_h^{(1)} u^{(1)} + y_h u^{(2)} \longrightarrow 2 y_h^{(1)}(0)\delta(t) + y_h(0)\delta^{(1)}(t)$$
$$h^{(3)} = y_h^{(3)} u + 3 y_h^{(2)} u^{(1)} + 3 y_h^{(1)} u^{(2)} + y_h u^{(3)} \longrightarrow 3 y_h^{(2)}(0)\delta(t) + 3 y_h^{(1)}(0)\delta^{(1)}(t) + y_h(0)\delta^{(2)}(t)$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$h(t) = y_h(t)u(t)$$

All the derivatives of $h(t)$ up to N incurs the derivatives of an impulse $\delta(t)$ up to $N-1$

$$h^{(N)}(t) = \frac{d^N}{dt^N}\{y_h(t)u(t)\} \longrightarrow K_1\delta^{(N-1)}(t) + K_2\delta^{(N-2)}(t) + \cdots + K_{N-1}\delta^{(1)}(t) + K_N\delta(t)$$

Three different h(t)'s

$h(t) = y_h(t)u(t)$ ⇒ All the derivatives of *h(t)* up to *N* incurs the derivatives of an impulse *δ(t)* up to *N-1*

$h^{(1)}(t) = y_h(t)u(t)$ ⇒ All the derivatives of *h(t)* up to *N* incurs the derivatives of an impulse *δ(t)* up to *N-2*

$h^{(2)}(t) = y_h(t)u(t)$ ⇒ All the derivatives of *h(t)* up to *N* incurs the derivatives of an impulse *δ(t)* up to *N-3*

Derivatives of three different h(t)'s

$h(t) = y_h(t)u(t)$

$h^{(N)}(t) \longrightarrow K_1\delta^{(N-1)}(t) + K_2\delta^{(N-2)}(t) + \cdots + K_{N-1}\delta^{(1)}(t) + K_N\delta(t)$

$h(t) = \int_{-\infty}^{t} y_h(t)u(t)\, dt$

$h^{(N)}(t) \longrightarrow K_1\delta^{(N-2)}(t) + K_2\delta^{(N-3)}(t) + \cdots + K_{N-1}\delta(t)$

$h(t) = \iint_{-\infty}^{t} y_h(t)u(t)\, dt\, dt$

$h^{(N)}(t) \longrightarrow K_1\delta^{(N-3)}(t) + \cdots + K_{N-2}\delta(t)$

All the derivatives of h(t) up to N

	$h^{(N)}(t)$	$+a_1 h^{(N-1)}(t)$	$+ \cdots$	$+a_{N-2} h^{(2)}(t)$	$+a_{N-1} h^{(1)}(t)$	$+a_N h^{(0)}(t)$
(a)	$\frac{d^N}{dt^N}\{y_h u\}$	$\frac{d^{N-1}}{dt^{N-1}}\{y_h u\}$		$\frac{d^2}{dt^2}\{y_h u\}$	$\frac{d}{dt}\{y_h u\}$	$y_h u$
(b)	$\frac{d^{N-1}}{dt^{N-1}}\{y_h u\}$	$\frac{d^{N-2}}{dt^{N-2}}\{y_h u\}$		$\frac{d}{dt}\{y_h u\}$	$y_h u$	$\int_{-\infty}^{t} y_h u \, dt$
(c)	$\frac{d^{N-2}}{dt^{N-2}}\{y_h u\}$	$\frac{d^{N-3}}{dt^{N-3}}\{y_h u\}$		$y_h u$	$\int_{-\infty}^{t} y_h u \, dt$	$\int_{-\infty}^{t}\int_{-\infty}^{t} y_h u \, dt\, dt$

$(N = M+1)$ $\quad (N-1 = M)$ $\quad \delta^{(N-1)} = \delta^{(M)}$ $\quad (M-N+1 = 0)$

$(N = M+2)$ $\quad (N-2 = M)$ $\quad \delta^{(N-2)} = \delta^{(M)}$ $\quad (M-N+1 = -1)$

$(N = M+3)$ $\quad (N-3 = M)$ $\quad \delta^{(N-3)} = \delta^{(M)}$ $\quad (M-N+1 = -2)$

Negative powers denote integration

(a) $h(t) = y_h(t)u(t)$ ⇒ $h(t) = y_h(t)u(t)$ = $g(t)$

(b) $h^{(1)}(t) = y_h(t)u(t)$ ⇒ $h(t) = \int_{-\infty}^{t} y_h(t)u(t)\, dt$ = $g^{(-1)}(t) \equiv \int_{-\infty}^{t} g(t)\, dt$

(c) $h^{(2)}(t) = y_h(t)u(t)$ ⇒ $h(t) = \iint_{-\infty}^{t} y_h(t)u(t)\, dt\, dt$ = $g^{(-2)}(t) \equiv \int_{-\infty}^{t} \int_{-\infty}^{t} g(t)\, dt\, dt$

Derivative of three different h(t) : g(t), $g^{(1)}(t)$, $g^{(2)}(t)$

	$h^{(N)}(t)$	$+a_1 h^{(N-1)}(t)$	$+ \cdots$	$+a_{N-2} h^{(2)}(t)$	$+a_{N-1} h^{(1)}(t)$	$+a_N h^{(0)}(t)$
(a)c	$g^{(N)}(t)$	$g^{(N-1)}(t)$		$g^{(2)}(t)$	$g^{(1)}(t)$	$g(t)$
(b)	$g^{(N-1)}(t)$	$g^{(N-2)}(t)$		$g^{(1)}(t)$	$g(t)$	$g^{(-1)}(t)$
(c)	$g^{(N-2)}(t)$	$g^{(N-3)}(t)$		$g(t)$	$g^{(-1)}(t)$	$g^{(-2)}(t)$

$g(t) = y_h(t) \cdot u(t)$

$(N = M+1)$ $(M-N+1 = 0)$

$(N = M+2)$ $(M-N+1 = -1)$

$(N = M+3)$ $(M-N+1 = -2)$

$h(t) = g^{(M-N+1)}(t) = g^{(0)}(t)$

$h(t) = g^{(M-N+1)}(t) = g^{(-1)}(t)$

$h(t) = g^{(M-N+1)}(t) = g^{(-2)}(t)$

Impulse response $h(t)$ in terms of $y_h(t) \cdot u(t)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$(N = M+1)$ $(N-1 = M)$ $\delta^{(N-1)} = \delta^{(M)}$ $(M-N+1 = 0)$ $h(t) = g^{(M-N+1)}(t) = g^{(0)}(t)$

$(N = M+2)$ $(N-2 = M)$ $\delta^{(N-2)} = \delta^{(M)}$ $(M-N+1 = -1)$ $h(t) = g^{(M-N+1)}(t) = g^{(-1)}(t)$

$(N = M+3)$ $(N-3 = M)$ $\delta^{(N-3)} = \delta^{(M)}$ $(M-N+1 = -2)$ $h(t) = g^{(M-N+1)}(t) = g^{(-2)}(t)$

$$g(t) = y_h(t)u(t)$$

$$h(t) = g^{(M-N+1)}(t) = \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} y_h(t)u(t)\,dt \cdots dt \quad (N > M)$$

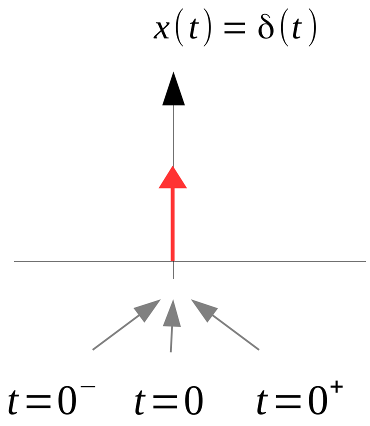
$$h(t) = g(t) + m_0 \delta(t) \quad (N = M)$$

$$h(t) = g(t) + m_0 \delta(t) + m_1 \dot{\delta}(t) + \cdots + m_{M-N} \delta^{(M-N)}(t) \quad (N < M)$$

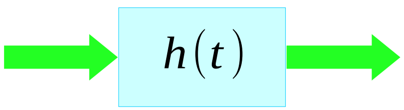
Finding Impulse Response from Diff Equations

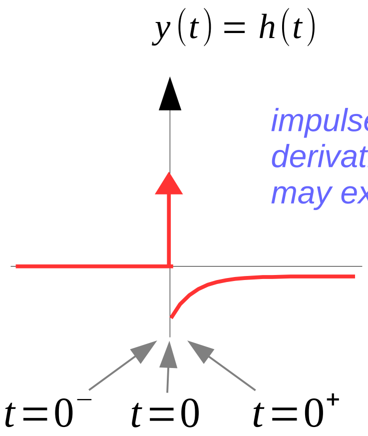
- Impulse Matching Method
- Simplified Impulse Matching Method
- Green's Function

Impulse Response and Other System Responses

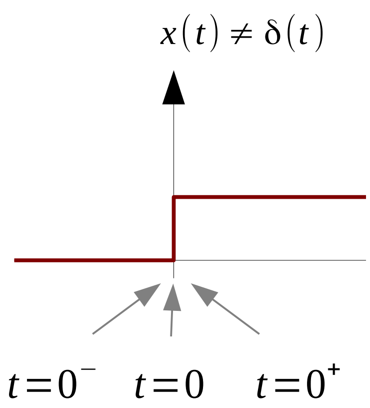


Impulse Response

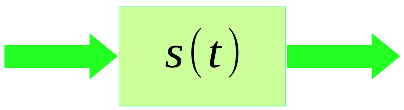


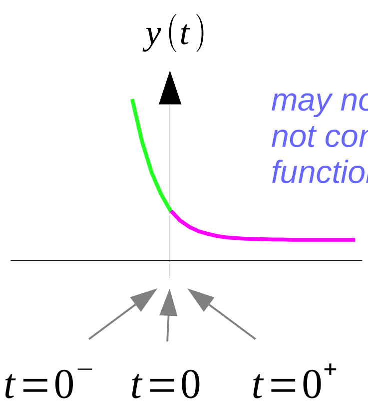


impulse or its derivative functions may exist at t=0



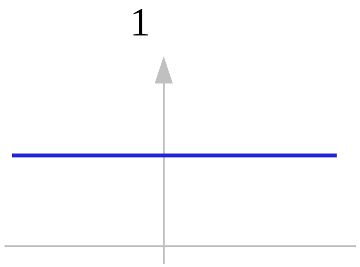
Step Response



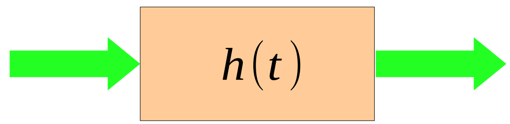


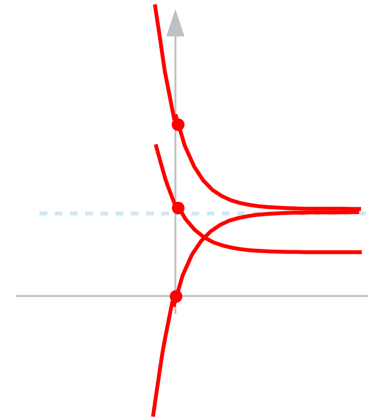
may not be continuous but not contain an impulse function at t=0

x(t) = 1

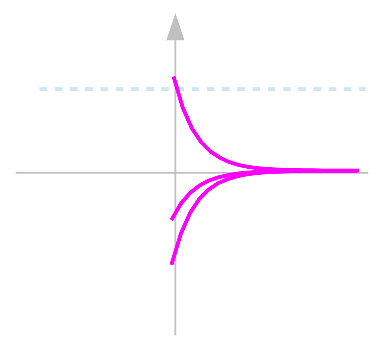


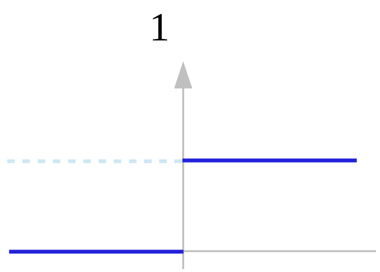
$$y'(t) + y(t) = 1$$



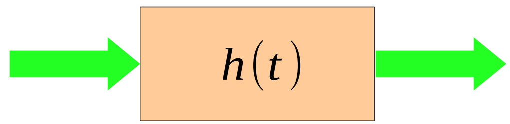


$y(0) = 2$	$y(t) = 1 + e^{-t}$	$y'(t) = -e^{-t}$
$y(0) = 1$	$y(t) = 0.5(1 + e^{-t})$	$y'(t) = -0.5\,e^{-t}$
$y(0) = 0$	$y(t) = 1 - e^{-t}$	$y'(t) = +e^{-t}$





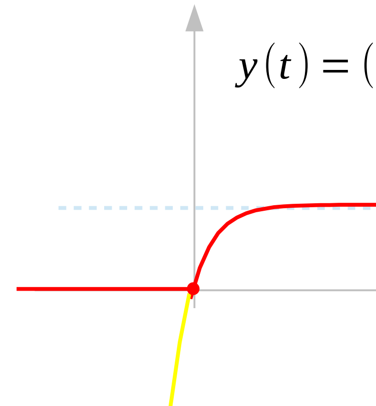
$$y'(t) + y(t) = 1$$

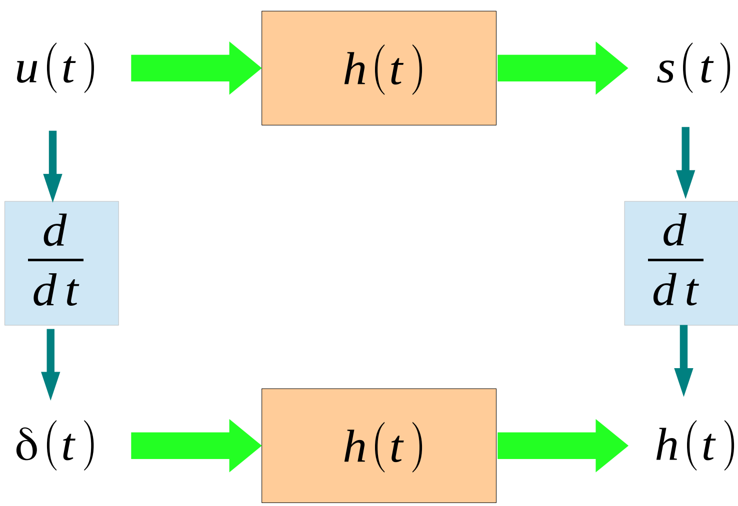


Step Response

$$y(t) = (1 - e^{-t})u(t)$$

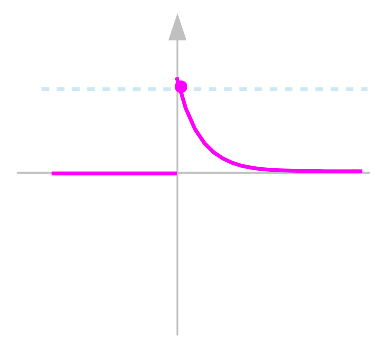
$$y(0) = 0$$



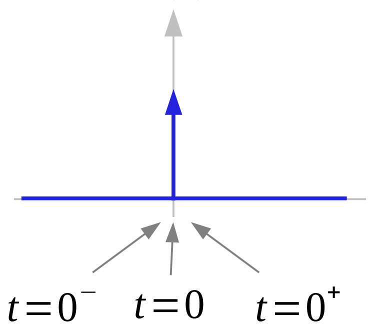


$$y'(t) = [e^{-t}u(t) + (1 - e^{-t})\delta(t)]$$

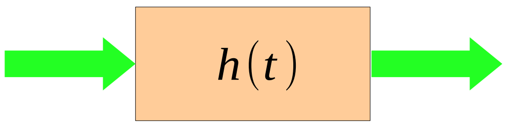
$$y'(t) = e^{-t}u(t)$$



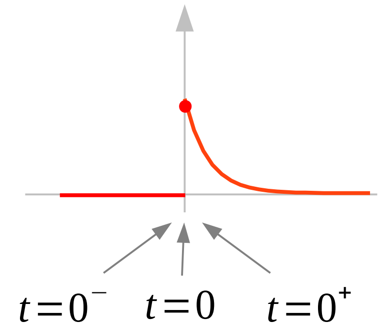
x(t) = δ(t)



$$y'(t) + a\,y(t) = x(t)$$



$$h(t) = e^{-at}\,u(t)$$



$$y(0^-) = 0$$

All initial conditions are zero at $t=0^-$

Generates energy storage creates nonzero initial condition at $t=0^+$

$$y(0) = 1$$

$$h'(t) = -a\,e^{-at}\,u(t) + e^{-at}\,\delta(t)$$

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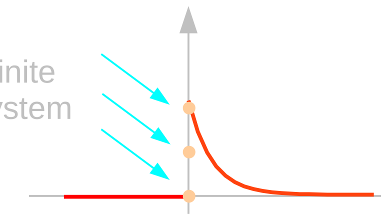
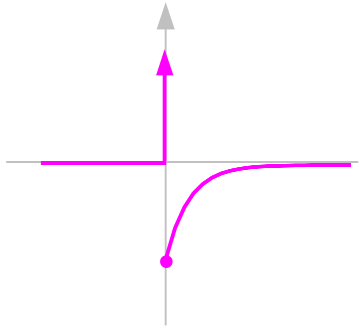
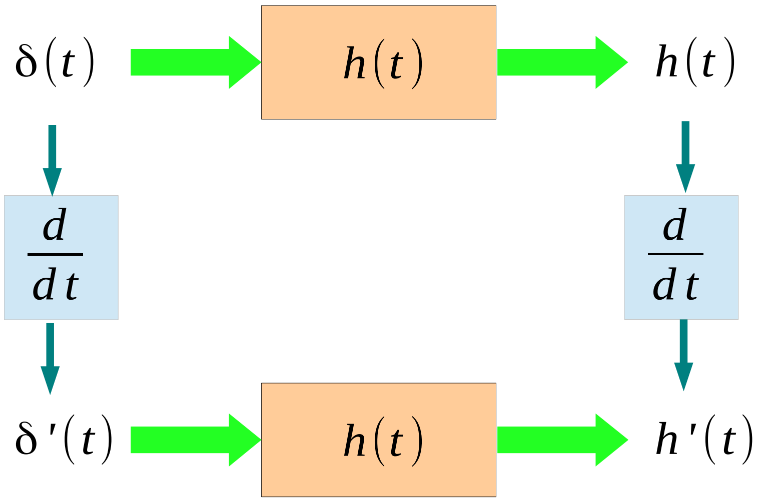
Any two functions that have finite values everywhere and **differ** in value only at a finite number of points are **equivalent** in the system response or transform





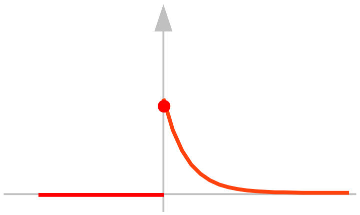
x(t) & x'(t) forcing functions

$$y'(t) + a\,y(t) = x(t)$$



$$y'(t) + a\,y(t) = x'(t)$$

$$h(t) = e^{-at}\,u(t)$$



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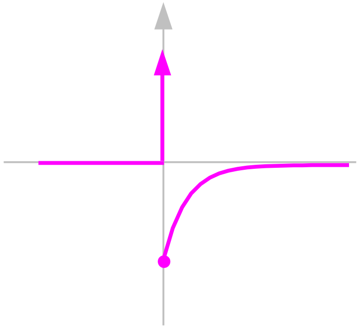


Initial Conditions & Total Response

$\delta(t) \rightarrow \boxed{h(t)} \rightarrow b_0\delta(t) + \text{char mode terms} \quad t \geq 0$

$\dot{\delta}(t) \rightarrow \boxed{h(t)} \rightarrow b_0\dot{\delta}(t) + \frac{d}{dt}\text{char mode terms} \quad t \geq 0$

$\ddot{\delta}(t) \rightarrow \boxed{h(t)} \rightarrow b_0\ddot{\delta}(t) + \frac{d}{dt}\text{char mode terms} \quad t \geq 0$

linear combination of an **impulse** and **its unique derivatives** (the doublet, the triplet, etc) : all these exist at time t = 0

$u(t) \rightarrow \boxed{h(t)} \rightarrow c + \int \text{char mode terms}\, dt \quad t \geq 0$

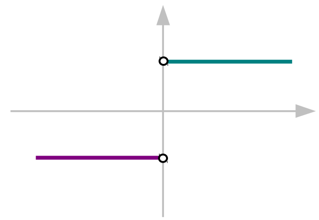
$r(t) \rightarrow \boxed{h(t)} \rightarrow c + \int\int \text{char mode terms}\, dt\, dt \quad t \geq 0$

no **impulse** and **its unique derivatives** at time t = 0

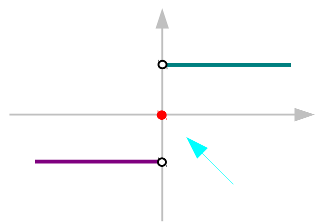
Generally, the interval of interest for y(t) is t > 0

Unit Step Function

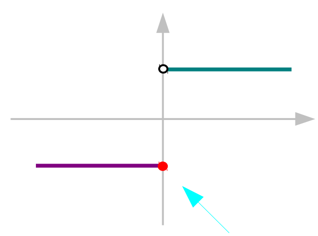
$$g_1(t) = \begin{cases} y_1 & (t < 0) \\ \textit{undefined} & (t = 0) \\ y_2 & (t > 0) \end{cases}$$



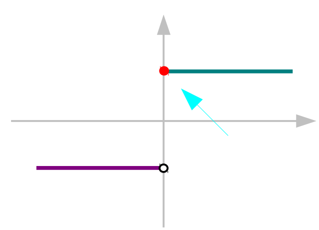
$$g_2(t) = \begin{cases} y_1 & (t < 0) \\ (y_1 + y_2)/2 & (t = 0) \\ y_2 & (t > 0) \end{cases}$$



$$g_3(t) = \begin{cases} y_1 & (t < 0) \\ y_1 & (t = 0) \\ y_2 & (t > 0) \end{cases}$$



$$g_4(t) = \begin{cases} y_1 & (t < 0) \\ y_2 & (t = 0) \\ y_2 & (t > 0) \end{cases}$$



$$\int_{0^-}^{0^+} g_i(t)\,dt = 0 \qquad (i = 1,2,3,4)$$

The area under a single point is zero, regardless of the point's value , if it is finite

- The system response is the same
- The transform is the same also

Any two functions that have finite values everywhere and **differ** in value only at a finite number of points are **equivalent** in the system response or transform

$$\int_{\alpha}^{\beta} g_i(t)\,dt = \int_{\alpha}^{\beta} g_j(t)\,dt \qquad (i \neq j)$$

* Superposition of Input Functions

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)
[4] M.J. Roberts, Fundamentals of Signals and Systems