Background – Transfer Function (4A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Valid Interval of ZIR & ZSR IVPs

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(0^-) = y_0$$
$$y'(0^-) = y_1$$

$$y = y_h + y_p$$

$$0^- < t < \infty$$

$$y(0^-) = y_h(0^-) + y_p(0^-) = y_0 + 0 = y_0$$
$$y'(0^-) = y_h'(0^-) + y_p'(0^-) = y_1 + 0 = y_1$$

$$y'' + P(x)y' + Q(x)y = 0$$
$$y(0^-) = y_0$$
$$y'(0^-) = y_1$$

$$y_h$$

Nonzero Initial Conditions

Zero Input Response

Response due to the initial conditions

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(0^-) = 0$$
$$y'(0^-) = 0$$

$$y_h + y_p$$

Zero Initial Conditions

Zero State Response

Response due to the forcing function f

ZIR & ZSR

$$y'' + 3y' + 2y = x(t) \qquad y(0^-) = k_1, \quad y'(0^-) = k_2$$

$$[s^2 Y(s) - s\,y(0^-) - y'(0^-)] + 3[s\,Y(s) - y(0^-)] + 2Y(s) = X(s)$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1 + X(s)$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{X(s)}{(s+1)(s+2)}$$

Zero Input Response

$$x(t) = 0$$

$$Y(s) = \frac{s+5}{(s+1)(s+2)}$$

$$= +4\frac{1}{(s+1)} - 3\frac{1}{(s+2)}$$

$$\leftrightarrow \quad y = 4e^{-t} - 3e^{-2t}$$

Zero State Response

$$y(0^-) = 0, \quad y'(0^-) = 0$$

$$Y(s) = \frac{X(s)}{(s+1)(s+2)} \qquad x(t) = e^{+t}$$

$$= +\frac{1}{3}\frac{1}{(s+2)} - \frac{1}{2}\frac{1}{(s+1)} + \frac{1}{6}\frac{1}{(s-1)}$$

$$\leftrightarrow \quad y = -\frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t} + \frac{1}{6}e^{+t}$$

Transfer Function

$$y'' + 3y' + 2y = x(t) \qquad y(0^-) = k_1, \quad y'(0^-) = k_2$$

$$[s^2Y(s) - s\,y(0^-) - y'(0^-)] + 3[s\,Y(s) - y(0^-)] + 2Y(s) = X(s)$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1 + X(s)$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{X(s)}{(s+1)(s+2)}$$

Zero State Response

$$y(0^-) = 0, \quad y'(0^-) = 0$$

$$Y(s) = \frac{X(s)}{(s+1)(s+2)}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{(s+1)(s+2)}$$

Transfer Function

Transfer Function & Impulse Response

$$y'' + 3y' + 2y = x(t) \qquad y(0^-) = k_1, \quad y'(0^-) = k_2$$

$$[s^2 Y(s) - s\,y(0^-) - y'(0^-)] + 3[s\,Y(s) - y(0^-)] + 2Y(s) = X(s)$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1 + X(s)$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{X(s)}{(s+1)(s+2)}$$

Transfer Function

$$\frac{Y(s)}{X(s)} = H(s)$$

$$Y(s) = H(s)X(s) \quad \Longleftrightarrow \quad y(t) = h(t) * x(t)$$

$$H(s) \quad \Longleftrightarrow \quad h(t)$$

Transfer Function — **Impulse Response**

Zero State Response

$$y(0^-) = 0, \quad y'(0-) = 0$$

$$Y(s) = \frac{X(s)}{(s+1)(s+2)}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{(s+1)(s+2)}$$

Transfer Function

Transfer Function and an ODE

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) \;=\; b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$(s^N + a_1 s^{N-1} + \cdots + a_{N-1} s + a_N) \cdot Y(s) \;=\; (b_0 s^M + b_1 s^{M-1} + \cdots + b_{N-1} s + b_N) \cdot X(s)$$

$$H(s) \;=\; \frac{Y(s)}{X(s)} \;=\; \frac{(b_0 s^M + b_1 s^{M-1} + \cdots + b_{N-1} s + b_N)}{(s^N + a_1 s^{N-1} + \cdots + a_{N-1} s + a_N)}$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) \;=\; (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \;=\; (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)$$

$$Q(s) \;=\; (s^N + a_1 s^{N-1} + \cdots + a_{N-1} s + a_N)$$

$$P(D) \;=\; (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N)$$

$$P(s) \;=\; (b_0 s^M + b_1 s^{M-1} + \cdots + b_{N-1} s + b_N)$$

$$Q(s) \cdot Y(s) \;=\; P(s) \cdot X(s)$$

$$H(s) \;=\; \frac{Y(s)}{X(s)} \;=\; \frac{P(s)}{Q(s)}$$

* *Everlasting Exponential Inputs*
* *Causal Exponential Inputs*

Exponential and Sinusoid Functions

exponential function

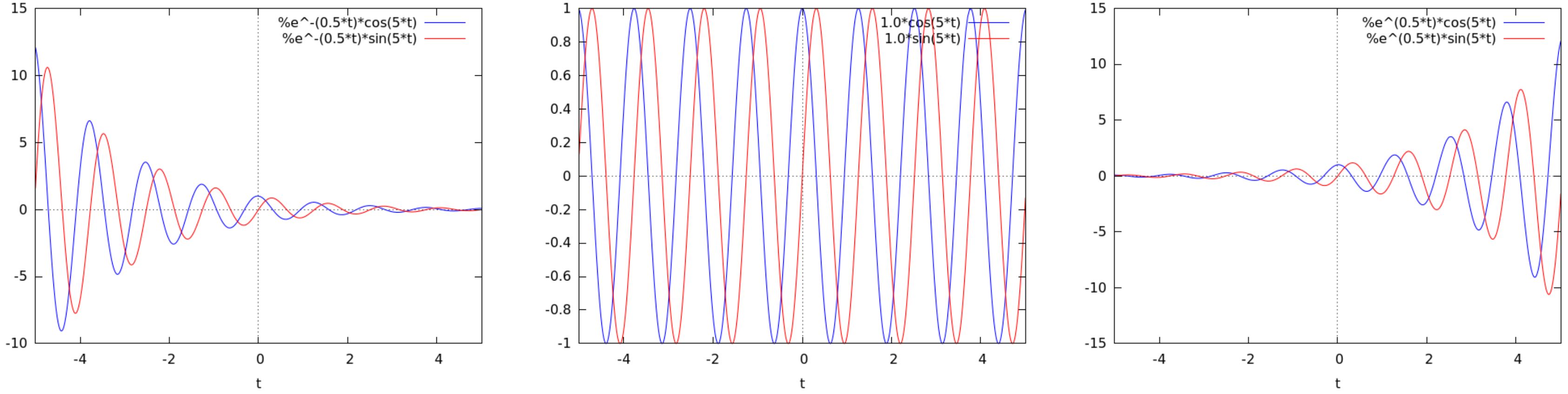
$$e^{st} = e^{\sigma t + i\omega t} \qquad (s = \sigma + i\omega)$$

$$= e^{\sigma t}(\cos(\omega t) + i\sin(\omega t))$$

sinusoid function

$$e^{\zeta t} = e^{i\omega t} \qquad (\zeta = i\omega)$$

$\Re\{s\} < 0 \ (\sigma < 0)$ $\qquad$ $\Re\{s\} = 0 \ (\sigma = 0)$ $\qquad$ $\Re\{s\} > 0 \ (\sigma > 0)$



Everlasting & Causal Function

• **exponential** function

$$\left(s = \sigma + i\omega\right)$$

• **sinusoid** function

$$\left(\zeta = i\omega\right)$$

• everlasting **exponential** function

applied at $t = -\infty$

e^{st}

• everlasting **sinusoid** function

applied at $t = -\infty$

$e^{\zeta t}$

• causal **exponential** function

applied at $t = 0$ applied at $t = 0$

$e^{st}\,u(t)$

• causal **sinusoid** function

$e^{\zeta t}\,u(t)$

Sinusoidal Inputs and States

- **everlasting sinusoid function**

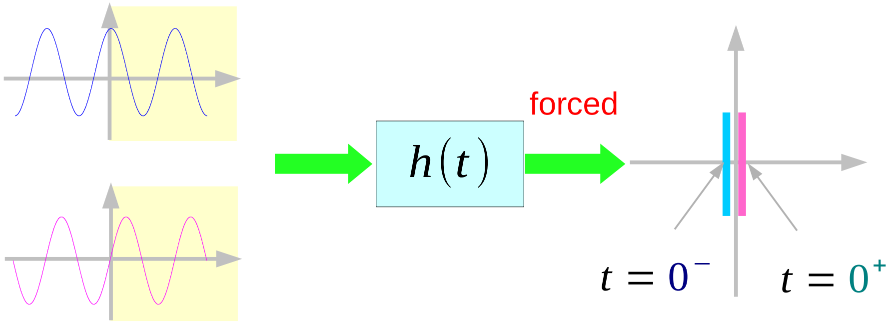
  state at t = 0– = state at t = 0+

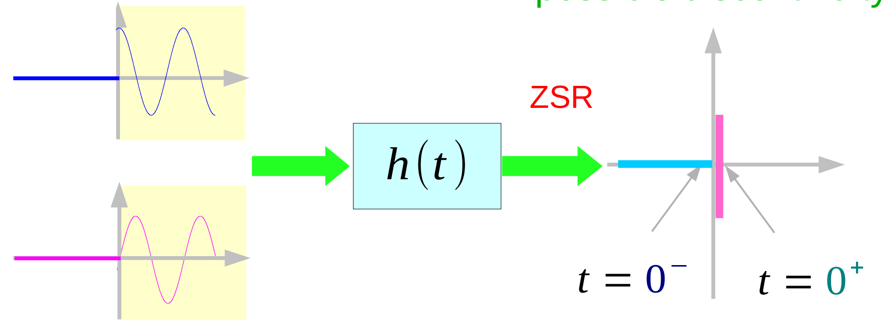
  continuous state between t = 0– & 0+

- **causal sinusoid function**

  zero state at t = 0–

  non-zero state at t = 0+





non-causal

ZIR + ZSR

natural + forced

ZIR + ZSR

natural + forced

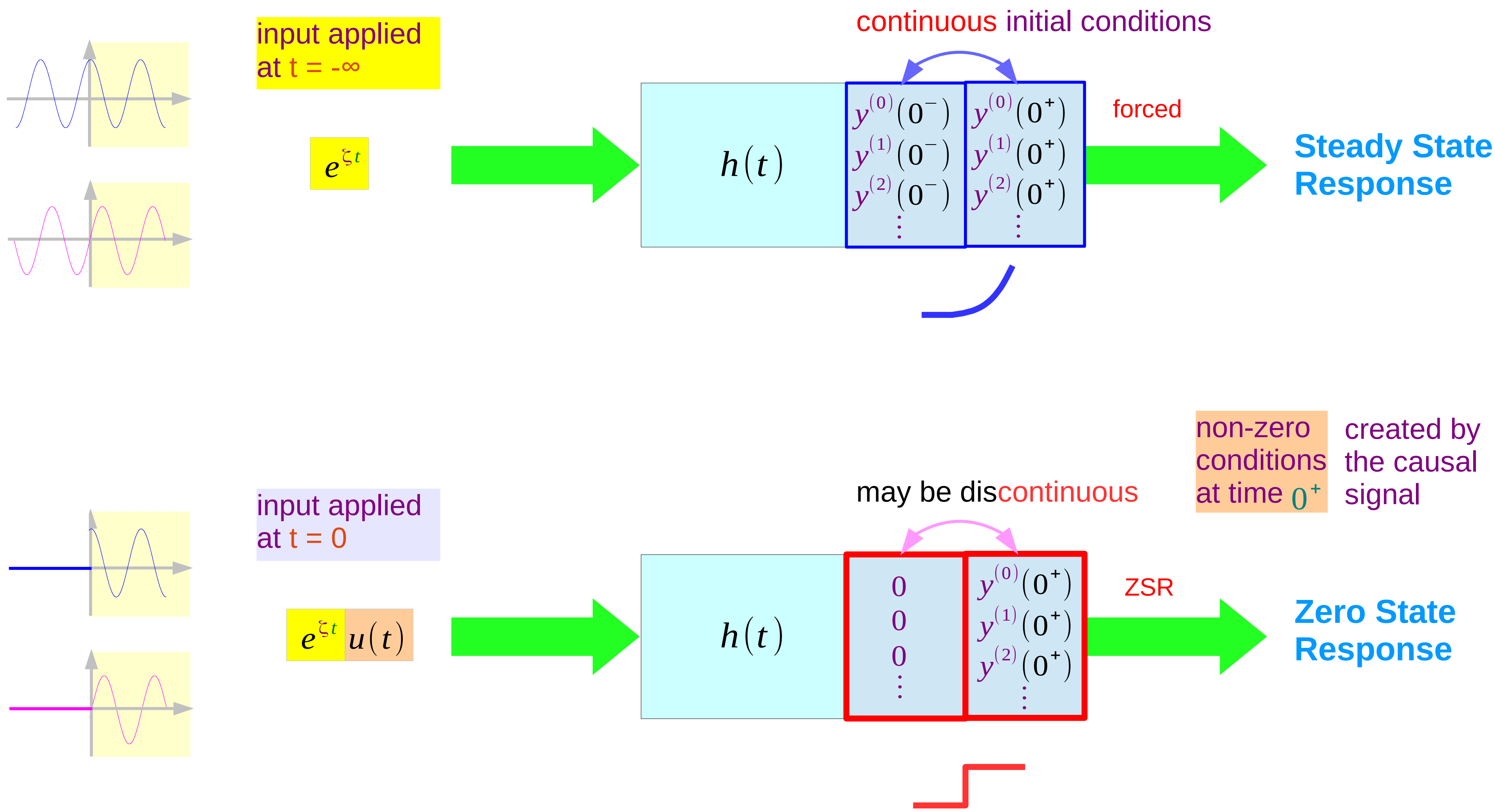
input applied at t = -∞

$\lim_{t \to \infty} y_p$

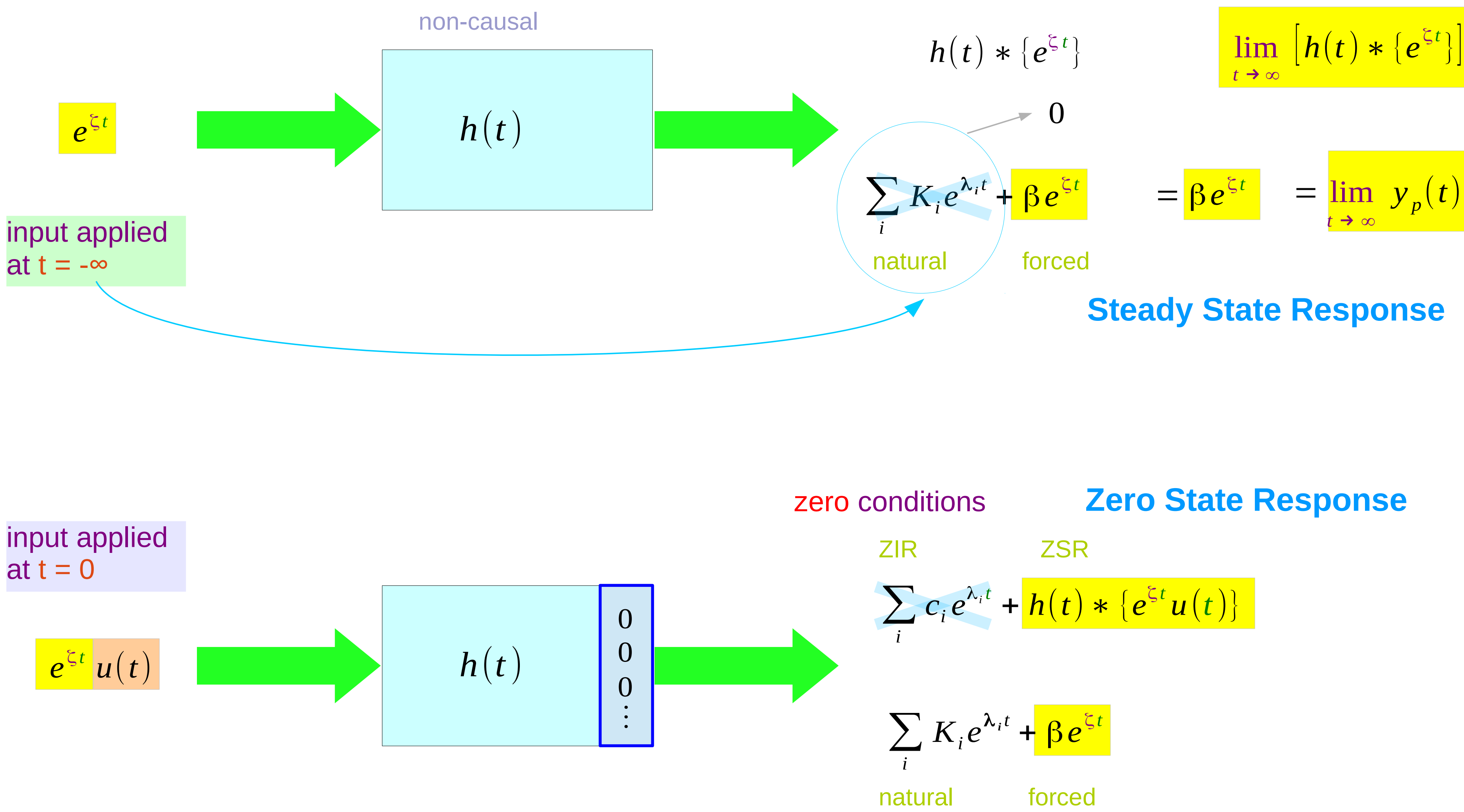
input applied at t = 0

$h(t) * \{e^{\xi t} u(t)\}$

States at t=0+ and t=0-



Steady State and Zero State Responses



Steady State Responses

$h(t) * \{e^{\zeta t}\}$

$\lim_{t \to \infty} [h(t) * \{e^{\zeta t}\}]$

$\lim_{t \to \infty} [h(t) * \{e^{\zeta t}\}]$

0

$\sum_i K_i e^{\lambda_i t} + \beta e^{\zeta t} = \beta e^{\zeta t} = \lim_{t \to \infty} y_p(t)$

natural forced

$\lim_{t \to \infty} \beta e^{\zeta t} = \beta e^{\zeta t} = \lim_{t \to \infty} y_p(t)$

Steady State Response

zero conditions

Zero State Response

h(t) can contain mode terms which approach to zero

ZIR ZSR

$\sum_i c_i e^{\lambda_i t} + h(t) * \{e^{\zeta t} u(t)\}$

$\lim_{t \to \infty} [h(t) * \{e^{\zeta t} u(t)\}]$

$\sum_i K_i e^{\lambda_i t} + \beta e^{\zeta t}$

natural forced

$\lim_{t \to \infty} \left[\sum_i K_i e^{\lambda_i t} + \beta e^{\zeta t}\right] = \beta e^{\zeta t} = \lim_{t \to \infty} y_p(t)$

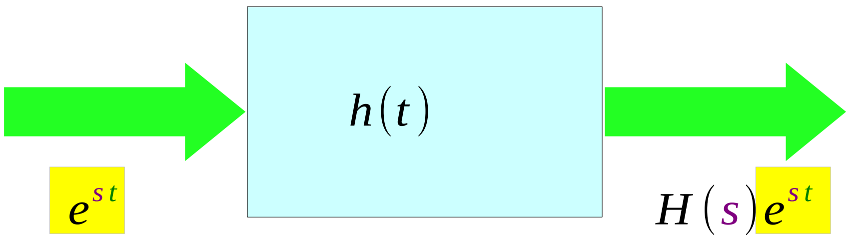
natural forced

for a pure imaginary ζ

Total Response to an <u>everlasting</u> exponential input

input applied
at $t = -\infty$

non-causal



$$
\begin{aligned}
y(t) &= h(t) * e^{st} \\
&= \int_{-\infty}^{+\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \\
&= e^{st} \cdot \boxed{\int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau}\, d\tau} \qquad \begin{matrix} h(t) = 0 \\ (t < 0) \end{matrix} \\
&= e^{st} \cdot \boxed{H(s)}
\end{aligned}
$$

convolution works for
non-causal input x(t) also

But causal system is assumed

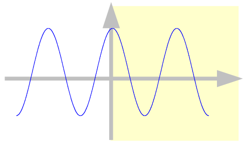
$$
\boxed{H(s) = \int_{0}^{+\infty} \boxed{h(\tau)}\, e^{-s\tau}\, d\tau}
$$

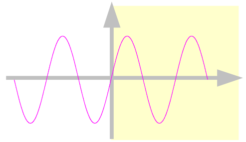
$$h(t) \leftrightarrow H(s)$$

Total Response to an everlasting exponential input

$$(b_0,\ b_1,\ \cdots,\ b_{N-1},\ b_N)$$

$$(1,\ a_1,\ \cdots,\ a_{N-1},\ a_N)$$





$$Q(D)\,y(t) = P(D)\,x(t)$$

$$Q(D)[h(t) * x(t)] = P(D)\,x(t)$$

$$Q(D)[e^{st} H(s)] = P(D)\,e^{st}$$

$$H(s)\,Q(D)\,e^{st} = P(D)\,e^{st}$$

$$D^r e^{st} = \frac{d^r}{dt^r} e^{st} = s^r e^{st}$$

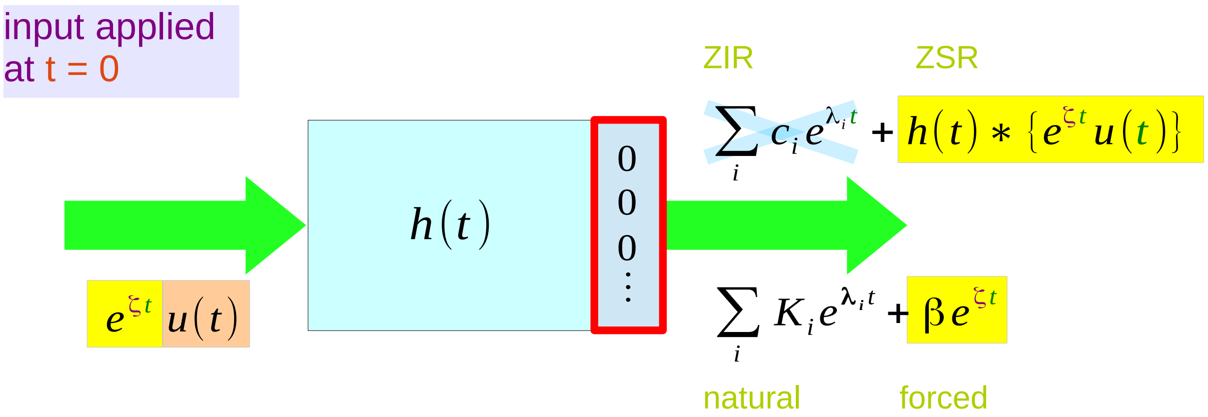
$$Q(D)\,e^{st} = Q(s)\,e^{st}$$

$$P(D)\,e^{st} = P(s)\,e^{st}$$

$$H(s)\,Q(s)\,e^{st} = P(s)\,e^{st}$$

$$H(s) = \frac{P(s)}{Q(s)}$$

Forced Response to a causal exponential input



natural　　forced

$$y(t) = \sum_i K_i e^{\lambda_i t} + \beta e^{\zeta t} \qquad (t > 0)$$

$$Y(s) = \sum_i \frac{K_i}{(s - \lambda_i)} + H(\zeta) X(\zeta)$$

For a forced response　　*any complex* ζ

For a steady response　　*a pure imaginary* ζ

Steady State Response

$$\lim_{t \to \infty} y_p(t) = \beta e^{\zeta t}$$

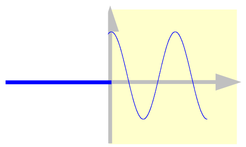
Transient Response

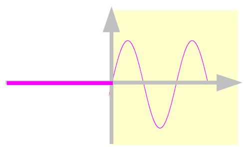
$$\{y(t) - \lim_{t \to \infty} y_p(t)\} = \sum_i K_i e^{\lambda_i t}$$

Forced Response to a causal exponential input

$$(b_0,\ b_1,\ \cdots,\ b_{N-1},\ b_N)$$

$$(1,\ a_1,\ \cdots,\ a_{N-1},\ a_N)$$





$$Q(D)\,y(t) = P(D)\,x(t)$$

$$Q(D)[\beta e^{\zeta t}] = P(D)\,e^{\zeta t}$$

$$\beta Q(D)\,e^{\zeta t} = P(D)\,e^{\zeta t}$$

$$D^r e^{\zeta t} = \frac{d^r}{dt^r} e^{\zeta t} = \zeta^r e^{\zeta t}$$

$$Q(D)\,e^{\zeta t} = Q(\zeta)\,e^{\zeta t}$$

$$P(D)\,e^{\zeta t} = P(\zeta)\,e^{\zeta t}$$

ζ : **NOT** a characteristic mode

$$\beta = \frac{P(\zeta)}{Q(\zeta)}$$

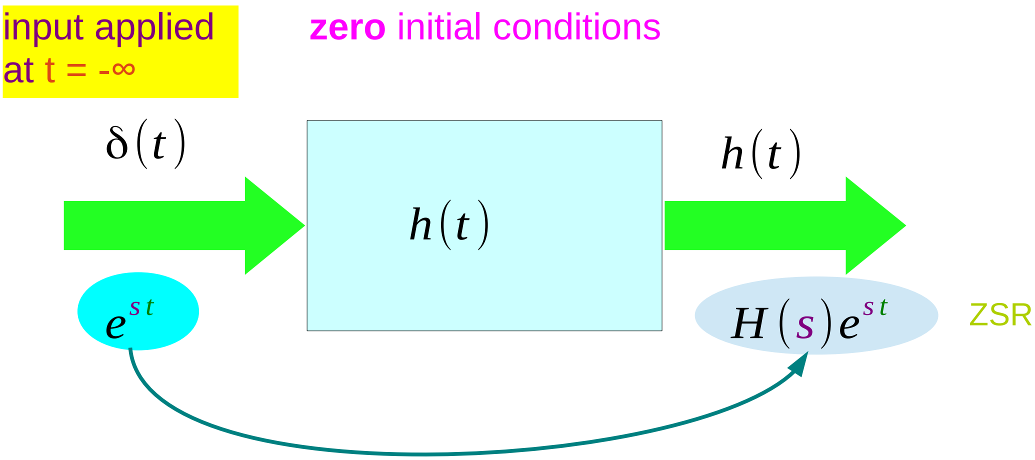
Transfer Function Definitions

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

ZSR

$$Q(s) \cdot Y(s) = P(s) \cdot X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)}$$



Transfer function (s-domain)

$$H(s) = \frac{Y(s)}{X(s)}$$

Transfer function (t-domain)

$$H(s) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{st}}$$

Polynomials of Differential Equation

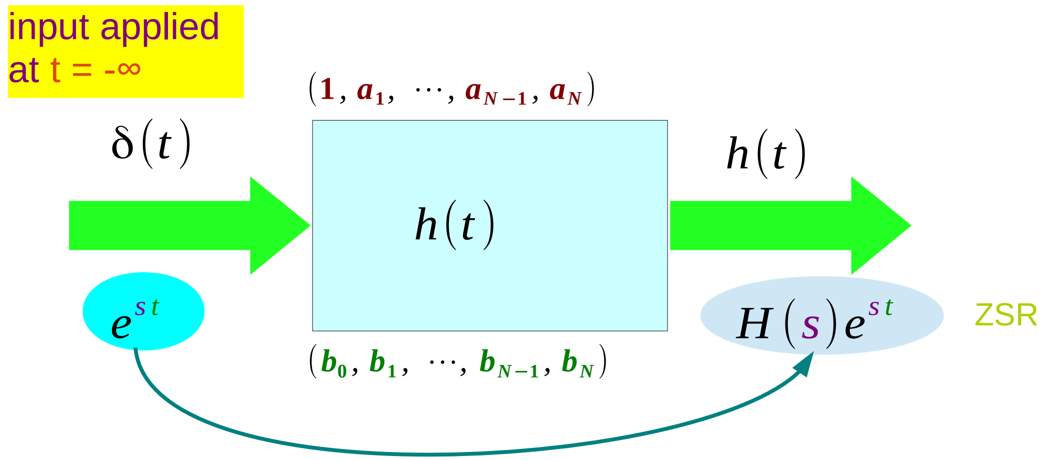
$$H(s) = \frac{P(s)}{Q(s)}$$

$(b_0, b_1, \cdots, b_{N-1}, b_N)$

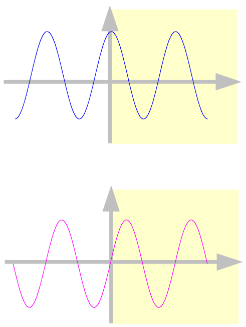
$(1, a_1, \cdots, a_{N-1}, a_N)$

Laplace Transform of h(t)

$$H(s) = \int_0^{+\infty} h(\tau) e^{-s\tau} d\tau$$



for a given $s = \zeta$



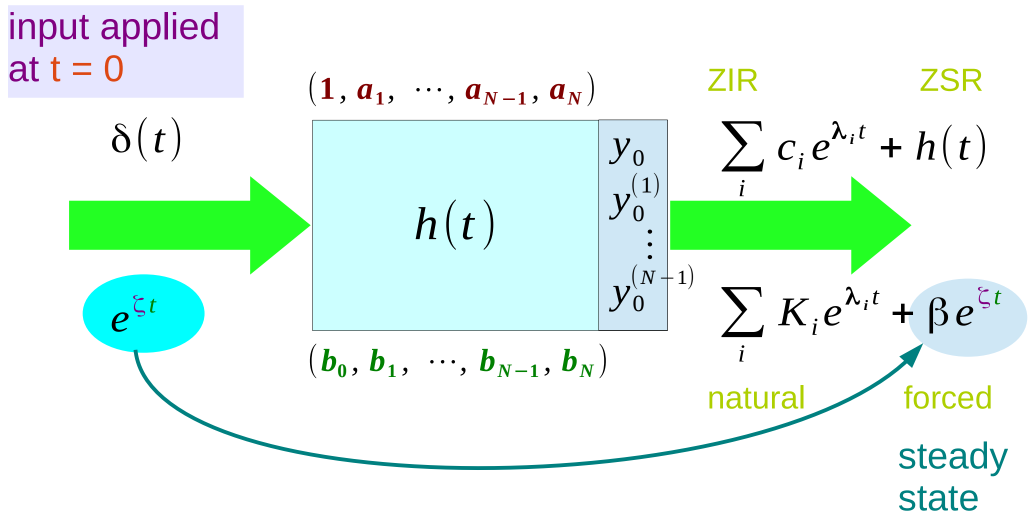
ZSR

$$y(t) = H(\zeta)e^{\zeta t} \qquad -\infty < t < +\infty$$

$$y(t) = H(\zeta)x(t) \qquad x(t) = e^{\zeta t}$$

$$Y(s) = H(s)X(s)$$

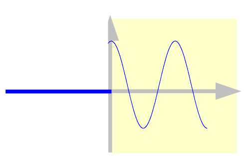
Causal Exponential Total Response



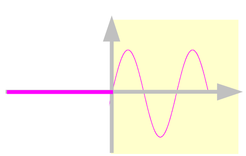
natural forced

for a given $s = \zeta$

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) e^{\zeta t} \qquad t \geq 0$$



$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) x(t) \qquad x(t) = e^{\zeta t}$$



$$Y(s) = \sum_i \frac{K_i}{(s - \lambda_i)} + H(s) X(s)$$

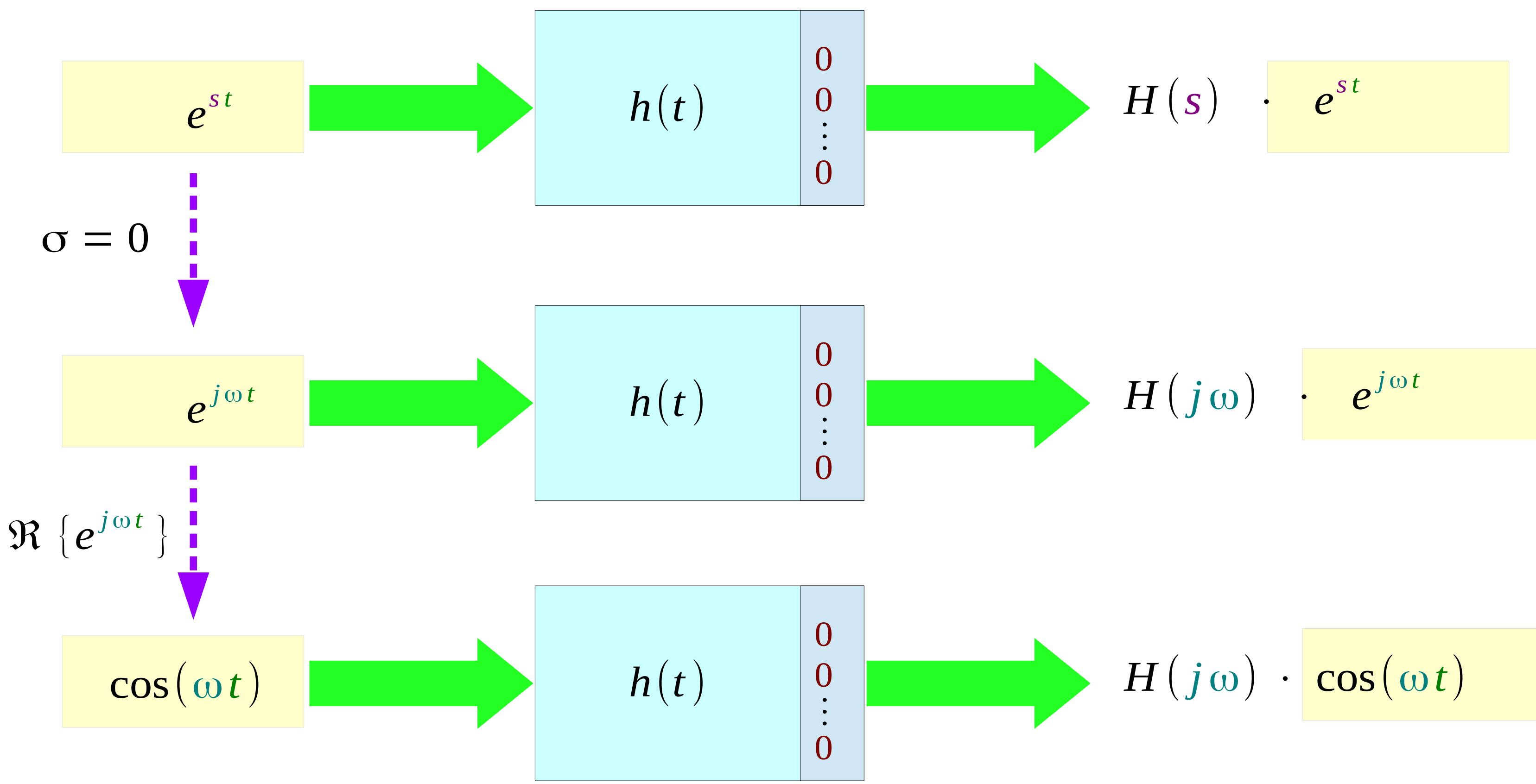
Special cases of H(s)

$s = \sigma + j\omega$

input applied at t = -∞

general cases

restricted cases



Frequency Response

Laplace Transform of h(t)

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \qquad s = \sigma + j\omega$$

Polynomials of Differential Equation

$$H(s) = \frac{P(s)}{Q(s)} \qquad s = \sigma + j\omega$$

Transfer function (t-domain)

$$H(s) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{st}} \qquad s = \sigma + j\omega$$

Transfer function (s-domain)

$$H(s) = \frac{Y(s)}{X(s)} \qquad s = \sigma + j\omega$$

Fourier Transform of h(t)

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau \qquad s = j\omega$$

Polynomials of Differential Equation

$$H(j\omega) = \frac{P(j\omega)}{Q(j\omega)} \qquad s = j\omega$$

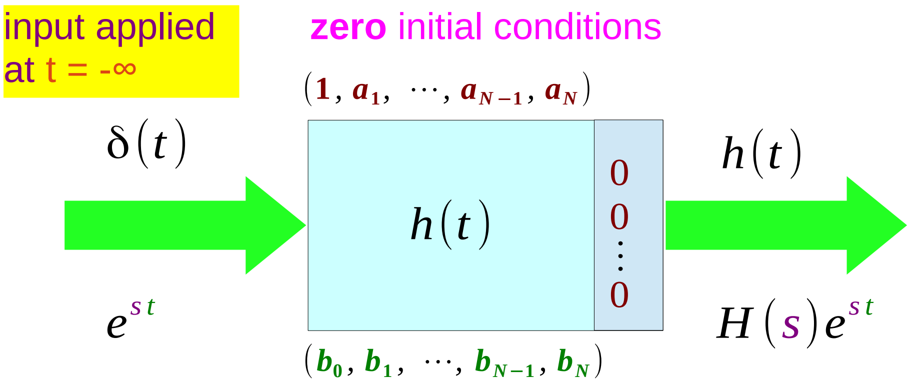
Frequency response (t-domain)

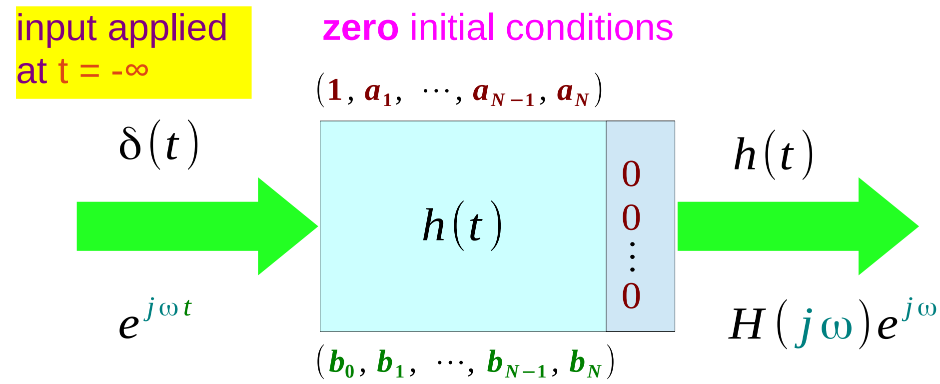
$$H(j\omega) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{j\omega t}} \qquad s = j\omega$$

Frequency response (ω-domain)

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \qquad s = j\omega$$

Transfer Function & Frequency Response





$s = \sigma + j\omega$

$$
\begin{aligned}
y(t) &= h(t) * e^{st} \\
&= \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= e^{st} \cdot \boxed{\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau} \\
&= e^{st} \cdot \boxed{H(s)}
\end{aligned}
$$

$h(t) = 0$ $(t < 0)$

$$\boxed{H(s) = \int_{0}^{+\infty} h(\tau) e^{-s\tau} d\tau}$$

Transfer function

$s = j\omega$

$$
\begin{aligned}
y(t) &= h(t) * e^{j\omega t} \\
&= \int_{-\infty}^{+\infty} h(\tau) e^{j\omega(t-\tau)} d\tau \\
&= e^{j\omega t} \cdot \boxed{\int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau} \\
&= e^{j\omega t} \cdot \boxed{H(j\omega)}
\end{aligned}
$$

$h(t) = 0$ $(t < 0)$

$$\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau}$$

Frequency response

Total Response to everlasting sinusoidal inputs

$$e^{+j\omega t} \rightarrow h(t) \rightarrow H(+j\omega)e^{+j\omega t} = |H(+j\omega)|\, e^{+j\,arg\{H(+j\omega)\}} \cdot e^{+j\omega t}$$

$$e^{-j\omega t} \rightarrow h(t) \rightarrow H(-j\omega)e^{-j\omega t} = |H(-j\omega)|\, e^{+j\,arg\{H(-j\omega)\}} \cdot e^{-j\omega t}$$

$$e^{+j\omega t} + e^{-j\omega t} \rightarrow h(t) \rightarrow |H(+j\omega)|\, e^{[+j\omega t + arg\{H(+j\omega)\}]} + |H(-j\omega)|\, e^{[-j\omega t + arg\{H(-j\omega)\}]}$$

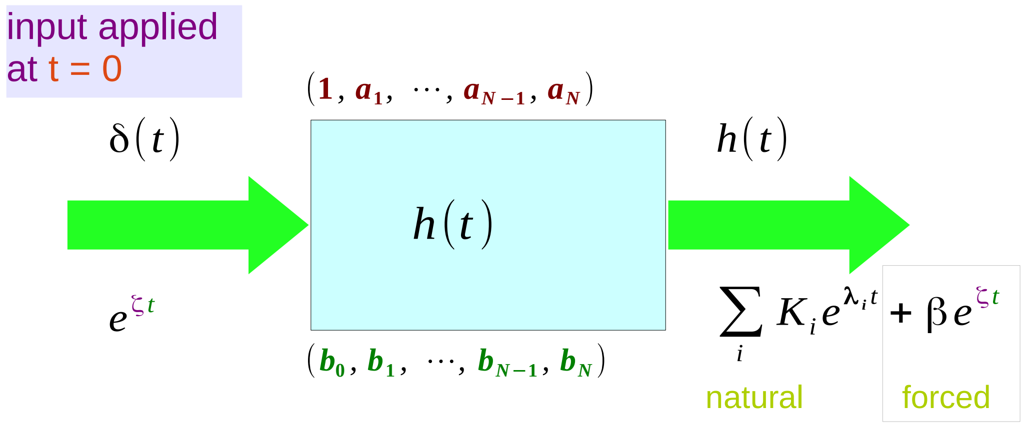
$$= |H(+j\omega)| \{ e^{[+j\omega t + arg\{H(+j\omega)\}]} + e^{[-j\omega t - arg\{H(+j\omega)\}]} \}$$

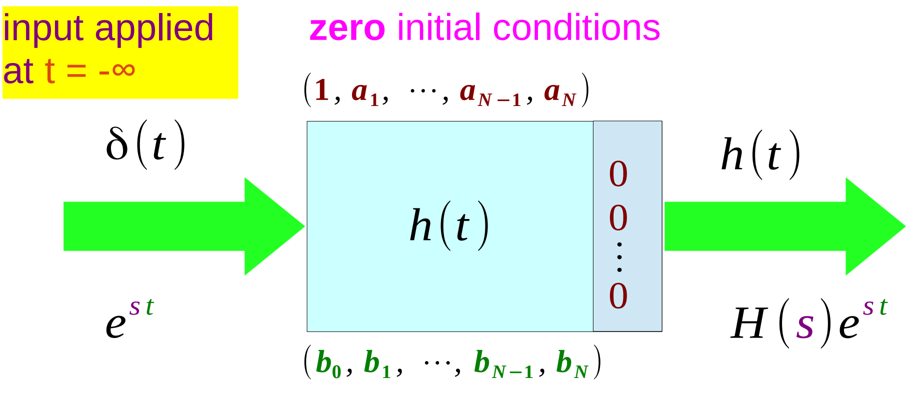
$$2\cos(\omega t) \rightarrow h(t) \rightarrow |H(j\omega)|\, 2\cos(\omega t + arg\{H(j\omega)\})$$

$$A\cos(\omega t + \alpha) \rightarrow h(t) \rightarrow A|H(j\omega)| \cos(\omega t + \alpha + arg\{H(j\omega)\})$$

$$A\sin(\omega t + \alpha) \rightarrow h(t) \rightarrow A|H(j\omega)| \sin(\omega t + \alpha + arg\{H(j\omega)\})$$

Sinusoidal Steady State Response (1)





total response

natural forced

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) e^{\zeta t} \qquad t \geq 0$$

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) x(t) \qquad x(t) = e^{\zeta t}$$

$$Y(s) = \sum_i \frac{K_i}{(s - \lambda_i)} + H(s) X(s)$$

steady state response

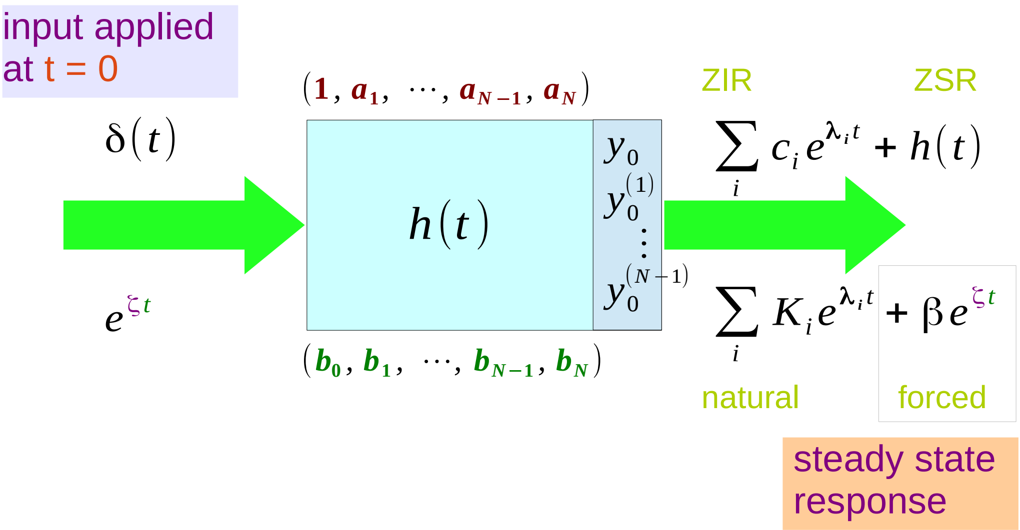
forced

$$y_{ss}(t) = H(\zeta) e^{\zeta t} \qquad t \to \infty$$

$$y_{ss}(t) = H(\zeta) x(t) \qquad x(t) = e^{\zeta t}$$

$$Y_{ss}(s) = H(s) X(s)$$

Sinusoidal Steady State Response (2)

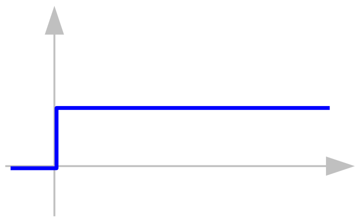


$x(t) = A$, $\xi = 0$ → $h(t)$ → $y_{ss}(t) = H(0) \cdot A e^{0t} = A \cdot H(0)$

$x(t) = A e^{j\omega t}$, $\xi = j\omega$ → $h(t)$ → $y_{ss}(t) = H(j\omega) \cdot A e^{j\omega t} = A \cdot H(j\omega) e^{j\omega}$

$x(t) = A\cos(\omega t)$, $\xi = j\omega$ → $h(t)$ → $y_{ss}(t) = H(j\omega) \cdot A\cos(\omega t) = A \cdot H(j\omega)\cos(\omega t)$

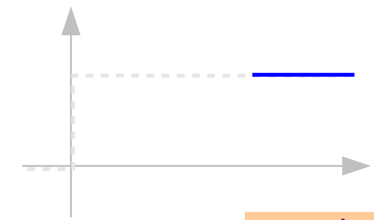
Sinusoidal Steady State Response (3)



$$y_{ss}(t) = H(0)\cdot A\,e^{0t} = A\cdot H(0)$$

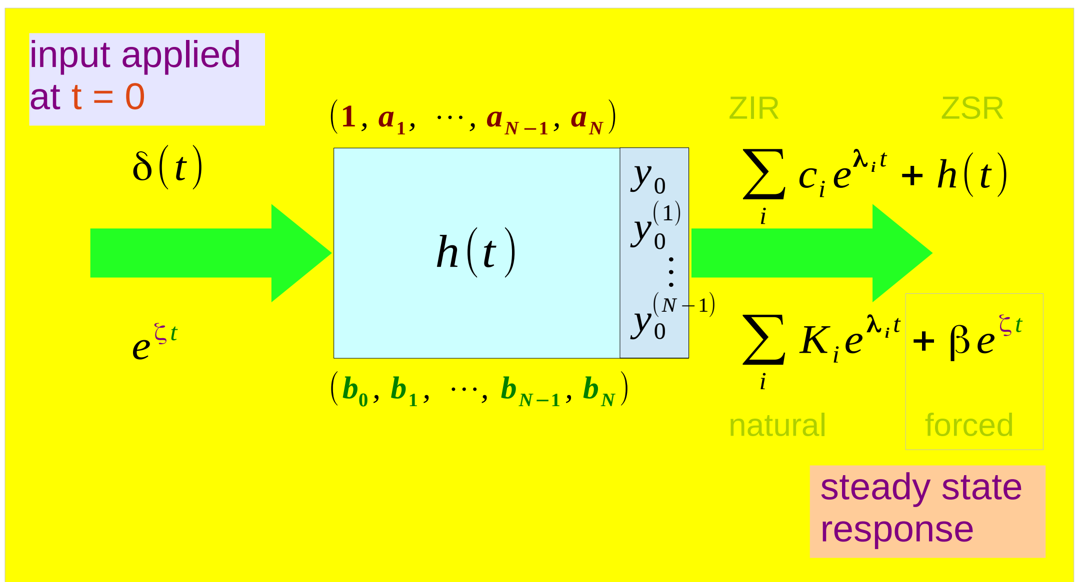
$x(t) = A$

$\xi = 0$



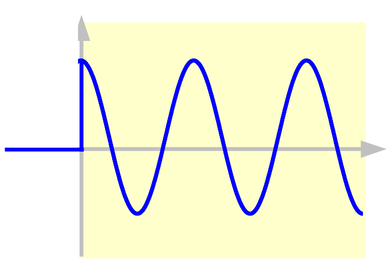
steady state response

Forced response



Forced response

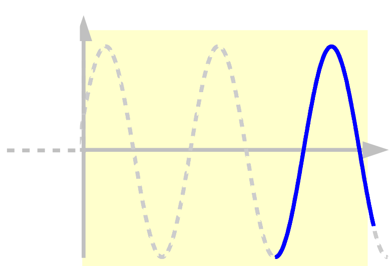
steady state response



$$y_{ss}(t) = H(j\omega)\cdot A\cos(\omega t) = A\cdot H(j\omega)\cos(\omega t)$$

$x(t) = A\cos(\omega t)$

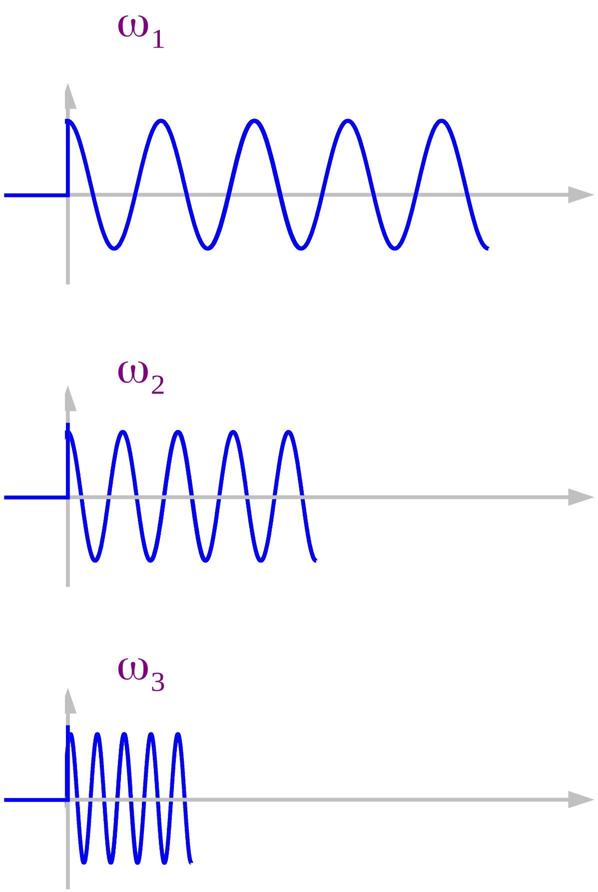
$\xi = j\omega$

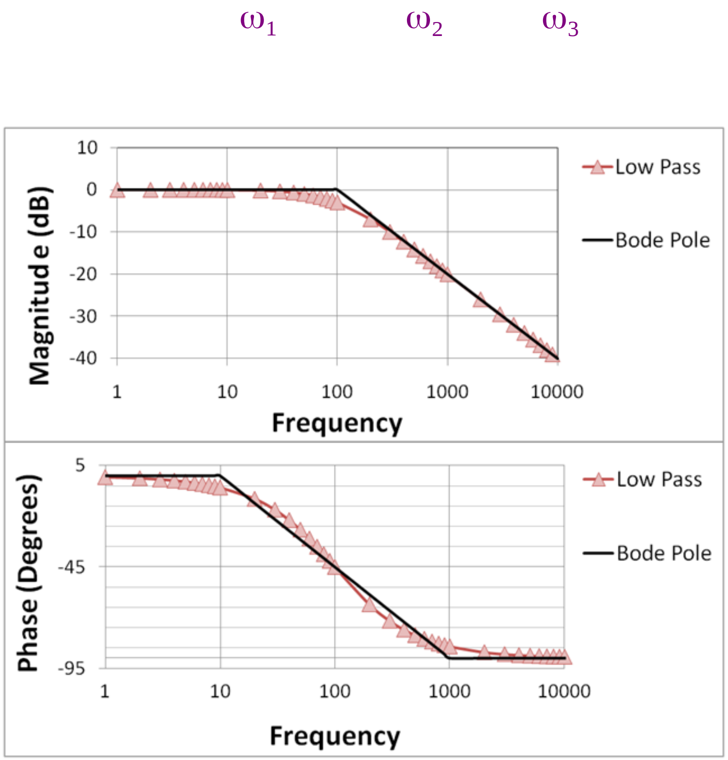


Frequency Response

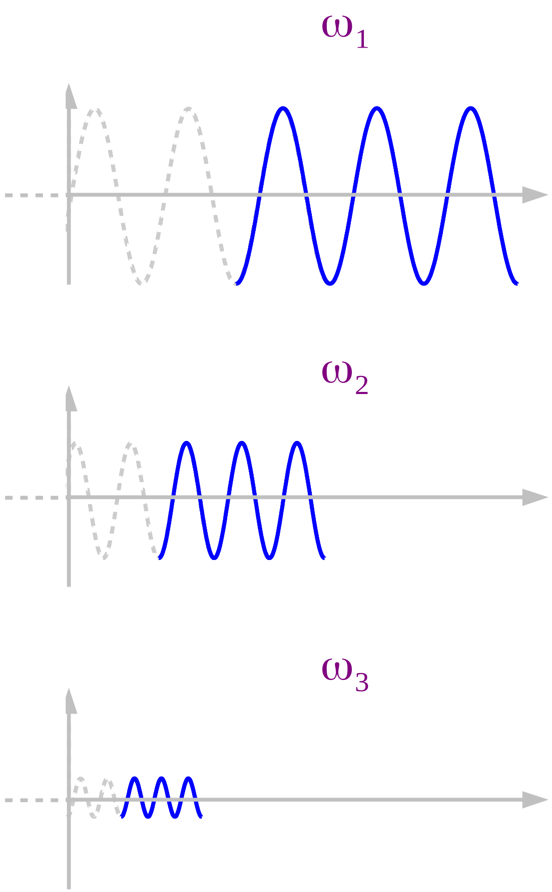
$$y_{ss}(t) = H(j\omega)\cdot A\cos(\omega t)$$
$$= A\cdot H(j\omega)\cos(\omega t)$$

$$x(t) = A\cos(\omega t)$$
$$\xi = j\omega$$

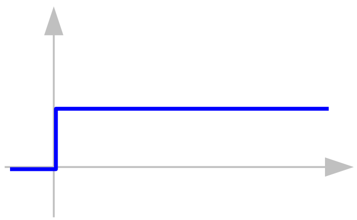




http://en.wikipedia.org/wiki/File:Bode_Low-Pass.PNG



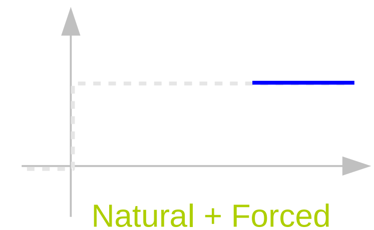
Transient Response



$$y_{ss}(t) = H(0) \cdot A e^{0t} = A \cdot H(0)$$

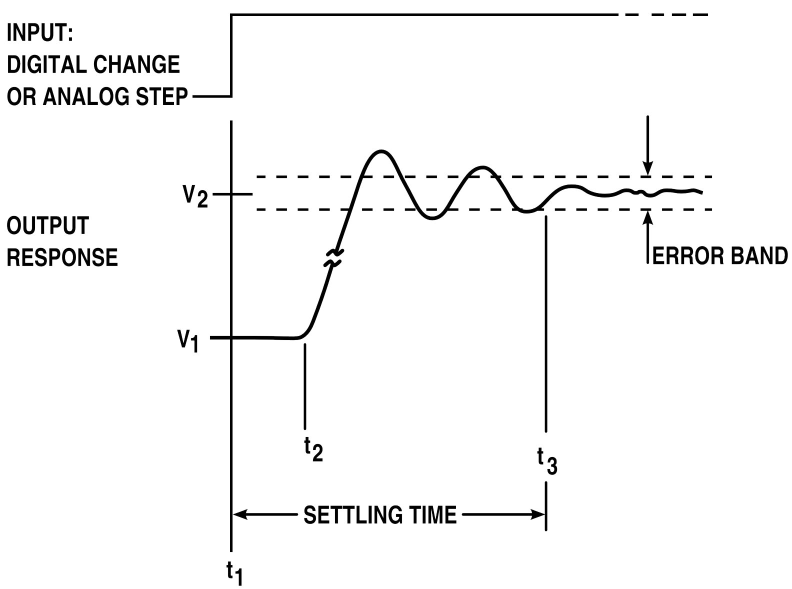
$$x(t) = A$$

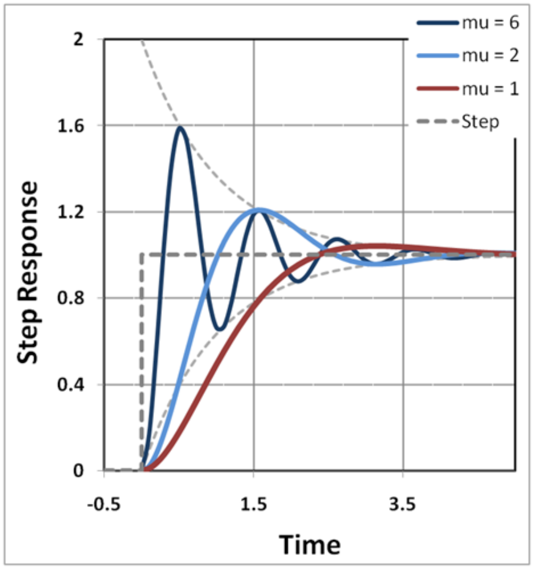
$$\xi = 0$$



Natural + Forced response

transient response





http://en.wikipedia.org/wiki/File:High_accuracy_settling_time_measurements_figure_1.png
http://en.wikipedia.org/wiki/File:Step_response_for_two-pole_feedback_amplifier.PNG

Frequency Response in Control Theory (1)

$$\sqrt{A^2+B^2}\cos(\omega t-\theta) \rightarrow G(s) \rightarrow |G(j\omega)|\sqrt{A^2+B^2}\cos(\omega t-\theta+arg\{G(j\omega)\})$$

$$A\cos(\omega t)+B\sin(\omega t)$$
$$=\sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\cos(\omega t)+\frac{B}{\sqrt{A^2+B^2}}\sin(\omega t)\right]$$
$$=\sqrt{A^2+B^2}\left[\cos(\theta)\cos(\omega t)+\sin(\theta)\sin(\omega t)\right]$$
$$=\sqrt{A^2+B^2}\cos(\theta-\omega t)$$
$$=\sqrt{A^2+B^2}\cos(\omega t-\theta)$$

$$A\cos(\omega t)+B\sin(\omega t)$$
$$=\sqrt{A^2+B^2}\cos(\omega t-\theta)$$

$$\cos(\theta)=\frac{A}{\sqrt{A^2+B^2}}$$

$$\sin(\theta)=\frac{B}{\sqrt{A^2+B^2}}$$

Frequency Response in Control Theory (2)

$$\sqrt{A^2+B^2}\cos(\omega t-\theta) \;\rightarrow\; \boxed{G(s)} \;\rightarrow\; |G(j\omega)|\sqrt{A^2+B^2}\cos\left(\omega t-\theta+arg\{G(j\omega)\}\right)$$

$$A\cos(\omega t)+B\sin(\omega t) = \sqrt{A^2+B^2}\cos(\omega t-\theta) \quad\longleftrightarrow\quad \frac{A\,s}{s^2+\omega^2}+\frac{B\,\omega}{s^2+\omega^2} = \frac{A\,s+B\,\omega}{s^2+\omega^2}$$

$$Y(s) = \frac{A\,s+B\,\omega}{s^2+\omega^2}G(s) = \frac{A\,s+B\,\omega}{(s+j\,\omega)(s-j\,\omega)}G(s) = \frac{K_1}{s+j\,\omega}+\frac{K_2}{s-j\,\omega}+F(s)$$

Partial Fraction

$$K_1 = \left[\frac{A\,s+B\,\omega}{s^2+\omega^2}(s+j\,\omega)G(s)\right]_{s=-j\,\omega} = \left[\frac{A\,s+B\,\omega}{(s-j\,\omega)}G(s)\right]_{s=-j\,\omega} = \frac{-A\,j\,\omega+B\,\omega}{-2\,j\,\omega}G(-j\,\omega) = \frac{1}{2}(A+j\,B)G(-j\,\omega)$$

$$K_2 = \left[\frac{A\,s+B\,\omega}{s^2+\omega^2}(s-j\,\omega)G(s)\right]_{s=+j\,\omega} = \left[\frac{A\,s+B\,\omega}{(s+j\,\omega)}G(s)\right]_{s=+j\,\omega} = \frac{A\,j\,\omega+B\,\omega}{+2\,j\,\omega}G(+j\,\omega) = \frac{1}{2}(A-j\,B)G(+j\,\omega)$$

Frequency Response in Control Theory (3)

$\sqrt{A^2+B^2}\cos(\omega t-\theta)$ → $G(s)$ → $|G(j\omega)|\sqrt{A^2+B^2}\cos\left(\omega t-\theta+arg\{G(j\omega)\}\right)$

$$Y(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} + F(s) \qquad K_1 = \frac{1}{2}(A+jB)G(-j\omega) \qquad K_2 = \frac{1}{2}(A-jB)G(+j\omega)$$

$$A \pm jB = \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}} \pm j\frac{B}{\sqrt{A^2+B^2}}\right]$$

$$= \sqrt{A^2+B^2}\,[\cos\theta \pm j\sin\theta]$$

$$= \sqrt{A^2+B^2}\,e^{\pm j\theta}$$

Stable System

system modes

$e^{p_i} \to 0 \qquad e^{\sigma_i} \to 0 \qquad p_i<0,\ \sigma_i<0$

$$\lim_{t\to\infty} f(t) = 0 \qquad \lim_{s\to\infty} sF(s) = 0$$

Ignore $F(s)$

$$Y_{ss}(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} \qquad K_1 = \frac{1}{2}\sqrt{A^2+B^2}\,e^{+j\theta}G(-j\omega) \qquad K_2 = \frac{1}{2}\sqrt{A^2+B^2}\,e^{-j\theta}G(-j\omega)$$

Frequency Response in Control Theory (4)

$\sqrt{A^2+B^2}\cos(\omega t-\theta)$ → $G(s)$ → $|G(j\omega)|\sqrt{A^2+B^2}\cos(\omega t-\theta+arg\{G(j\omega)\})$

$$Y_{ss}(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} = \frac{1}{2}(A+jB)G(-j\omega)\frac{1}{s+j\omega} + \frac{1}{2}(A-jB)G(+j\omega)\frac{1}{s-j\omega}$$

$$= \frac{1}{2}\sqrt{A^2+B^2}e^{+j\theta}G(-j\omega)\frac{1}{s+j\omega} + \frac{1}{2}\sqrt{A^2+B^2}e^{-j\theta}G(+j\omega)\frac{1}{s-j\omega}$$

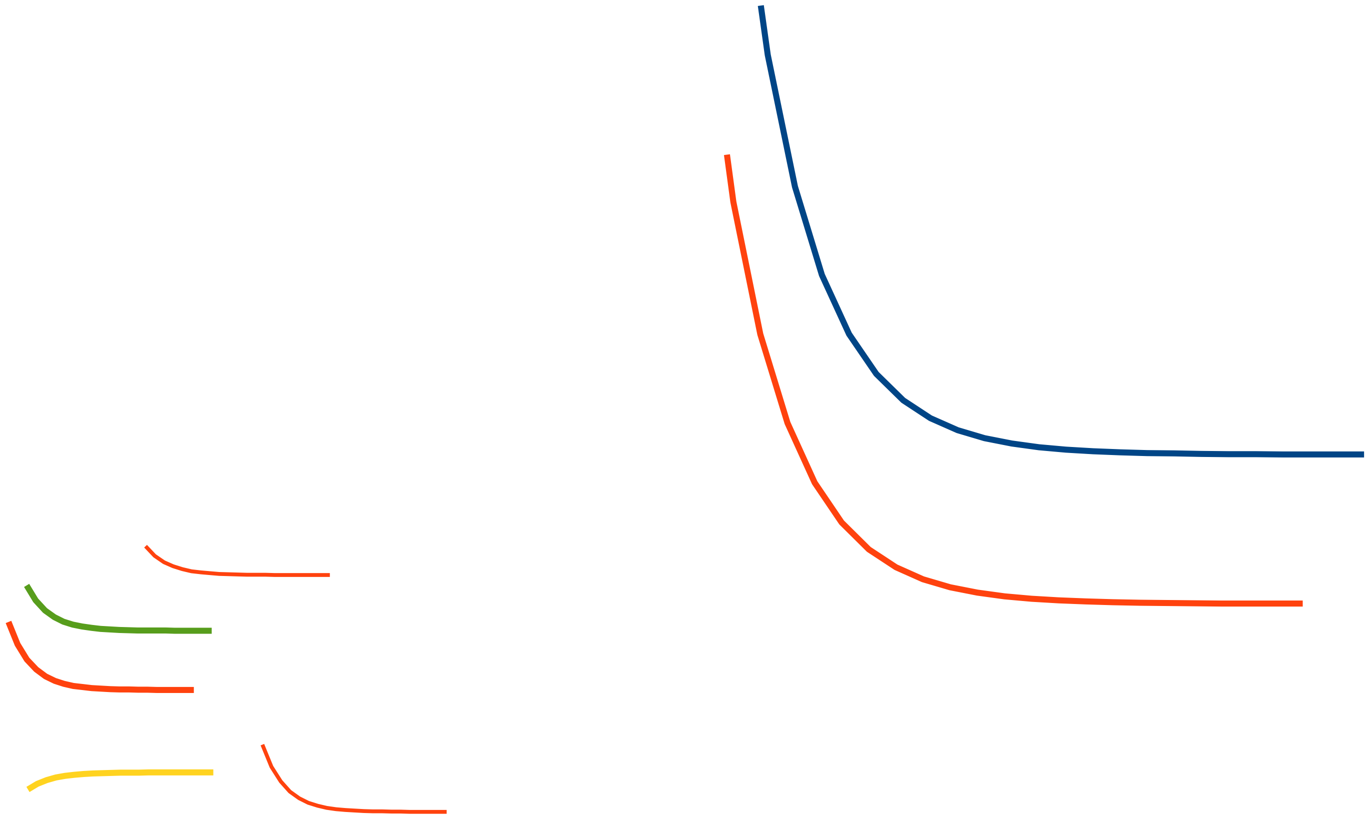
$$y_{ss}(t) = \frac{\sqrt{A^2+B^2}}{2}\left[G(-j\omega)e^{-j\omega t}e^{+j\theta} + G(+j\omega)e^{+j\omega t}e^{-j\theta}\right]$$

$$= \frac{\sqrt{A^2+B^2}}{2}\left[G(+j\omega)e^{-j(\omega t-\theta)} + G(+j\omega)e^{+j(\omega t-\theta)}\right]$$

$$= \sqrt{A^2+B^2}\ \Re\{G(+j\omega)e^{+j(\omega t-\theta)}\}$$

$$= \sqrt{A^2+B^2}\ |G(+j\omega)|\cos(\omega t-\theta+arg\{G(+j\omega)\})$$

Impulse Response h(t)



References

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[4] S. J. Orfanidis, Introduction to Signal Processing
[5] B. P. Lathi, Signals and Systems