# CLTI Impulse Response (5A)

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# Impulse Response & Particular Solution

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,x(t)}{dt} + b_M x(t)$$

$y(t) = h(t)$ : impulse response

when $x(t) = \delta(t)$ : forcing function

causal system

valid interval

forcing function

| | $t < 0$ | $t = 0$ | $t > 0$ |
|---|---|---|---|
| $x(t) =$ | 0 | $\delta(t)$ | 0 |

| | | $t < 0$ | $t = 0$ | $t > 0$ |
|---|---|---|---|---|
| particular solution | $y_p(t)$ | 0 | $b_0\delta(t)$ | 0 |
| homogeneous solution | $y_h(t)$ | 0 | $y_h(t)$ | $y_h(t)$ |

impulse response

$$\begin{cases} h(t) = y_h(t)u(t) & (N>M) \quad b_0=0 \\ h(t) = y_h(t)u(t) + b_0\delta(t) & (N=M) \quad b_0\neq 0 \end{cases}$$

# Impulse Response of Differential Equations

**(t > 0)** Homogeneous solution

$$a_0 \frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = 0$$

$y_h(t)$ **homogeneous solution**

– *characteristic modes only*

$$\begin{cases} h(t) = y_h(t)u(t) & (N>M) \\ h(t) = y_h(t)u(t) + b_0\delta(t) & (N=M) \end{cases}$$

linear combination of all the derivatives of $h(t)$ must add to <u>zero</u> for any time <u>$t \neq 0$</u>

**(t = 0)** Particular solution

$$a_0 \frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t)$$

$$= b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$y_p(t)$ **particular solution**

– linear combination of the *forcing function* $x(t)$ and *all its unique derivatives* $x^{(i)}(t)$

$$y_p(t) \Leftarrow \begin{cases} y(t) = b_0\delta(t) \\ y^{(i)}(t) = d_i\delta^{(i)}(t) \end{cases} \Leftarrow \begin{cases} x(t) = \delta(t) \\ x^{(i)}(t) = \delta^{(i)}(t) \end{cases}$$

linear combination of an **impulse** $\delta$ and **its unique derivatives** (the doublet, the triplet, etc) : all these exist at time <u>$t = 0$</u>

# Particular solutions for x(t)=δ(t) at $t=0$

**Finding particular solution**

Particular solution

$$a_0 \frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = \boxed{b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)}$$

h(t) is differentiated up to *n* times ⇐ must match ⇒ δ(t) is differentiated up to *m* times

We can determine whether h(t) can contain an impulse or its unique derivatives

$$\int_{0^-}^{0^+} h(t)\,dt = 0$$

when h(t) does not have an impulse or its derivatives

$$\int_{0^-}^{0^+} h(t)\,dt \neq 0$$

Otherwise

# h(t) can have at most a δ(t)

$$\frac{d^N h(t)}{dt^N} + \mathbf{a_1} \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + \mathbf{a_{N-1}} \frac{dh(t)}{dt} + \mathbf{a_N} h(t) = \mathbf{b_0} \frac{d^N \delta(t)}{dt^N} + \mathbf{b_1} \frac{d^{N-1} \delta(t)}{dt^{N-1}} + \cdots + \mathbf{b_{N-1}} \frac{d\delta(t)}{dt} + \mathbf{b_N} \delta(t)$$

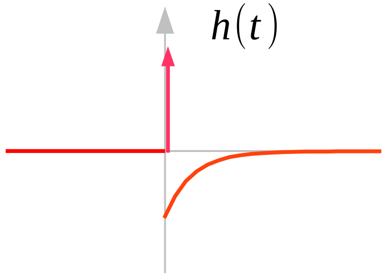
$\boxed{M = N}$ $\quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N) \delta(t)$

If $\delta^{(1)}(t)$ is included in $h(t)$, then the highest order term

$D^N \delta^{(1)}(t) \longrightarrow \boxed{\delta^{(N+1)}(t)}$ $\quad \mathbf{b_0} D^N \delta(t) \longrightarrow \boxed{\delta^{(N)}(t)}$

can't match

*h(t)* cannot contain $\delta^{(i)}$(t) at all ➡ *h(t)* can contain at most δ(t) $\quad M = N$




$h(t) = \boxed{b_0 \delta(t)} + char\ mode\ terms \quad t \ge 0$ $\quad M = N$

# Particular solution at $t=0$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,x(t)}{dt} + b_M\, x(t)$$

**General System S**

when x(t) = δ(t) : forcing function

$$y_p(t) = b_0 \delta^{(M)}(t) + b_1 \delta^{(M-1)}(t) + \cdots + b_{M-1} \delta^{(1)}(t) + b_M \delta(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = x(t)$$

**Base System S0**

when x(t) = δ(t) : forcing function

$$y_p(t) = \delta(t)$$

- Requirements of an Impulse Response

# Solutions of Differential Equations : h(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,x(t)}{dt} + b_M\, x(t)$$

**requirement at time t = 0**

All the derivatives of h(t) up to ***N*** must match a corresponding derivatives of the impulse up to ***M*** at time t=0

**requirement at time t ≠ 0**

The linear combination of all the derivatives of h(t) must add to zero for any time t≠0

$y_h(t)u(t)$ is such a function

$y_h(t)$ is the homogeneous solution

**Case 1 N > M**

The derivatives of the $y_h(t)u(t)$ provide all the singularity functions necessary to match the impulse and derivatives of the impulse on the right side and no other terms need to be added

**Case 2 N = M**

Need to add an impulse term $K_0\delta(t)$.. and solve for $K_0$ by matching coefficients of impulses on both sides

**Case 3 N < M**

The ***N****-th* derivative of the function we add to $y_h(t)u(t)$ must have a term that matches the ***M****-th* derivative of the unit impulse. We have to add these terms

$$K_{m-n} u_{m-n}(t) + \cdots + K_1 u_1(t) + K_0 u_0(t)$$

$$= K_{m-n}\delta^{(m-n)}(t) + \cdots + K_1\delta^{(1)}(t) + K_0\delta^{(0)}(t)$$

# Integrals of $y_h(t) \cdot u(t)$

$$\int_{-\infty}^{t} \cdots \int_{-\infty}^{t} y_h(t) u(t)\, dt \cdots dt \;\rightarrow\; y_h(t) u(t)$$

*linear equation with constant coefficients*

$$g(t) = y_h(t) u(t)$$

$$\begin{cases} h(t) = g^{(M-N+1)}(t) = \displaystyle\int_{-\infty}^{t} \cdots \int_{-\infty}^{t} y_h(t) u(t)\, dt \cdots dt = y_h(t) u(t) & (N > M) \\ h(t) = g(t) + m_0 \delta(t) & (N = M) \\ h(t) = g(t) + m_0 \delta(t) + m_1 \dot{\delta}(t) + \cdots + m_{M-N} \delta^{(M-N)}(t) & (N < M) \end{cases}$$

$$\begin{cases} h(t) = y_h(t) u(t) & (N > M) \\ h(t) = y_h(t) u(t) + m_0 \delta(t) & (N = M) \\ h(t) = y_h(t) u(t) + m_0 \delta(t) + m_1 \dot{\delta}(t) + \cdots + m_{M-N} \delta^{(M-N)}(t) & (N < M) \end{cases}$$

# Case Examples

$(N>M)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d h(t)}{dt} + a_N h(t) \ =$$

$$b_0 \frac{d^{N-1} \delta(t)}{dt^{N-1}} + b_1 \frac{d^{N-2} \delta(t)}{dt^{N-2}} + \cdots + b_{N-2} \frac{d \delta(t)}{dt} + b_{N-1} \delta(t)$$

$M = N-1$

in most systems N ≥ M

$h(t) \ =$

$y_h(t) u(t)$

$(N=M)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d h(t)}{dt} + a_N h(t) \ =$$

$$b_0 \frac{d^N \delta(t)}{dt^N} + b_1 \frac{d^{N-1} \delta(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{d \delta(t)}{dt} + b_N \delta(t)$$

$M = N$

$h(t) \ =$

$y_h(t) u(t) + m_0 \delta(t)$

$(N<M)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d h(t)}{dt} + a_N h(t) \ = \ \delta(t)$$

$$b_0 \frac{d^{N+1} \delta(t)}{dt^{N+1}} + b_1 \frac{d^N \delta(t)}{dt^N} + \cdots + b_N \frac{d \delta(t)}{dt} + b_{N+1} \delta(t)$$

$M = N+1$

seldom used N < M

$h(t) \ =$

$y_h(t) u(t) + m_0 \delta(t) + m_1 \dot{\delta}(t)$

- An Impulse Response of a Causal LTI System

# ODE's and Causal LTI Systems

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, \delta(t)}{dt} + b_M \delta(t)$$

N > M : (N-M) integrator

N < M : (M-N) differentiator – magnify high frequency components of noise (*seldom used*)

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$\boxed{N \geq M}$ *in most systems*

$$h(t) = y_h(t)u(t) \qquad (N > M)$$

$$h(t) = y_h(t)u(t) + m_0 \delta(t) \qquad (N = M)$$

# h(t) can have *at most* a δ(t) for most systems

$y(t) \leftarrow \boxed{h(t)} \leftarrow x(t)$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

$h(t) \leftarrow \boxed{h(t)} \leftarrow \delta(t)$

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h = b_0 \delta^{(N)} + b_1 \delta^{(N-1)} + \cdots + b_{N-1} \delta^{(1)} + b_N \delta$$

$if\ h\ contain\ \delta$ $\quad C\delta^{(N)} \quad \cdots \quad \cdots \quad = \quad b_0 \delta^{(N)}$

the highest order derivatives of δ(t)

cannot be matched

$if\ h\ contain\ \delta^{(1)}$ $\quad C\delta^{(N+1)} \quad \cdots \quad \cdots \quad \neq \quad b_0 \delta^{(N)}$

the highest order derivatives of δ(t)

cannot be matched

$if\ h\ contain\ \delta^{(2)}$ $\quad C\delta^{(N+2)} \quad \cdots \quad \cdots \quad \neq \quad b_0 \delta^{(N)}$

the highest order derivatives of δ(t)

$t = 0$

h(t) can have at most an impulse $b_0 \delta(t)$
no derivatives of $\delta(t)$ possible at all

in most systems $N \geq M$

# h(t) can have at most a δ(t) (N ≥ M)

$N = M \qquad Q(D)\,y(t) = P(D)\,x(t)$

$N \qquad\qquad M \quad (N \geq M)$

$$\underline{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\,h(t)} = \boxed{(b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\,\delta(t)}$$

the highest order term
in the LHS
if $\delta'(t)$ is included in $h(t)$

the highest order term
in the RHS

$\frac{d^N h}{dt^N} \to \delta^{(N+1)}(t)$ $\boxed{\delta^{(N+1)}(t)}$ $\neq$ $\boxed{\delta^{(N)}(t)}$ ➡ contradiction

*h(t)* cannot contain $\delta^{(i)}(t)$ at all ➡ *h(t)* can contain *at most* δ(t) only when $N = M$

- An Impulse Response and System Responses

# Causality

Causal Signals:
the input signals
starts at time $t = 0$

Causal System:
the response $h(t)$ cannot
begin before the input

Causal Signals:
the output signals
starts at time $t = 0$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$t \geq 0$ $t \geq 0$ $t \geq 0$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau)\, d\tau$$

$x(\tau) = 0 \quad (\tau < 0)$ causal signal: $x(t)=0 \quad (t < 0)$

$h(t-\tau) = 0 \quad (t-\tau < 0)$ causal system: $h(t)=0 \quad (t < 0)$

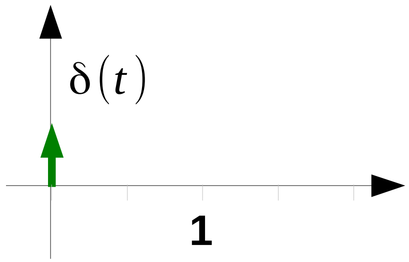
causal signal: $y(t)=0 \quad (t < 0)$

$$y(t) = x(t) * h(t) = \int_{0^-}^{t} x(\tau) h(t-\tau)\, d\tau$$

$$= \int_{0^-}^{t} x(t-\tau) h(\tau)\, d\tau$$

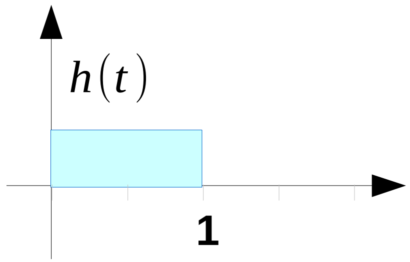
$0^-$ to include an impulse function $\delta(t)$ in $h(t)$

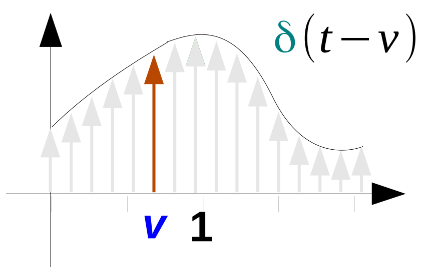
# h(t) as a ZSR (t ≥ 0)




t ≥ 0      t ≥ 0      t ≥ 0

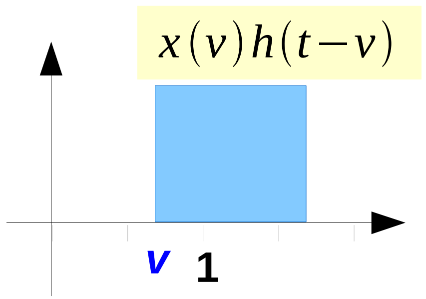
$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$



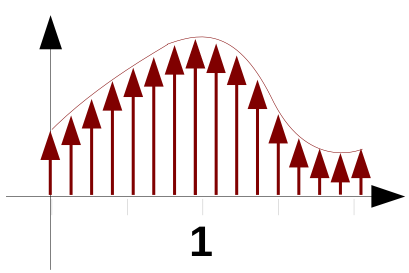




$$\delta(t-v) \rightarrow h(t-v)$$

$$x(v)\,\delta(t-v) \rightarrow x(v)\,h(t-v)$$

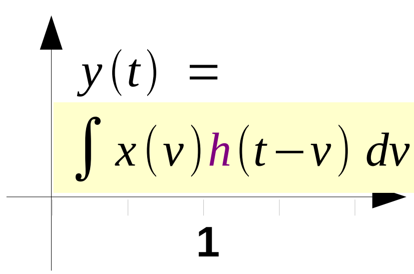



input value at time v → x(v)
delayed impulse response → x(v) h(t - v)




$$x(t) = \int x(v)\delta(t-v)\,dv$$

*No initial condition is used to compute the output*




$$y(t) = \int x(v)h(t-v)\,dv$$

**Zero State (initially at rest at t=0⁻ )**

# h(t) as a ZIR (t > 0)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$h(t) \;=\; y_h(t)u(t)$$

*The solution of the IVP with the following I.C.*

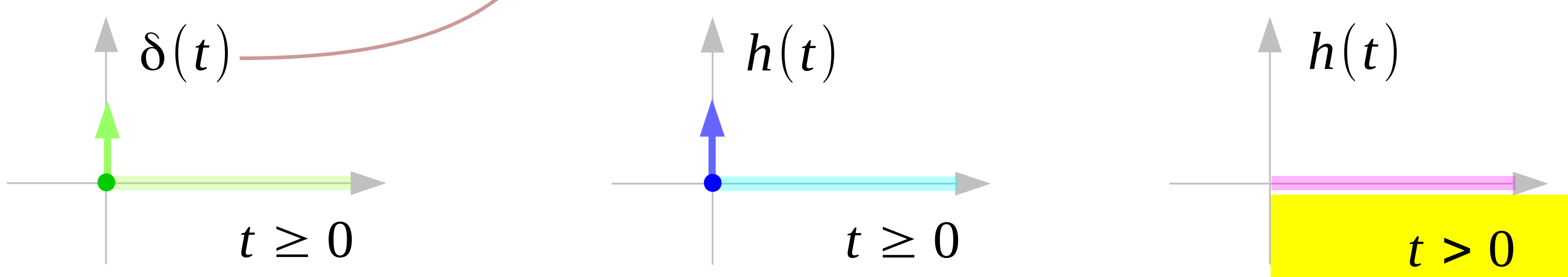
## Interval of Validity : (t > 0)

**Zero Input (no input for t > 0 )**

$\delta(t) = 0 \quad for \quad t > 0$

creates nonzero i. c

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

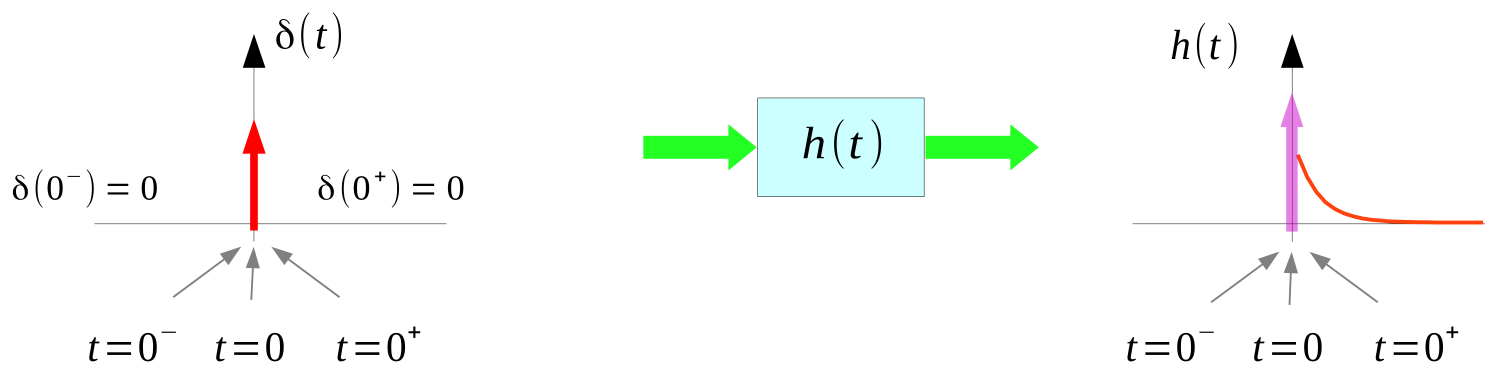



*an impulse may be present*

*No impulse is assumed*

**ZIR (no input for t > 0 )**

# Impulse Response h(t)




* general **y(t)** cannot be a **ZIR** for a general input **x(t)** (generally x(t) ≠ 0 for t > 0)

* but an impulse input vanishes (x(t) = δ(t) = 0 for t > 0 )

$(N \geq M)$

**h(t) : ZIR** with the newly created I.C. (t > 0)

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

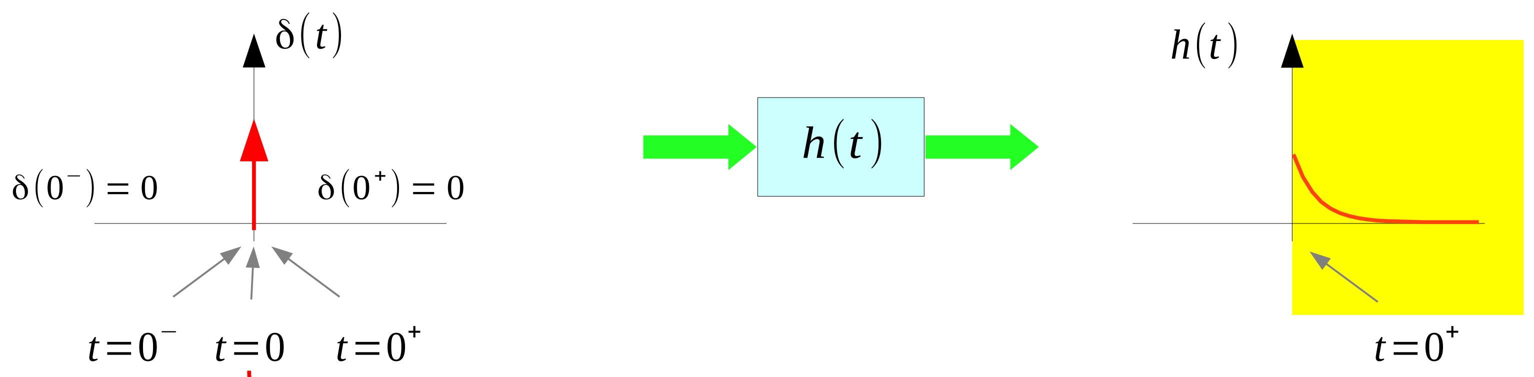
non-zero i. c.

$\exists\, i, \quad k_i \neq 0$

| | |
|---|---|
| $t \geq 0^+$ $(t \neq 0)$ | $h(t) = \textit{char mode terms}$ |
| $t = 0$ | $h(t) = b_0 \delta(t)$ at most an impulse |
| $t \geq 0$ | $h(t) = b_0 \delta(t) + \textit{char mode terms}$ |

**h(t)** : **ZSR** to an impulse input

# Assumed Finite Jumps




- generates energy storage
- creates nonzero i. c. at $t=0^+$

$$\begin{aligned} h^{(N-1)}(0^-) &= 0 \\ h^{(N-2)}(0^-) &= 0 \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^-) &= 0 \\ h(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} h^{(N)}(t) &= k_{N-1}\delta + \sum_i k_i \delta^{(i)} \\ h^{(N-1)}(t) &= k_{N-2}\delta + \sum_i k_i \delta^{(i)} \\ \vdots \quad & \quad \vdots \\ h^{(2)}(t) &= k_1 \delta + \sum_i k_i \delta^{(i)} \\ h^{(1)}(t) &= k_0 \delta + \sum_i k_i \delta^{(i)} \end{aligned}$$

*Impulse matching*

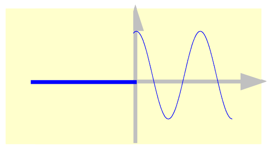
$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

non-zero initial conditions
$\exists\, i, \quad k_i \neq 0$

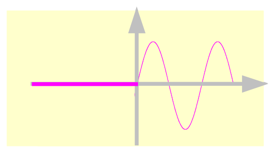
Assumed finite jumps

# System Responses and Valid Intervals

**Causal Inputs**

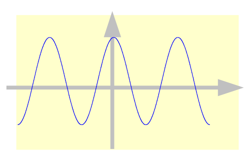


input applied at $t = 0$

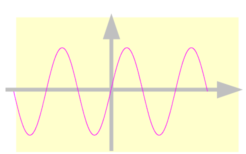


| | | |
|---|---|---|
| **ZIR + ZSR** | expression assumes | $-\infty < t < +\infty$ |
| | already suppressed | $t < 0$ *part* |
| | **valid interval** | $[-\infty, \ +\infty]$ |
| $y_n + y_p$ | expression assumes | $-\infty < t < +\infty$ |
| | must disregard | $t < 0$ *part* |
| | **valid interval** | $[0, \ +\infty]$ |

**Non-causal Inputs**

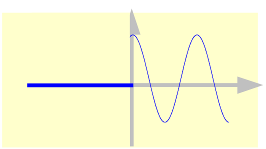
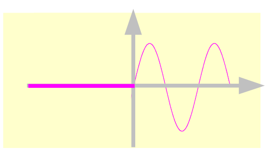


input applied at $t = -\infty$



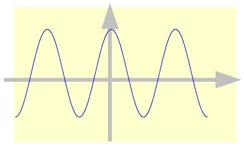
| | | |
|---|---|---|
| **ZIR + ZSR** | expression assumes | $-\infty < t < +\infty$ |
| | not valid | $t < 0$ *part* |
| | **valid interval** | $[0, \ +\infty]$ |
| $y_n + y_p$ | expression assumes | $-\infty < t < +\infty$ |
| | valid also for | $t < 0$ *part* |
| | **valid interval** | $[-\infty, \ +\infty]$ |

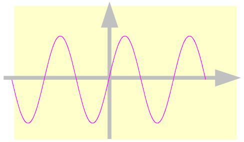
# Initial Conditions for Causal Sinusoidal Inputs

| | $[-\infty, \ 0^-]$ | | $[0^+, \ +\infty]$ |
|---|---|---|---|
| zero input response | $y_{zi}(0^-) \neq 0$ | $=$ | $y_{zi}(0^+)$ |
| + | + | | + |
| zero state response | $y_{zs}(0^-) = 0$ | $\neq$ | $y_{zs}(0^+)$ |
|  | $\Vert$ | | $\Vert$ |
| | $y(0^-)$ | $\neq$ | $y(0^+)$ |
|  | $\not\Vert$ | | $\Vert$ |
| natural response | $y_h(0^-)$ | $=$ | $y_h(0^+)$ |
| + | + | | + |
| forced response | $y_p(0^-)$ | $=$ | $y_p(0^+)$ |

# Initial Conditions for Everlasting Sinusoidal Inputs

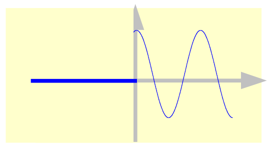
| | $[-\infty, \ 0^-]$ | | $[0^+, \ +\infty]$ |
|---|---|---|---|
| zero input response | $y_{zi}(0^-) \neq 0$ | $=$ | $y_{zi}(0^+)$ |
| + | + | | + |
| zero state response | $y_{zs}(0^-) = 0$ | $\neq$ | $y_{zs}(0^+)$ |
| | $\Vert$ | | $\Vert$ |
| | $y(0^-)$ | $=$ | $y(0^+)$ |
| | $\Vert$ | | $\Vert$ |
| natural response | $y_h(0^-) = 0$ | $=$ | $y_h(0^+) = 0$ |
| + | + | | + |
| forced response | $y_p(0^-) \neq 0$ | $=$ | $y_p(0^+)$ |



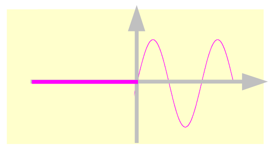


# Sinusoidal Inputs and System Responses

**Causal Inputs**



input applied at t = 0

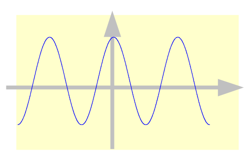


$\rightarrow h(t) \rightarrow$ ZSR

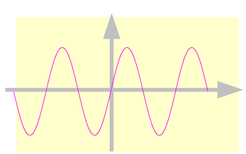
**ZIR + ZSR** **(O)**

**$y_n + y_p$** **(Δ)**

**Non-causal Inputs**



input applied at t = -∞



$\rightarrow h(t) \rightarrow$ forced

**ZIR + ZSR** **(X)**

**$y_n + y_p$** **(O)**

# Total Response for Causal Inputs

zero input response

+

zero state response

$[-\infty, \ 0^-]$

$y(t) = y_{zi}(t) \Leftarrow t \le 0^-$

because the input has not started yet

continuous at $t = 0$

$$y(0^-) = \boxed{y_{zi}(0^-)} = y_{zi}(0^+)$$
$$\dot{y}(0^-) = \boxed{\dot{y}_{zi}(0^-)} = \dot{y}_{zi}(0^+)$$

$[0^+, \ +\infty]$

$y(t) = y_{zi}(t) + y_{zs}(t)$

$y(0^+) \ne y(0^-)$

possible discontinuity at $t = 0$

$$\begin{cases} y(0^+) = \boxed{y_{zi}(0^+) + y_{zs}(0^+)} \\ \dot{y}(0^+) = \boxed{\dot{y}_{zi}(0^+) + \dot{y}_{zs}(0^+)} \end{cases}$$

natural response

+

forced response

$$\begin{cases} y_h(0^-) \ne \boxed{y_{zi}(0^-)} \\ \dot{y}_h(0^-) \ne \boxed{\dot{y}_{zi}(0^-)} \end{cases}$$

$$\begin{cases} y_p(0^-) \ne \boxed{y_{zi}(0^-)} \\ \dot{y}_p(0^-) \ne \boxed{\dot{y}_{zi}(0^-)} \end{cases}$$

$[0^+, \ +\infty]$

$y(t) = y_h(t) + y_p(t)$

$$\begin{cases} y(0^+) = \boxed{y_h(0^+) + y_p(0^+)} \\ \dot{y}(0^+) = \boxed{\dot{y}_h(0^+) + \dot{y}_p(0^+)} \end{cases}$$

Interval of validity $t > 0$

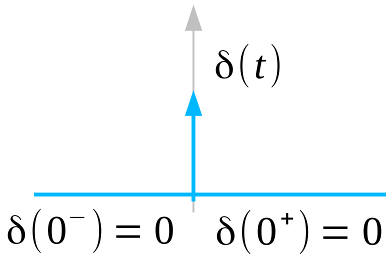
# Impulse In, Impulse Out

$$\frac{d\,h(t)}{dt} + \boldsymbol{a_1} h(t) = \boldsymbol{b_1} \delta(t)$$

$$\frac{d\,h(t)}{dt} + \boldsymbol{a_1} h(t) = \boldsymbol{b_0} \frac{d\,\delta(t)}{dt} + \boldsymbol{b_1} \delta(t)$$

$$\delta(t) \in \quad h(t) \leftarrow \boxed{h(t)} \leftarrow \delta(t)$$

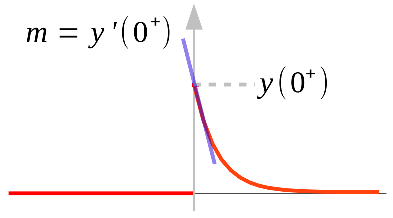
$$\delta(t) \notin \quad h(t) \leftarrow \boxed{h(t)} \leftarrow \delta(t)$$




$(\boldsymbol{0}, \boldsymbol{b_1})$

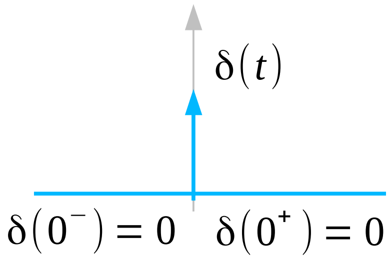
$$\rightarrow \boxed{h(t)} \rightarrow$$

$(1, \boldsymbol{a_1})$




Case **N>M** ($b_0$=0)

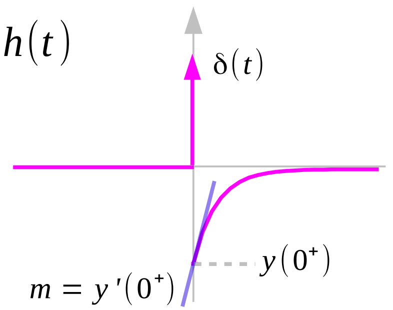
No delta function




$(\boldsymbol{b_0}, \boldsymbol{b_1})$

$$\rightarrow \boxed{h(t)} \rightarrow$$

$(1, \boldsymbol{a_1})$




Case **N=M** ($b_0$≠0)

delta function

# Impulse input creates initial conditions

$$\frac{d\,h(t)}{dt} + a_1 h(t) = b_1 \delta(t)$$

$$\delta(t) \in h(t) \leftarrow \boxed{h(t)} \leftarrow \delta(t)$$

$$\frac{d\,h(t)}{dt} + a_1 h(t) = b_0 \frac{d\,\delta(t)}{dt} + b_1 \delta(t)$$

$$\delta(t) \notin h(t) \leftarrow \boxed{h(t)} \leftarrow \delta(t)$$

All initial conditions are zero at $t = 0^-$

generates energy storage creates nonzero initial condition at $t = 0^+$

All zero Initial Conditions

~~All zero Initial Conditions~~

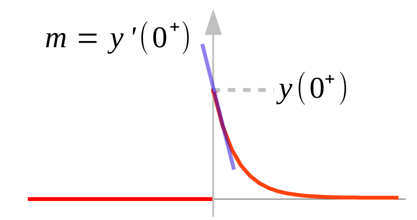
$$y(0^-) = 0 \qquad y'(0^-) = 0$$

$$y(0^+) = K_1 \qquad y'(0^+) = K_2$$

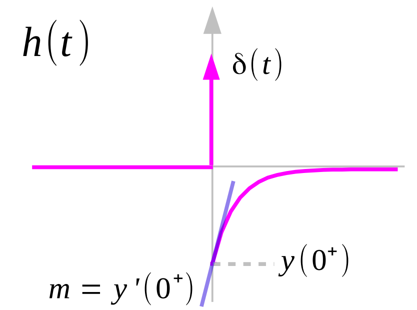
IVP (Initial Value Problem)

$(0, b_1)$ $(1, a_1)$

$(b_0, b_1)$ $(1, a_1)$ $\xrightarrow{\text{determines}}$ $K_1$ $K_2$

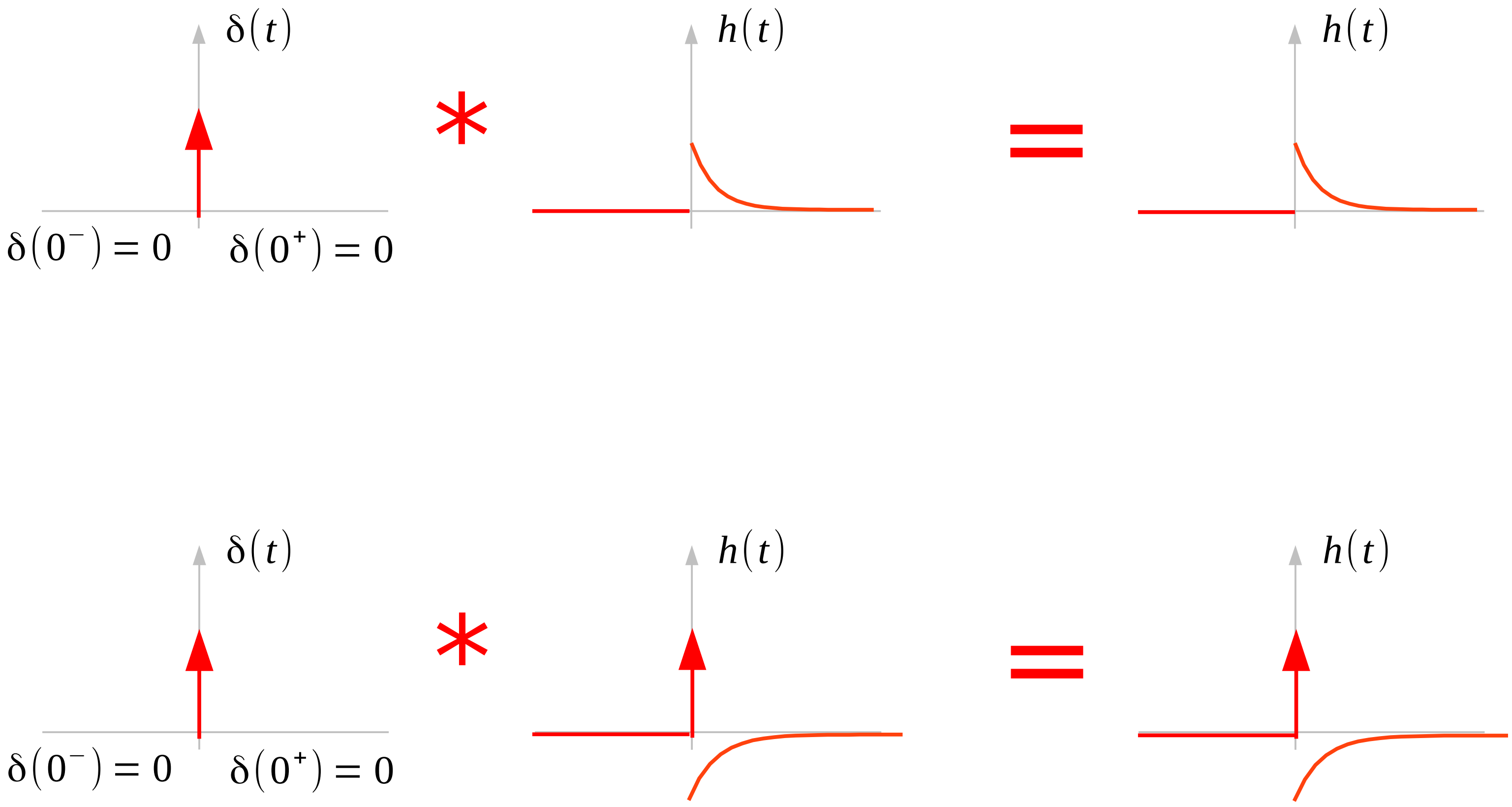



Case **N>M** ($b_0$=0)
No delta function




Case **N=M** ($b_0$≠0)
delta function

# Convolution with an impulse input



# Derivatives of a $\delta(t)$

$$\frac{d\,y(t)}{dt} + a_1\,y(t) = + b_0\frac{d\,x(t)}{dt} + b_1\,x(t) \quad \Rightarrow \quad \frac{d\,\mathbf{h}(t)}{dt} + a_1\,\mathbf{h}(t) = + b_0\frac{d\,\boldsymbol{\delta}(t)}{dt} + b_1\,\boldsymbol{\delta}(t)$$

if $h(t)$ contains $\delta^{(1)}(t) \quad (b_0 \neq 0)$

$$\delta^{(2)}(t) + \cdots \neq b_0\,\delta^{(1)}(t) + b_1\,\delta(t)$$

the impulse response $h(t)$ can contain at most $\delta(t) \qquad (b_0 \neq 0)$

the impulse response $h(t)$ cannot contain any derivatives of $\delta(t)$

N ≥ M

$$\delta(t) \rightarrow \boxed{h(t)} \rightarrow \delta(t),\ \cancel{\dot{\delta}(t)},\ \cancel{\ddot{\delta}(t)},\ \cancel{\dddot{\delta}(t)},\ \cdots$$

$h(t) = b_0\delta(t) + char\ mode\ terms \quad t \geq 0$

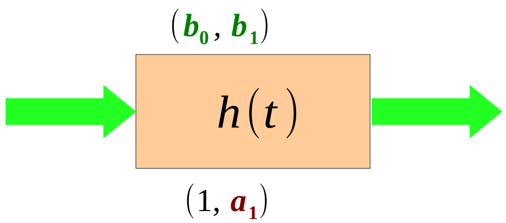
| | |
|---|---|
| $t \geq 0^+$ $(t \neq 0)$ | *h*(*t*) characteristic mode terms only |
| $t = 0$ | *h*(*t*) can have at most an impulse $b_0\delta(t)$ and some finite jumps |

# Finding an impulse response h(t)

**General System S**

$$\frac{d\,h(t)}{dt} + a_1 h(t) = b_0 \frac{d\,\delta(t)}{dt} + b_1 \delta(t)$$

$$h(t) = b_0 \delta(t) + \left(\sum_i c_i e^{-\lambda_i t}\right) u(t)$$




**Impulse Matching**

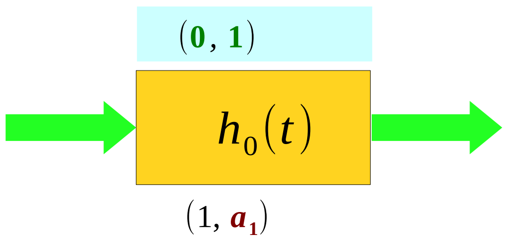
-determine $y(0^+),\ y'(0^+)$

-then determine $c_{0,}\ c_1$

**Base System S0**

$$\frac{d\,h(t)}{dt} + a_1 h(t) = \delta(t)$$

$$h(t) = b_0 \delta(t) + \left[(b_0 D^2 + b_1 D + b_2)\left(\sum_i e_i e^{-\lambda_i t}\right)\right] u(t)$$




**Simplified Impulse Matching**

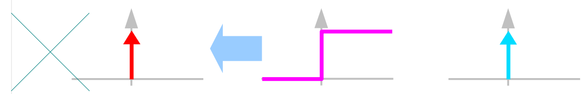
-use $y(0^+) = 0,\ y'(0^+) = 1$

-then determine $e_{0,}\ e_1$

# Simplified Impulse Matching

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \quad \cdots \quad + a_{N-1} y_n^{(1)}(t) + a_N y_n(t) = \delta(t)$$

**Base System S0**



$y_n(t)$ **must** continuous

$$y_n(0^+) = y_n(0^-) = 0$$

If $y_n(t)$ is discontinuous, then there should exist derivatives of delta : $\delta^{(i)}(t)$

$y_n^{(1)}(t)$ **must** continuous

$$y_n^{(1)}(0^+) = y_n^{(1)}(0^-) = 0$$

If $y_n^{(1)}(t)$ is discontinuous, then there should exist derivatives of delta : $\delta^{(i)}(t)$

$y_n^{(N-1)}(t)$ **must** be discontinuous

$$y_n^{(N-1)}(0^+) = 1, \quad y_n^{(N-1)}(0^-) = 0$$

$y_n(t)$ contains no impulse, but characteristic modes only

$N > M \ (=0)$

## References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)