Background – ODEs (2A)

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Second Order ODEs

First Order ***Linear Equations***

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x)$$

$$a_1(x)\,y' + a_0(x)\,y = g(x)$$

Second Order ***Linear Equations***

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x)$$

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = g(x)$$

Second Order ***Linear Equations*** ***with Constant Coefficients***

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0\,y = g(x)$$

$$a_2\,y'' + a_1\,y' + a_0\,y = g(x)$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = g(x)$$

$$a\,y'' + b\,y' + c\,y = g(x)$$

Background – ODEs (2A)

Second Order ODEs

First Order Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

Second Order Linear Equations

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

Second Order Linear Equations with Constant Coefficients

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = g(x)$$

$$a_2y'' + a_1y' + a_0y = g(x)$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x)$$

$$ay'' + by' + cy = g(x)$$

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Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$

$$a\frac{d^2}{dx^2}\{e^{mx}\} + b\frac{d}{dx}\{e^{mx}\} + c\{e^{mx}\} = 0$$

$$a\{m^2e^{mx}\} + b\{me^{mx}\} + c\{e^{mx}\} = 0$$

$$(am^2 + bm + c)\cdot e^{mx} = 0$$

$$a\{e^{mx}\}'' + b\{e^{mx}\}' + c\{e^{mx}\} = 0$$

$$a\{m^2e^{mx}\} + b\{me^{mx}\} + c\{e^{mx}\} = 0$$

$$(am^2 + bm + c)\cdot e^{mx} = 0$$

auxiliary equation

$$(am^2 + bm + c) = 0$$

$$(am^2 + bm + c) = 0$$

Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

auxiliary equation

try a solution $y = e^{mx}$ ⇒ $(am^2 + bm + c) = 0$

$$m_1 = \left(-b + \sqrt{b^2 - 4ac}\right)/2a$$

$$m_2 = \left(-b - \sqrt{b^2 - 4ac}\right)/2a$$

$y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$	**(A)**	$b^2 - 4ac > 0$	Real, distinct m_1, m_2
$y_1 = e^{m_1 x} = y_2 = e^{m_2 x}$	**(B)**	$b^2 - 4ac = 0$	Real, equal m_1, m_2
$y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$	**(C)**	$b^2 - 4ac < 0$	Conjugate complex m_1, m_2

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Linear Combination of Solutions

DEQ

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = 0$$

y_1

y_2

$C_1\,y_1 + C_2\,y_2$

$$a\,y_1'' + b\,y_1' + c\,y_1 = 0$$

$$a\,y_2'' + b\,y_2' + c\,y_2 = 0$$

$$a\left(y_1''+y_2''\right) + b\left(y_1'+y_2'\right) + c\left(y_1+y_2\right) = 0$$

$$a\left(y_1+y_2\right)'' + b\left(y_1+y_2\right)' + c\left(y_1+y_2\right) = 0$$

$$y_3 = y_1 + y_2$$

$$y_4 = y_1 - y_2$$

$$y_5 = y_3 + 2\,y_4$$

$$y_6 = y_3 - 2\,y_4$$

$$a\left(C_1 y_1''+C_2 y_2''\right) + b\left(C_1 y_1'+C_2 y_2'\right) + c\left(C_1 y_1+C_2 y_2\right) = 0$$

$$a\left(C_1 y_1+C_2 y_2\right)'' + b\left(C_1 y_1+C_2 y_2\right)' + c\left(C_1 y_1+C_2 y_2\right) = 0$$

Solutions of 2nd Order ODEs

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

y_1

y_2

$$C_1 y_1 + C_2 y_2$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \quad (D > 0)$$

$$\begin{cases} y_1 = e^{m_1 x} \end{cases} \quad (D = 0)$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \quad (D < 0)$$

$$\begin{cases} y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D > 0) \\ y = C_1 e^{m_1 x} \quad ? & (D = 0) \\ y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D < 0) \end{cases}$$

auxiliary equation

$$(am^2 + bm + c) = 0$$

$$m_1 = \left(-b + \sqrt{b^2 - 4ac}\right)/2a$$

$$m_2 = \left(-b - \sqrt{b^2 - 4ac}\right)/2a$$

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Fundamental Set of Solutions

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

y_1

y_2

$$C_1 y_1 + C_2 y_2$$

Functions y_1 and y_2 are either

- linearly independent functions or
- linearly dependent functions

$$\{y_1, \; y_2\}$$

Second Order

there can be at most **two** linearly independent functions

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

any *n* linearly independent solutions of the homogeneous linear *n-th* order differential equation

Fundamental Set of Solutions

(A) Real Distinct Roots Case

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$ ➡

$$(am^2 + bm + c) = 0$$

auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$y_1 = e^{m_1 x}$$

$$y_2 = e^{m_2 x}$$

$b^2 - 4ac > 0$	Real, distinct m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
$b^2 - 4ac = 0$	Real, equal m_1, m_2	$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$
$b^2 - 4ac < 0$	Conjugate complex m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

(B) Repeated Real Roots Case

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$ → $(am^2 + bm + c) = 0$ auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$
$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

→ $b^2 - 4ac = 0$ →

$$m_1 = -b/2a$$
$$m_2 = -b/2a$$

$$e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

$b^2 - 4ac > 0$	Real, distinct m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
$b^2 - 4ac = 0$	Real, equal m_1, m_2	$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$
$b^2 - 4ac < 0$	Conjugate complex m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

(C) Complex Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$ → $(am^2 + bm + c) = 0$ auxiliary equation

$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$ → $m_1 = (-b + \sqrt{4ac - b^2}\, i)/2a$ $y_1 = e^{m_1 x}$

$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$ → $m_2 = (-b - \sqrt{4ac - b^2}\, i)/2a$ $y_2 = e^{m_2 x}$

$b^2 - 4ac > 0$	Real, distinct m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
$b^2 - 4ac = 0$	Real, equal m_1, m_2	$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$
$b^2 - 4ac < 0$	Conjugate complex m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

Complex Exponential Conversion

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a \;\Rightarrow\; m_1 = (-b + \sqrt{4ac - b^2}\, i)/2a = \alpha + i\beta$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a \;\Rightarrow\; m_2 = (-b - \sqrt{4ac - b^2}\, i)/2a = \alpha - i\beta$$

$$y_1 = e^{m_1 x}$$

$$y_2 = e^{m_2 x}$$

$b^2 - 4ac < 0$ Conjugate complex m_1, m_2

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

Pick **two** homogeneous solution

$$y_1 = \{e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}\}/2 = e^{\alpha x}\cos(\beta x) \qquad (C_1 = +1/2, \quad C_2 = +1/2)$$

$$y_2 = \{e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}\}/2i = e^{\alpha x}\sin(\beta x) \qquad (C_1 = +1/2i, \quad C_2 = -1/2i)$$

$$y = C_3 e^{\alpha x}\cos(\beta x) + C_4 e^{\alpha x}\sin(\beta x) = e^{\alpha x}(C_3\cos(\beta x) + C_4\sin(\beta x))$$

Fundamental Set Examples (1)

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = 0$$

y_1 $e^{(\alpha+i\beta)x}$

y_2 $e^{(\alpha-i\beta)x}$

$$y_3 = \frac{1}{2}\,y_1 + \frac{1}{2}\,y_2 \qquad \left\{e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}\right\}/2 = e^{\alpha x}\cos(\beta x)$$

$$y_4 = \frac{1}{2i}\,y_1 - \frac{1}{2i}\,y_2 \qquad \left\{e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}\right\}/2i = e^{\alpha x}\sin(\beta x)$$

Fundamental Set Examples (2)

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$y_1 = y_3 + i\,y_4 \qquad e^{(\alpha+i\beta)x}$$

$$y_2 = y_3 - i\,y_4 \qquad e^{(\alpha-i\beta)x}$$

$$y_3 = e^{\alpha x}\cos(\beta x)$$

$$y_4 = e^{\alpha x}\sin(\beta x)$$

$$e^{\alpha x}[\cos(\beta x) + i\sin(\beta x)]$$

$$e^{\alpha x}[\cos(\beta x) - i\sin(\beta x)]$$

General Solution Examples

Second Order EQ

$$a \frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c\, y = 0$$

linearly independent

Fundamental Set of Solutions

$$\{y_1,\ \ y_2\} = \{e^{(\alpha + i\beta)x},\ \ e^{(\alpha - i\beta)x}\}$$

$$C_1\, y_1 + C_2\, y_2$$

$$C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

General Solution

linearly independent

Fundamental Set of Solutions

$$\{y_3,\ \ y_4\} = \{e^{\alpha x}\cos(\beta x),\ \ e^{\alpha x}\sin(\beta x)\}$$

$$c_3\, y_3 + c_3\, y_4$$

$$c_3 e^{\alpha x}\cos(\beta x) + c_4 e^{\alpha x}\sin(\beta x)$$
$$= e^{\alpha x}(c_3 \cos(\beta x) + c_4 \sin(\beta x))$$

General Solution

Finding a Particular Solution
- Undetermined Coefficients

Particular Solutions

DEQ

$$a\frac{d^2 y}{d x^2} + b\frac{d y}{d x} + c\, y = g(x)$$ ← y_p

particular solution
by a conjecture

(I) FORM Rule

(II) Multiplication Rule

When coefficients are constant

And

$g(x) =$

- ***A constant or*** ------------------------- k
- ***A polynomial or*** ------------------------- $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$
- ***An exponential function or*** ------------- $e^{\alpha x}$
- ***A sine and cosine functions or*** -------- $\sin(\beta x)$ $\cos(\beta x)$
- ***Finite sum and products of the above functions*** ------ $e^{\alpha x}\sin(\beta x) + x^2$

And

$$g(x) \neq \quad \ln x \quad \frac{1}{x} \quad \tan x \quad \sin^{-1} x$$

Form Rule

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x) \longleftarrow y_p$$

When coefficients are constant

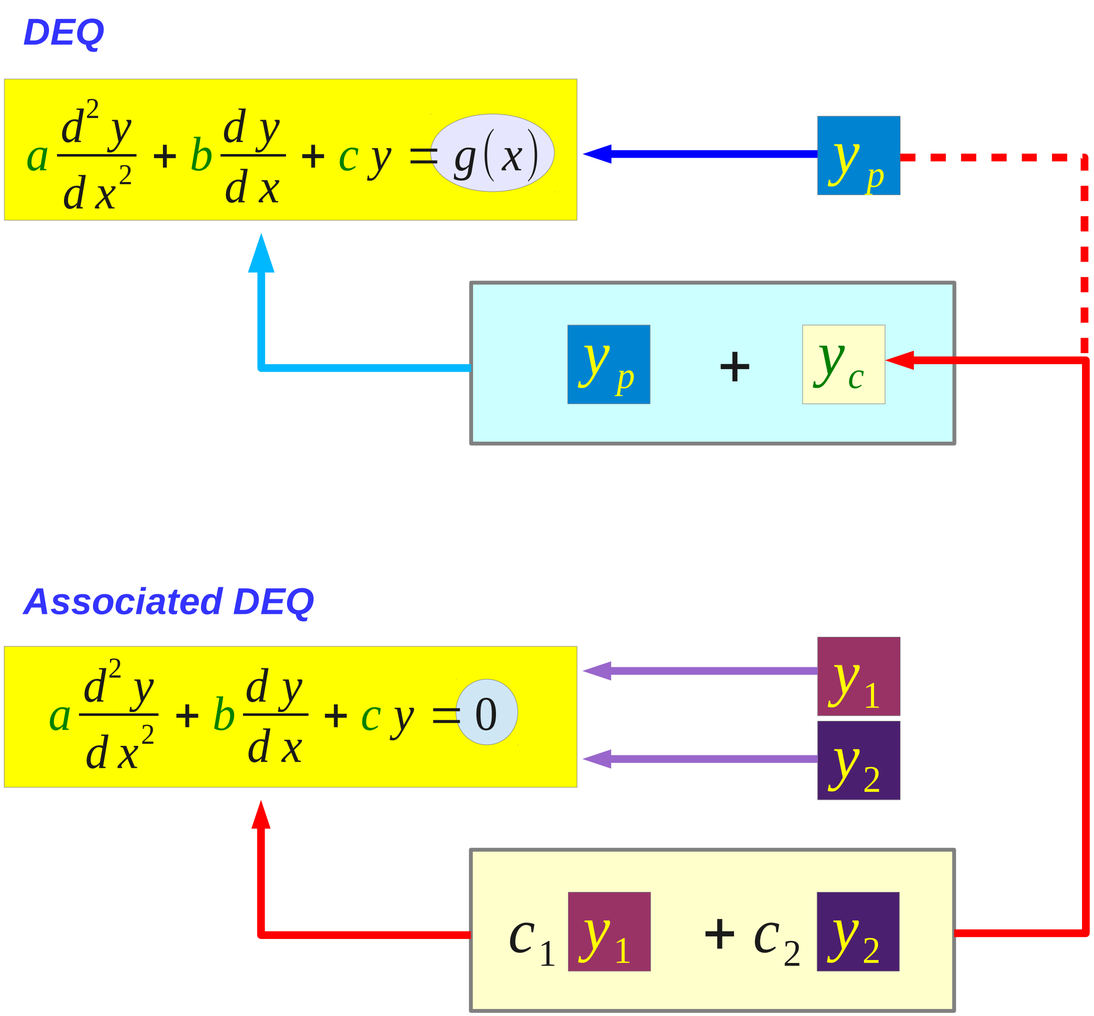
particular solution
by a conjecture

(I) FORM Rule

(II) Multiplication Rule

$g(x) = 2$	$y_p = A$
$g(x) = 3x+4$	$y_p = Ax+B$
$g(x) = 5x^2$	$y_p = Ax^2+Bx+C$
$g(x) = 6x^2-7$	$y_p = Ax^2+Bx+C$
$g(x) = \sin 8x$	$y_p = A\cos 8x + B\sin 8x$
$g(x) = \cos 9x$	$y_p = A\cos 9x + B\sin 9x$
$g(x) = e^{10x}$	$y_p = Ae^{10x}$
$g(x) = xe^{11x}$	$y_p = (Ax+B)e^{11x}$
$g(x) = e^{11x}\sin 12x$	$y_p = Ae^{11x}\sin 12x + Be^{11x}\cos 12x$

Multiplication Rule



use $y_p = x^n y_1$ $\quad$ $y_p = x^n y_2$

if $y_p = y_1$ $\quad$ $y_p = y_2$

Any y_p contains a term which is the same term in y_c

Use y_p multiplied by x^n

n is the smallest positive integer that eliminates the duplication

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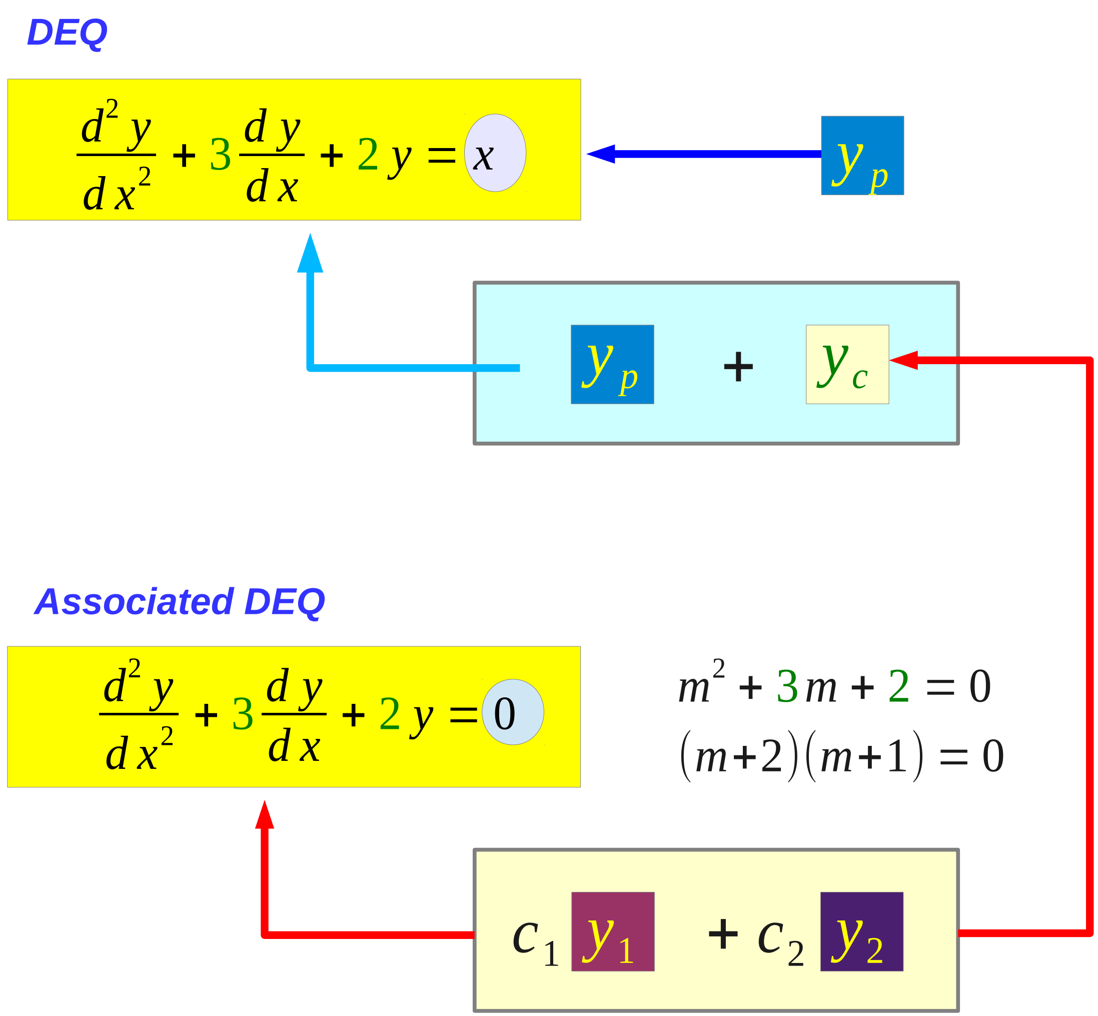
Example – Form Rule (1)



$$y_p = Ax + B$$
$$y_p' = A$$
$$y_p'' = 0$$

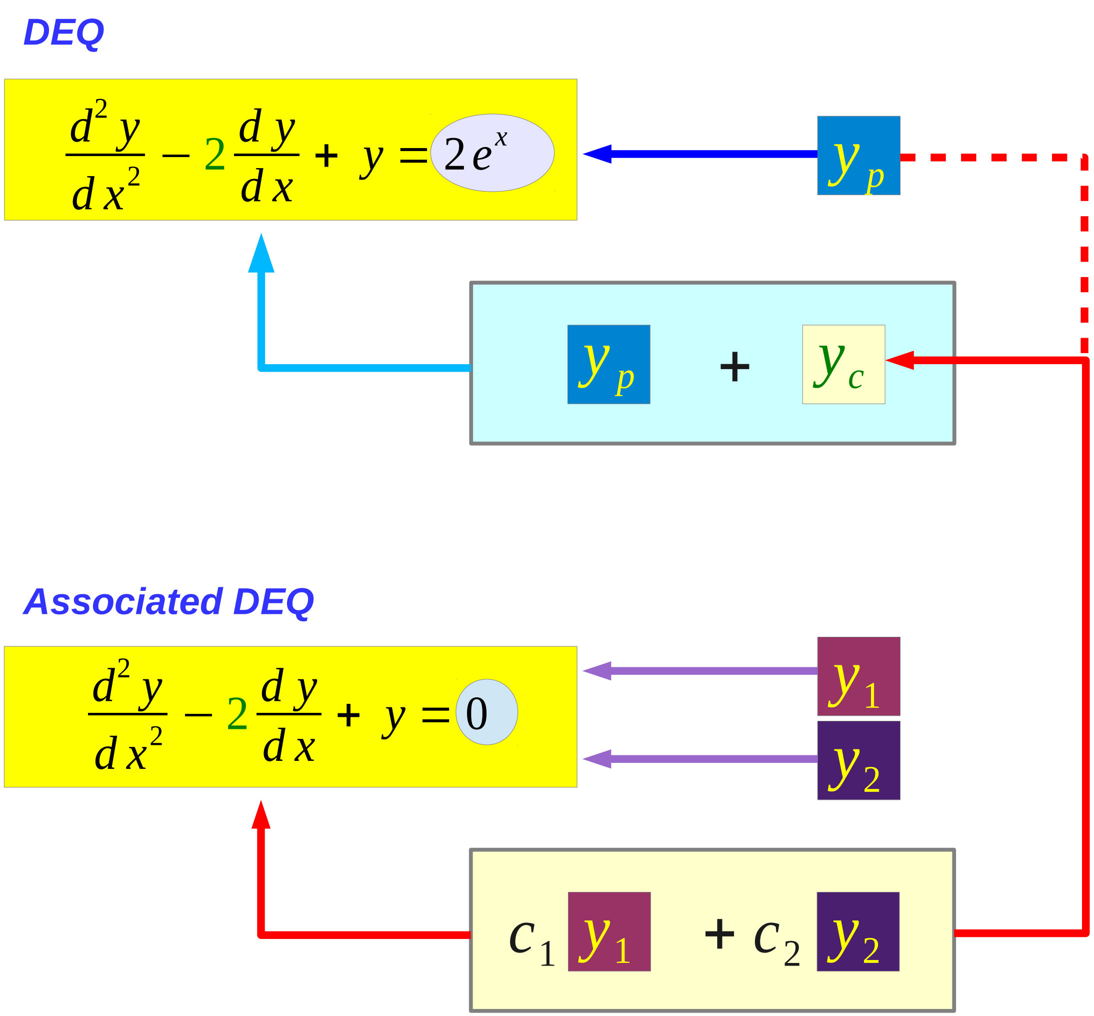
$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= 3A + 2(Ax + B) \\ &= 2Ax + 3A + 2B \\ &= x \end{aligned}$$

$$2A = 1 \qquad A = \tfrac{1}{2}$$
$$3A + 2B = 0 \qquad B = -\tfrac{3}{4}$$

$$y_p = \tfrac{1}{2}x - \tfrac{3}{4}$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \tfrac{1}{2}x - \tfrac{3}{4}$$

Example – Multiplication Rule (2)



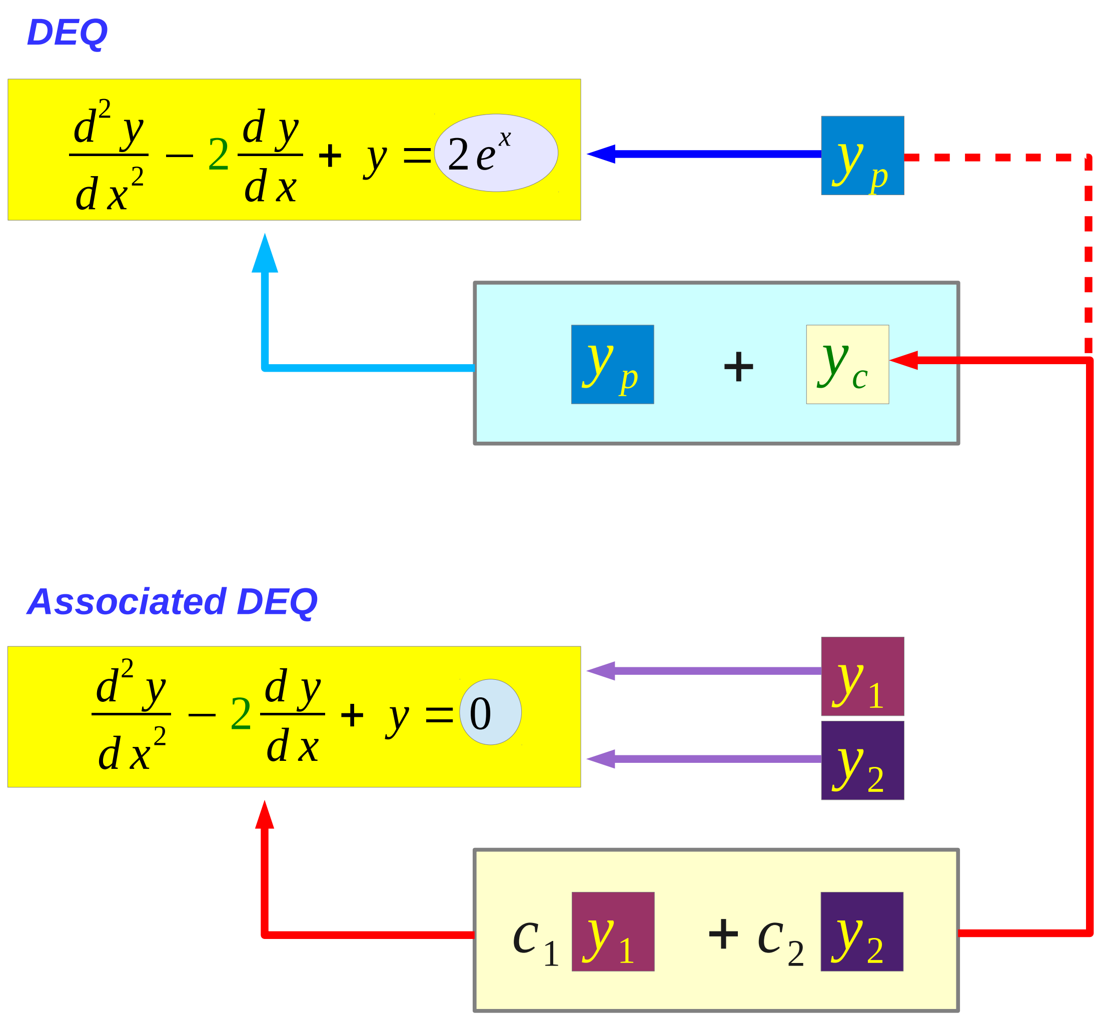
$$y'' - 2y' + y = 2e^x$$

$$y_p = \cancel{Ae^x} \Rightarrow \cancel{Axe^x} \Rightarrow Ax^2e^x$$

$$y_1 = e^x \qquad y_2 = xe^x$$

$$y'' - 2y' + y = 0$$

Example – Multiplication Rule (3)



$$y'' - 2y' + y = 6xe^x$$

$$y_p = Axe^x \Rightarrow Ax^2e^x \Rightarrow Ax^3e^x$$

$$LHS: 2Ae^x$$

$$y_1 = e^x \qquad y_2 = xe^x$$

$$y'' - 2y' + y = 0$$

Three Differential Equations

y_p **particular** solutions

EQ 1

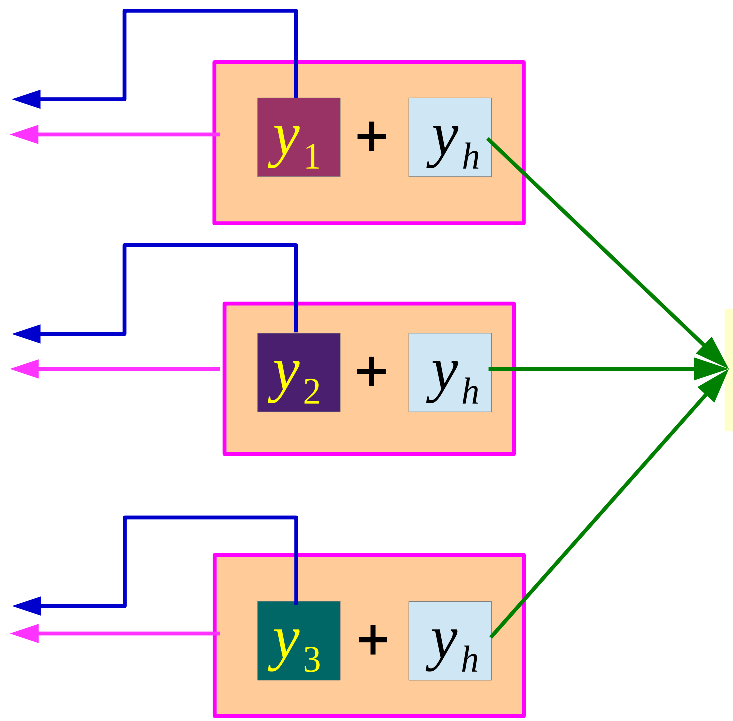
$$\frac{d^2 y(t)}{dt^2} + \boldsymbol{a_1} \frac{d\,y(t)}{dt} + \boldsymbol{a_2}\, y(t) = x_1(x)$$

EQ 2

$$\frac{d^2 y(t)}{dt^2} + \boldsymbol{a_1} \frac{d\,y(t)}{dt} + \boldsymbol{a_2}\, y(t) = x_2(x)$$

EQ 3

$$\frac{d^2 y(t)}{dt^2} + \boldsymbol{a_1} \frac{d\,y(t)}{dt} + \boldsymbol{a_2}\, y(t) = x_3(x)$$



$$\frac{d^2 y(t)}{dt^2} + \boldsymbol{a_1} \frac{d\,y(t)}{dt} + \boldsymbol{a_2}\, y(t) = 0$$

general solutions

y_h **homogeneous** solution

Superposition (1)

DEQ

$$\frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c y = 2 x^2 + 3 + \cos 8 x$$

$$\left(2 x^2 + 3\right) + \left(\cos 8 x\right)$$

$$\frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c y = 0 \quad \longleftarrow \quad y_c \qquad y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c y = 2 x^2 + 3 \quad \longleftarrow \quad y_{p1} \qquad y_{p1} = A x^2 + B x + C$$

$$\frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c y = \cos 8 x \quad \longleftarrow \quad y_{p2} \qquad y_{p2} = E \cos 8 x + F \sin 8 x$$

$$\frac{d^2}{d x^2}\left[y_c + y_{p1} + y_{p2}\right] + b \frac{d}{d x}\left[y_c + y_{p1} + y_{p2}\right] + c\left[y_c + y_{p1} + y_{p2}\right] = 2 x^2 + 3 + \cos 8 x$$

Superposition (2)

DEQ

$$a \frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c\, y = (2x^2 + 3) \cdot \cos 8x$$

$$\frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c\, y = 0 \quad \longleftarrow \quad y_c$$

$$\frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c\, y = (2x^2 + 3) \quad \longleftarrow \quad \cancel{y_{p1}}$$

$$\frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c\, y = \cos 8x \quad \longleftarrow \quad \cancel{y_{p2}}$$

$$y_p = (A x^2 + B x + C) \cdot (\cos 8x + \sin 8x)$$

$$\frac{d^2}{d x^2}[y_c + \cancel{y_{p1} \cdot y_{p2}}] + b \frac{d}{d x}[y_c + \cancel{y_{p1} \cdot y_{p2}}] + c\,[y_c + \cancel{y_{p1} \cdot y_{p2}}] = (2x^2 + 3) \cdot \cos 8x$$

Finite Number of Derivative Functions

$$y = x e^{mx}$$

$$\dot{y} = e^{mx} + m x e^{mx}$$

$$\ddot{y} = m e^{mx} + m\left(e^{mx} + m x e^{mx}\right) = 2 m e^{mx} + m^2 x e^{mx}$$

$$\dddot{y} = 2 m e^{mx} + m^2\left(e^{mx} + m x e^{mx}\right) = \left(m^2 + 2m\right) e^{mx} + m^3 x e^{mx}$$

-
-
-

$$\left\{e^{mx},\ \ x e^{mx}\right\}$$

$$y = 2x^2 + 3x + 4$$

$$\dot{y} = 4x + 3$$

$$\ddot{y} = 4$$

$$\dddot{y} = 0$$

$$\left\{2x^2 + 3x + 4,\ \ 4x + 3,\ \ 4\right\}$$

Infinite Number of Derivative Functions

$y = +x^{-1}$

$\dot{y} = -x^{-2}$

$\ddot{y} = +2x^{-3}$

$\dddot{y} = -6x^{-4}$

⋮

$y = \ln x$

$y = +x^{-1}$

$\dot{y} = -x^{-2}$

$\ddot{y} = +2x^{-3}$

$\dddot{y} = -6x^{-4}$

⋮

Complex Exponentials

$$c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t}$$

$$(c_1 + c_2) = c_3$$

$$i(c_1 - c_2) = c_4$$

$$= e^{-\sigma t}(c_1 e^{+i\omega t} + c_2 e^{-i\omega t})$$

$$= e^{-\sigma t}[c_1(\cos(\omega t) + i\sin(\omega t)) + c_2(\cos(\omega t) - i\sin(\omega t))]$$

$$= e^{-\sigma t}[(c_1 + c_2)\cos(\omega t) + i(c_1 - c_2)\sin(\omega t)]$$

$$= c_3 e^{-\sigma t}\cos(\omega t) + c_4 e^{-\sigma t}\sin(\omega t)$$

$$c_3 e^{-\sigma t}\cos(\omega t) + c_4 e^{-\sigma t}\sin(\omega t)$$

$$\frac{(c_3 - c_4 i)}{2} = c_1$$

$$\frac{(c_3 + c_4 i)}{2} = c_2$$

$$= c_3 e^{-\sigma t}(e^{+i\omega t} + e^{-i\omega t})/2 + c_4 e^{-\sigma t}(e^{+i\omega t} - e^{-i\omega t})/2i$$

$$= c_3 e^{-\sigma t}(e^{+i\omega t} + e^{-i\omega t})/2 + c_4 e^{-\sigma t}(-i e^{+i\omega t} + i e^{-i\omega t})/2$$

$$= \frac{(c_3 - c_4 i)}{2} e^{-\sigma t} e^{+i\omega t} + \frac{(c_3 + c_4 i)}{2} e^{-\sigma t} e^{-i\omega t}$$

$$= c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t}$$

Complex Exponentials

$$c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t} \rightarrow c_3 e^{-\sigma t} \cos(\omega t) + c_4 e^{-\sigma t} \sin(\omega t)$$

$$(c_1 + c_2) = c_3$$

$$i(c_1 - c_2) = c_4$$

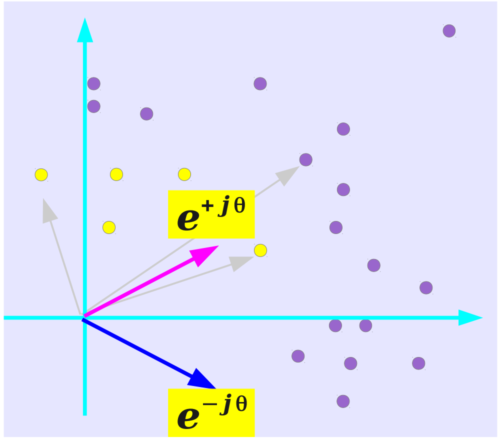
$$c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t} \leftarrow c_3 e^{-\sigma t} \cos(\omega t) + c_4 e^{-\sigma t} \sin(\omega t)$$

$$c_1 = \frac{(c_3 - c_4 i)}{2}$$

$$c_2 = \frac{(c_3 + c_4 i)}{2}$$

Basis of the Complex Plane

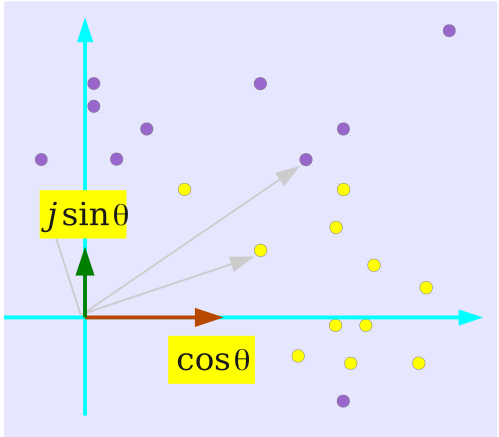
Basis : a set of linear independent spanning vectors



every complex number can be represented by

$$k_1 e^{+j\theta} + k_2 e^{+j\theta}$$

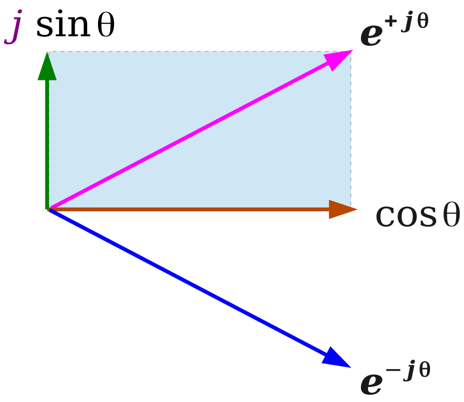
linear combination of $e^{+j\theta}$ and $e^{+j\theta}$
which are one set of linear independent
two vectors



every complex number can also be represented by

$$l_1 \cos\theta + l_2 j \sin\theta$$

$$l_1 \cos\theta + l_2 \; j\sin\theta$$



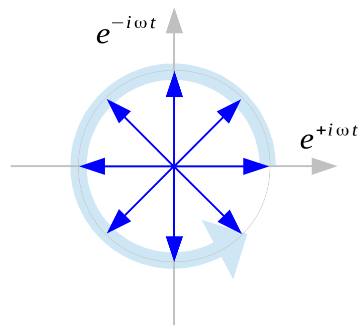
Real Coefficients C_1 & C_2

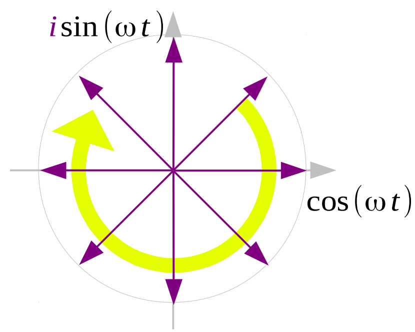
$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 \sin(\omega t)$

real number	$c_1 = (c_3 - c_4 i)/2$
real number	$c_2 = (c_3 + c_4 i)/2$

$c_3 = (c_1 + c_2)$	*real number*
$c_4 = i(c_1 - c_2)$	*imaginary number*

$(c_1 e^{-\sigma t},\ c_2 e^{-\sigma t})$





$(c_3 e^{-\sigma t},\ c_4 e^{-\sigma t})$

$1\cdot e^{+i\omega t} + 0\cdot e^{-i\omega t}$	$1\cdot\cos(\omega t) + 1i\cdot\sin(\omega t)$
$\frac{1}{\sqrt{2}}\cdot e^{+i\omega t} + \frac{1}{\sqrt{2}}\cdot e^{-i\omega t}$	$\sqrt{2}\cdot\cos(\omega t) + 0i\cdot\sin(\omega t)$
$0\cdot e^{+i\omega t} + 1\cdot e^{-i\omega t}$	$1\cdot\cos(\omega t) - 1i\cdot e^{-\sigma t}\sin(\omega t)$
$\frac{-1}{\sqrt{2}}\cdot e^{+i\omega t} + \frac{1}{\sqrt{2}}\cdot e^{-i\omega t}$	$0\cdot\cos(\omega t) - \sqrt{2}i\cdot\sin(\omega t)$
$-1\cdot e^{+i\omega t} + 0\cdot e^{-i\omega t}$	$-1\cdot\cos(\omega t) - 1i\cdot\sin(\omega t)$
$\frac{-1}{\sqrt{2}}\cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}}\cdot e^{-i\omega t}$	$-\sqrt{2}\cdot\cos(\omega t) - 0i\cdot\sin(\omega t)$
$0\cdot e^{+i\omega t} - 1\cdot e^{-i\omega t}$	$-1\cdot\cos(\omega t) + 1i\cdot\sin(\omega t)$
$\frac{1}{\sqrt{2}}\cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}}\cdot e^{-i\omega t}$	$0\cdot\cos(\omega t) + \sqrt{2}i\cdot\sin(\omega t)$

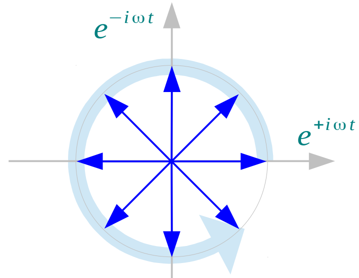
Complex Plane Basis

$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3' \cos(\omega t) + c_4' i \sin(\omega t)$

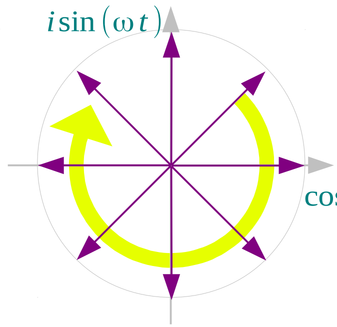
$$c_1 = (c_3' + c_4')/2 \quad \textit{real number}$$
$$c_2 = (c_3' - c_4')/2 \quad \textit{real number}$$

$$\mathbb{C}^1$$

$$c_3' = (c_1 + c_2) \quad \textit{real number}$$
$$c_4' = (c_1 - c_2) \quad \textit{real number}$$



complex plane



$1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$	$1 \cdot \cos(\omega t) + 1 i \cdot \sin(\omega t)$
$\frac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega t}$	$\sqrt{2} \cdot \cos(\omega t) + 0 i \cdot \sin(\omega t)$
$0 \cdot e^{+i\omega t} + 1 \cdot e^{-i\omega t}$	$1 \cdot \cos(\omega t) - 1 i \cdot \sin(\omega t)$
$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega t}$	$0 \cdot \cos(\omega t) - \sqrt{2} i \cdot \sin(\omega t)$
$-1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$	$-1 \cdot \cos(\omega t) - 1 i \cdot \sin(\omega t)$
$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$	$-\sqrt{2} \cdot \cos(\omega t) - 0 i \cdot \sin(\omega t)$
$0 \cdot e^{+i\omega t} - 1 \cdot e^{-i\omega t}$	$-1 \cdot \cos(\omega t) + 1 i \cdot \sin(\omega t)$
$\frac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$	$0 \cdot \cos(\omega t) + \sqrt{2} i \cdot \sin(\omega t)$

Real Coefficients C_3 & C_4

$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 \sin(\omega t)$

$c_1 = (c_3 - c_4 i)/2$
$c_2 = (c_3 + c_4 i)/2$

conjugate
complex number

R^2 $+2*real\ part$, $-2*imag\ part$

$c_3 = (c_1 + c_2)$ *real number*
$c_4 = i(c_1 - c_2)$ *real number*

$\frac{(+1-0i)}{2}\cdot e^{+i\omega t} + \frac{(+1+0i)}{2}\cdot e^{-i\omega t}$ — $1\cdot\cos(\omega t) + 0\cdot\sin(\omega t)$

$\frac{(+1-i)}{2\sqrt{2}}\cdot e^{+i\omega t} + \frac{(+1+i)}{2\sqrt{2}}\cdot e^{-i\omega t}$ — $\frac{1}{\sqrt{2}}\cdot\cos(\omega t) + \frac{1}{\sqrt{2}}\cdot\sin(\omega t)$

$\frac{(0-i)}{2}\cdot e^{+i\omega t} + \frac{(0+i)}{2}\cdot e^{-i\omega t}$ — $0\cdot\cos(\omega t) + 1\cdot\sin(\omega t)$

$\frac{(-1-i)}{2\sqrt{2}}\cdot e^{+i\omega t} + \frac{(-1+i)}{2\sqrt{2}}\cdot e^{-i\omega t}$ — $\frac{-1}{\sqrt{2}}\cdot\cos(\omega t) + \frac{1}{\sqrt{2}}\cdot\sin(\omega t)$

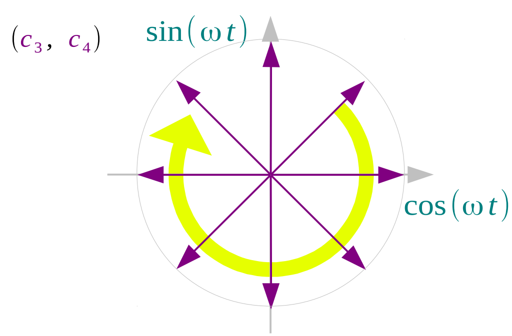
$\frac{(-1-0i)}{2}\cdot e^{+i\omega t} + \frac{(-1+0i)}{2}\cdot e^{-i\omega t}$ — $-1\cdot\cos(\omega t) + 0\cdot\sin(\omega t)$

$\frac{(-1+i)}{2\sqrt{2}}\cdot e^{+i\omega t} + \frac{(-1-i)}{2\sqrt{2}}\cdot e^{-i\omega t}$ — $\frac{-1}{\sqrt{2}}\cdot\cos(\omega t) + \frac{-1}{\sqrt{2}}\cdot\sin(\omega t)$

$\frac{(0+i)}{2}\cdot e^{+i\omega t} + \frac{(0-i)}{2}\cdot e^{-i\omega t}$ — $0\cdot\cos(\omega t) - 1\cdot\sin(\omega t)$

$\frac{(+1+i)}{2\sqrt{2}}\cdot e^{+i\omega t} + \frac{(+1-i)}{2\sqrt{2}}\cdot e^{-i\omega t}$ — $\frac{1}{\sqrt{2}}\cdot\cos(\omega t) - \frac{1}{\sqrt{2}}\cdot\sin(\omega t)$

real plane



Real Coefficients C_3 & C_4

$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t} \quad \Longleftrightarrow \quad c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

$$c_1 = (c_3 - c_4 i)/2$$
$$c_2 = (c_3 + c_4 i)/2$$

conjugate complex number

R^2 $+2*real\ part$, $-2*imag\ part$

$$c_3 = (c_1 + c_2)$$
$$c_4 = i(c_1 - c_2)$$

real number

real number

$$A\cos(\omega t - \varphi) \quad \Longleftrightarrow \quad c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

$$\sqrt{c_3^2 + c_4^2} = A$$

$$\frac{c_3}{\sqrt{c_3^2 + c_4^2}} = \cos(\varphi)$$

$$\frac{c_4}{\sqrt{c_3^2 + c_4^2}} = \sin(\varphi)$$

C^1 and R^2 Spaces

$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t} \quad \Longleftrightarrow \quad c_3 \cos(\omega t) + c_4 i \sin(\omega t)$$

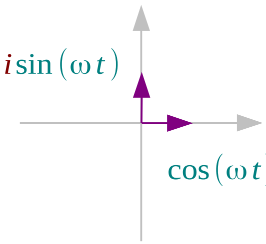
$c_1 = (c_3 + c_4)/2$ *real number*

$c_2 = (c_3 - c_4)/2$ *real number*

$c_3 = (c_1 + c_2)$ *real number*

$c_4 = (c_1 - c_2)$ *real number*

C^1



$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t} \quad \Longleftrightarrow \quad c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

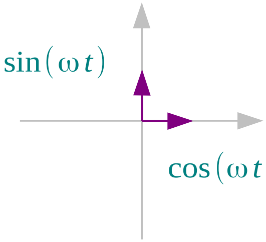
$c_1 = (c_3 - c_4 i)/2$ *conjugate*

$c_2 = (c_3 + c_4 i)/2$ *complex number*

$+2 * real\ part$ $\quad c_3 = (c_1 + c_2)$ *real number*

$-2 * imag\ part$ $\quad c_4 = i(c_1 - c_2)$ *real number*

R^2



Signal Spaces and Phasors

$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 \sin(\omega t)$

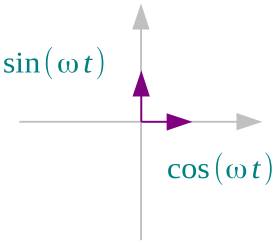
$$c_1 = (c_3 - c_4 i)/2$$
$$c_2 = (c_3 + c_4 i)/2$$

conjugate complex number

$+2*real\ part$: $c_3 = (c_1 + c_2)$ *real number*

$-2*imag\ part$: $c_4 = i(c_1 - c_2)$ *real number*

R^2



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