

# Background – Complex Analysis (1A)

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# *Complex Numbers*

# Complex Numbers

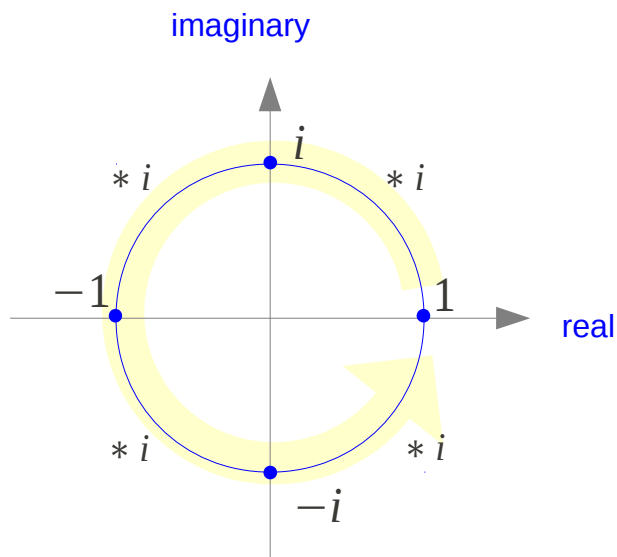
$$i = \sqrt{-1}$$
$$i^2 = -1$$

$$i^3 = -i$$

$$i^2 \cdot i = -i$$

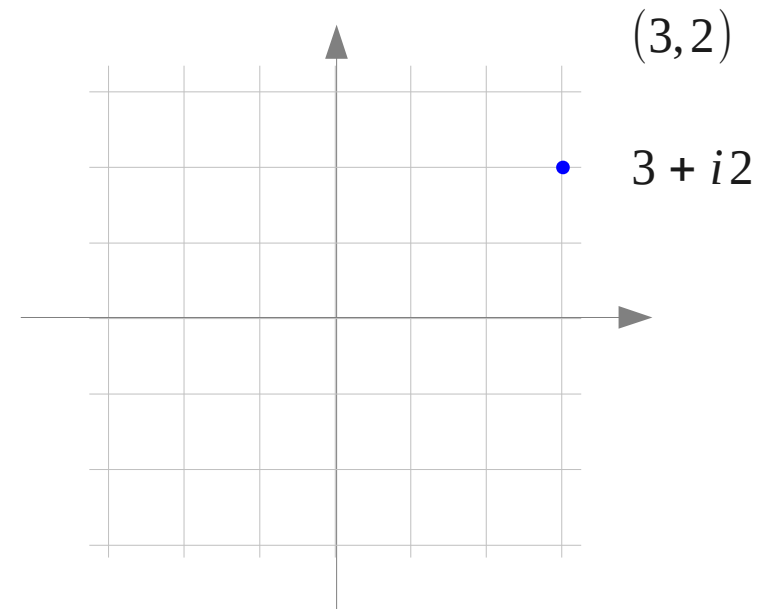
$$i^4 = +1$$

$$i^2 \cdot i^2 = +1$$

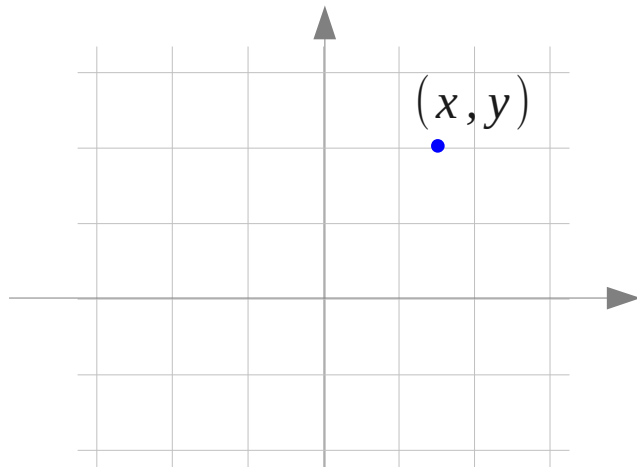


$(3,2)$  **two** real numbers  
2-d coordinate

$3 + i2$  **one** complex number  
with real part of 3  
and imaginary part of 2



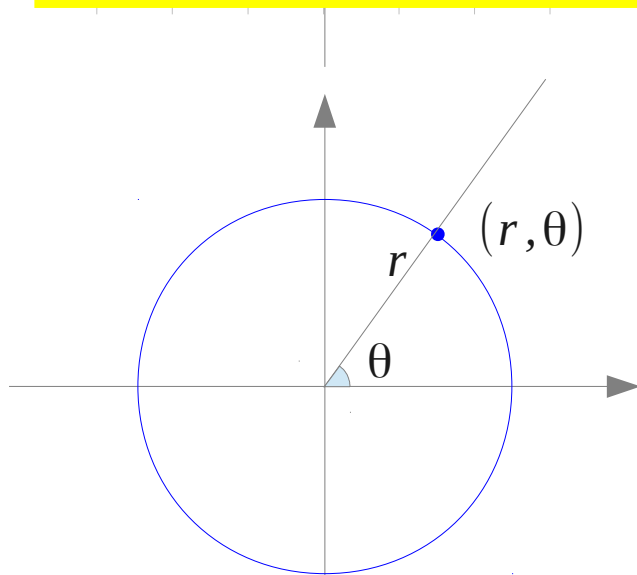
# Coordinate Systems



(a) Cartesian Coordinate System

$$x = r \cos \theta$$

$$y = r \sin \theta$$

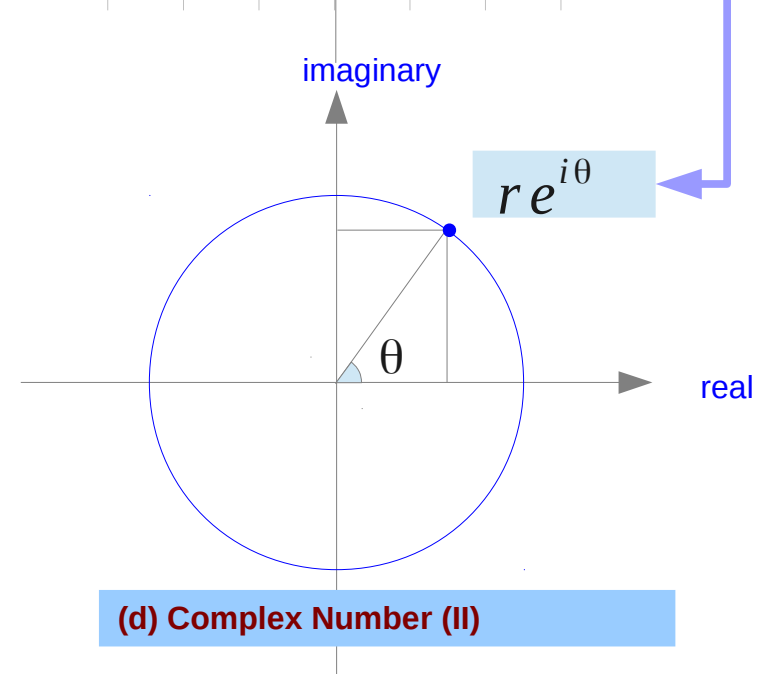
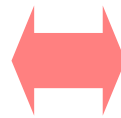
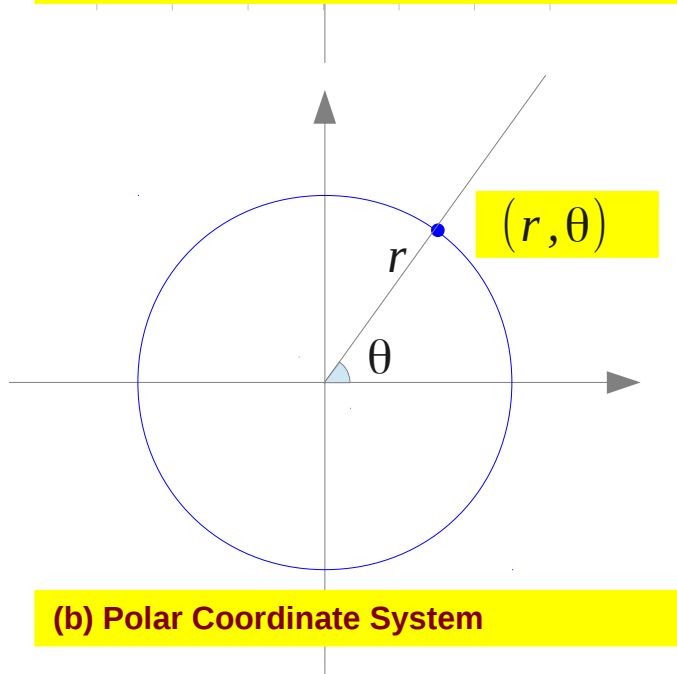
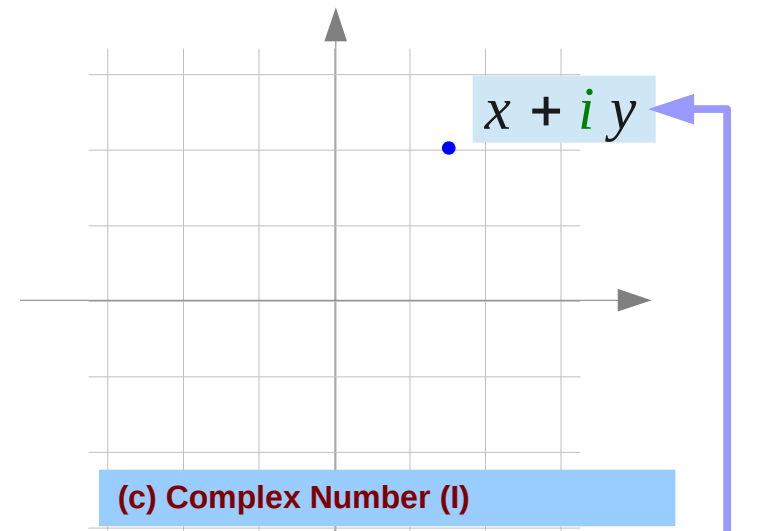
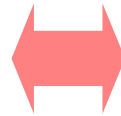
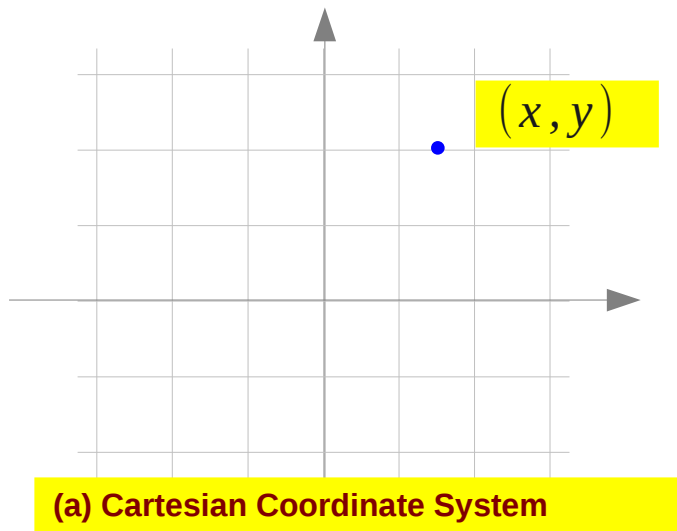


(b) Polar Coordinate System

$$r = \sqrt{x^2 + y^2}$$

$$\tan \theta = \frac{y}{x}$$

# Coordinate Systems and Complex Numbers



# Complex Numbers

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2}$$

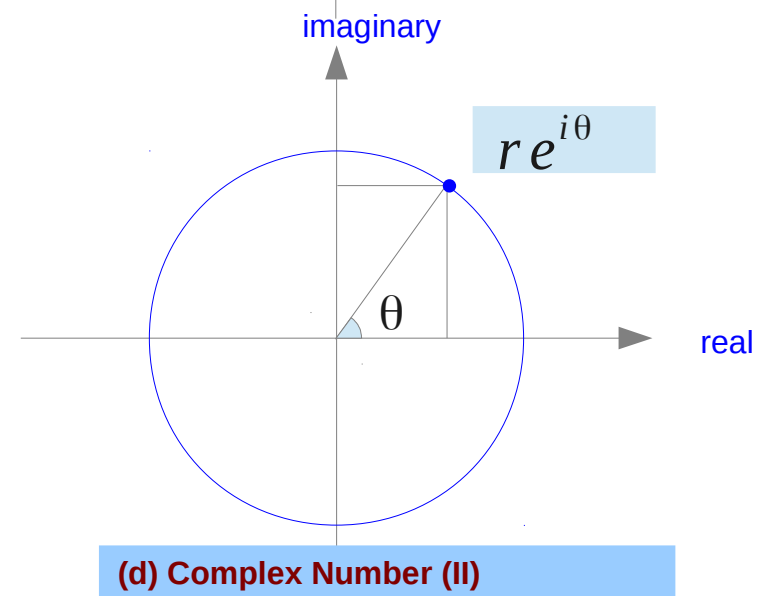
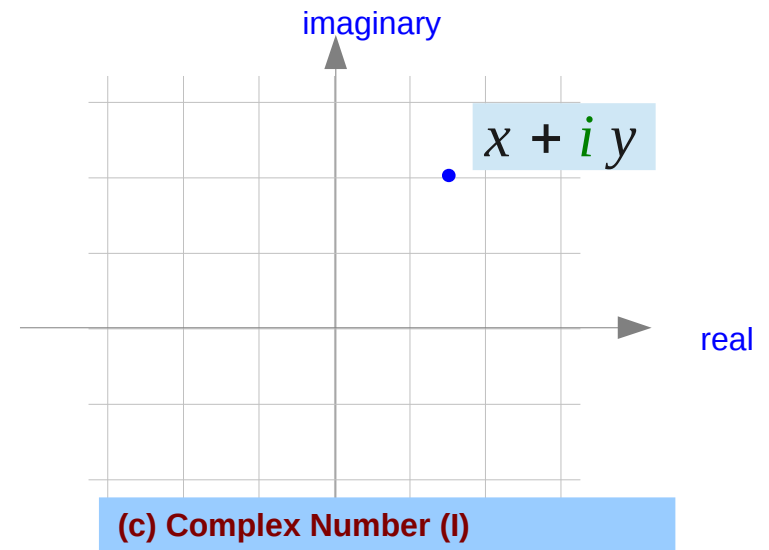
$$\tan \theta = \frac{y}{x}$$

$$x + iy = r \cos \theta + ir \sin \theta$$

$$= r(\cos \theta + i \sin \theta)$$

$$= r e^{i\theta}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$



# Euler's Formula (1)

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$x \quad \Rightarrow \quad \Re\{e^{i\theta}\} = \cos \theta$$

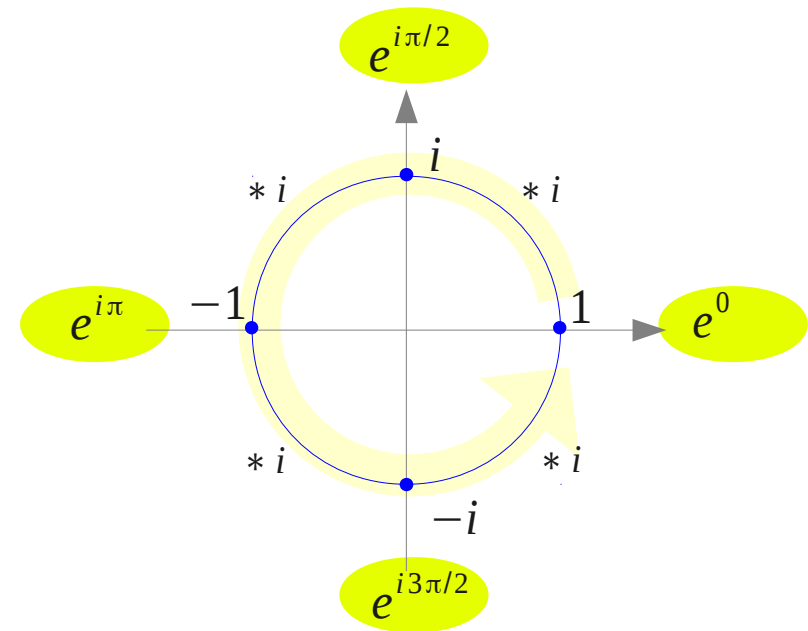
$$y \quad \Rightarrow \quad \Im\{e^{i\theta}\} = \sin \theta$$

$$e^0 = +1 + 0i$$

$$e^{i\pi/2} = 0 + 1i$$

$$e^{i\pi} = -1 + 0i$$

$$e^{i3\pi/2} = 0 - 1i$$



$$= +1 \quad = e^{-i2\pi}$$

$$= +i \quad = e^{-i3\pi/2}$$

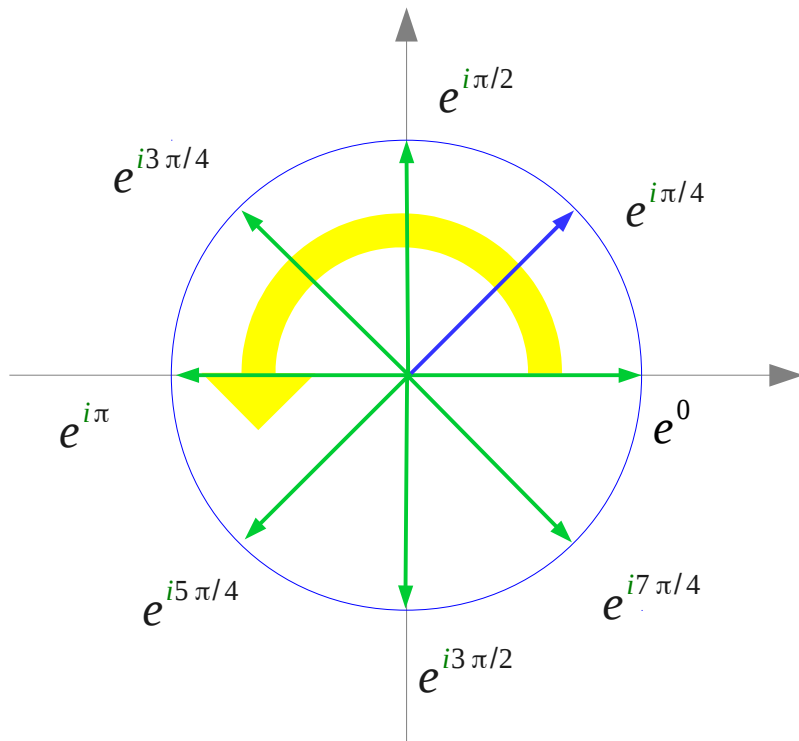
$$= -1 \quad = e^{-i\pi}$$

$$= -i \quad = e^{-i\pi/2}$$

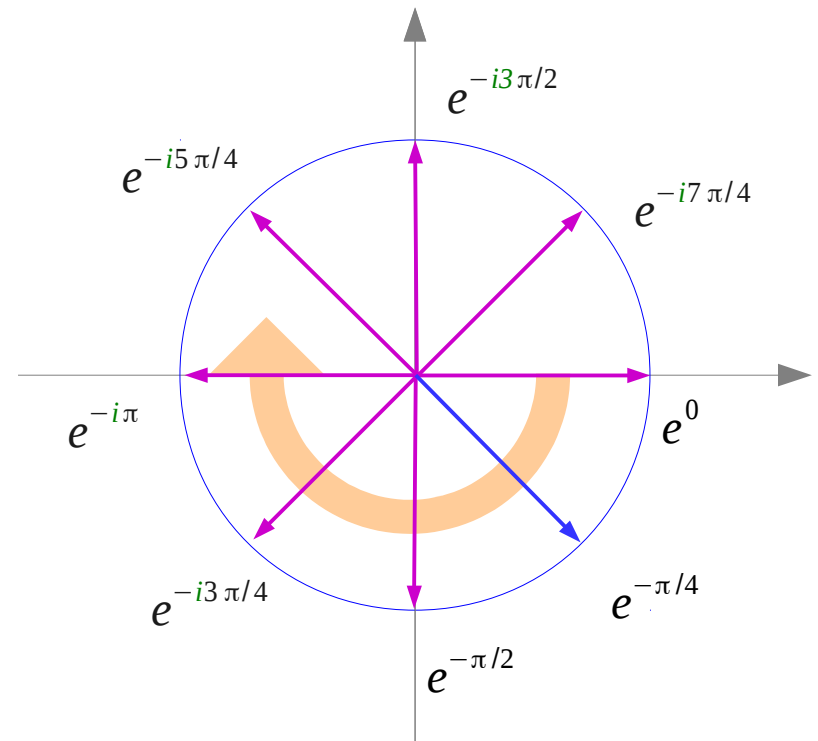


# Euler's Formula (2)

$$e^{+i\theta} = \cos \theta + i \sin \theta$$



$$e^{-i\theta} = \cos \theta - i \sin \theta$$



# The Euler constant $e$

$$\frac{d}{dx} a^x = \lim_{h \rightarrow 0} \frac{a^{(x+h)} - a^x}{h} = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \Rightarrow a^x \quad \text{such } a, \text{ we call } e$$

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1 \iff \lim_{h \rightarrow 0} \frac{a^h - a^0}{h - 0} = 1 \iff f'(0) = 1$$

$$\frac{d}{dx} e^x = e^x$$

$$e = 2.71828\dots$$

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

# The Euler constant $e$

$$\frac{d}{dx} e^x = e^x$$

$$e = 2.71828\dots$$

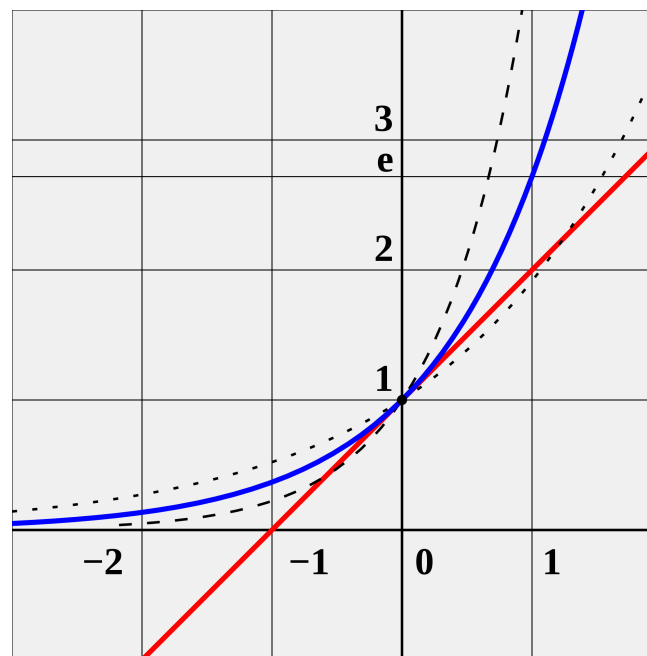
$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1 \quad \text{iif} \quad a = e$$

$$f'(0) = 1 \quad \lim_{h \rightarrow 0} \frac{a^h - a^0}{h - 0} = 1$$



Functions  $f(x) = a^x$  are shown for several values of  $a$ .  $e$  is the unique value of  $a$ , such that the derivative of  $f(x) = a^x$  at the point  $x = 0$  is equal to 1. The blue curve illustrates this case,  $\text{ex}$ . For comparison, functions  $2^x$  (dotted curve) and  $4^x$  (dashed curve) are shown; they are not tangent to the line of slope 1 and  $y$ -intercept 1 (red).

<http://en.wikipedia.org/wiki/Derivative>

# The Derivative of $a^x$

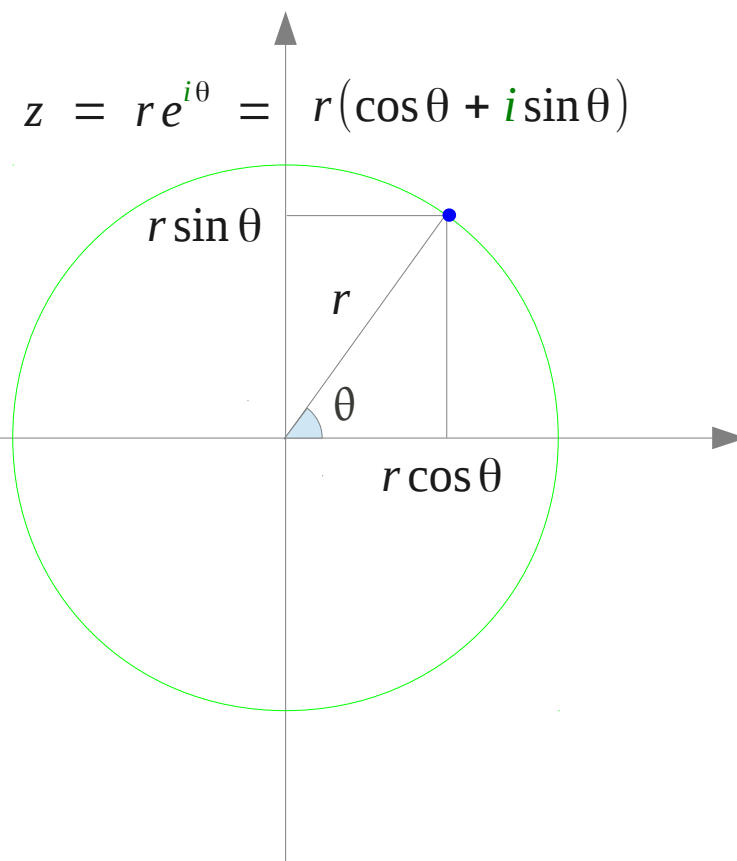
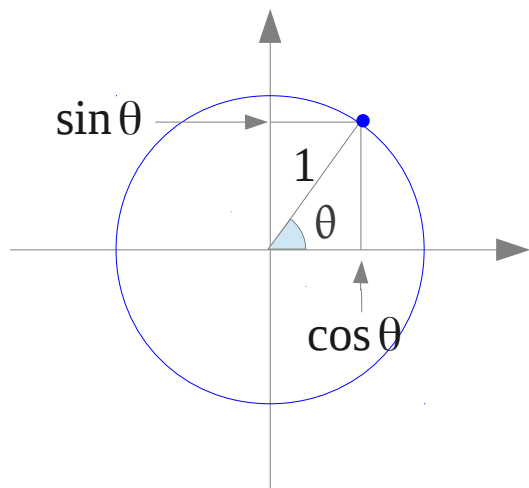
$$a^x = e^{\ln a^x} = e^{x \ln a}$$

$$\begin{aligned} \frac{d}{dx} \{a^x\} &= \frac{d}{dx} \{e^{x \ln a}\} \\ &= \{e^{x \ln a}\} \frac{d}{dx} \{x \ln a\} \end{aligned}$$

$$\frac{d}{dx} \{a^x\} = \{a^x\} \ln a$$

$$\frac{d}{dx} \{e^x\} = \{e^x\} \ln e = \{e^x\}$$

# Absolute Values and Arguments



**absolute value**

$$|z| = r$$

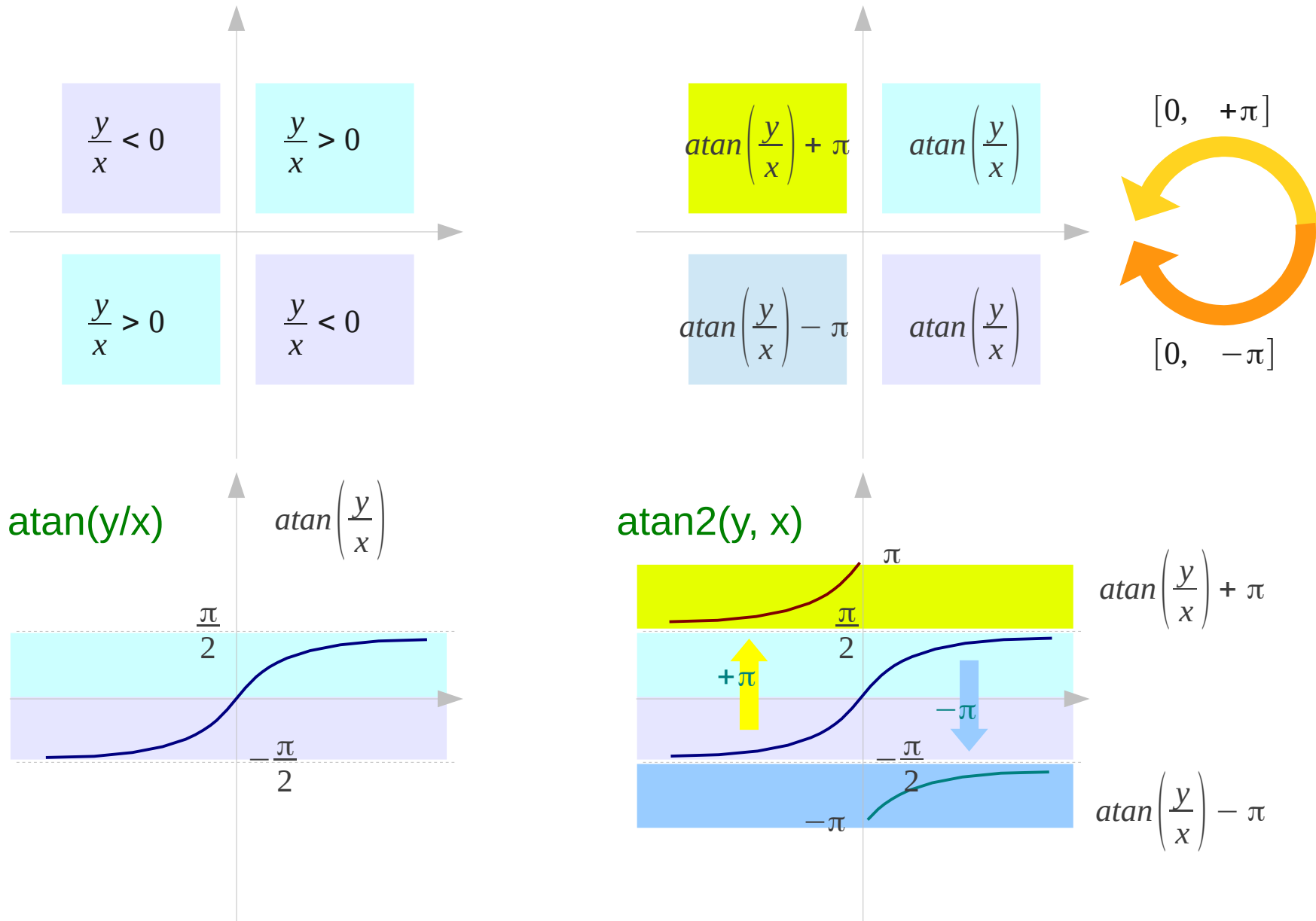
$$\Rightarrow |r e^{i\theta}| = |r| |e^{i\theta}| = r \sqrt{\cos^2 \theta + \sin^2 \theta}$$

**argument, phase**

$$\arg(z) = \theta$$

$$\Rightarrow \arg(r e^{i\theta})$$

# Computing Complex Number Argument



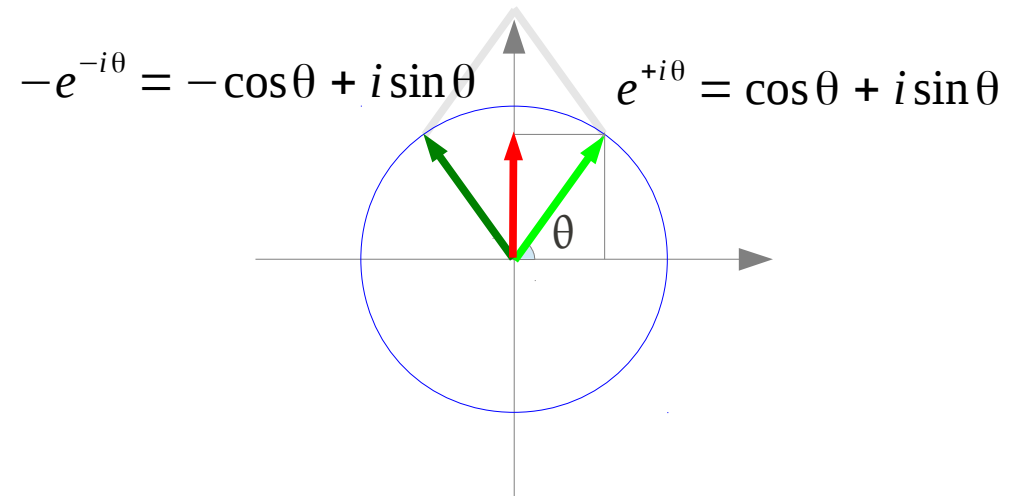
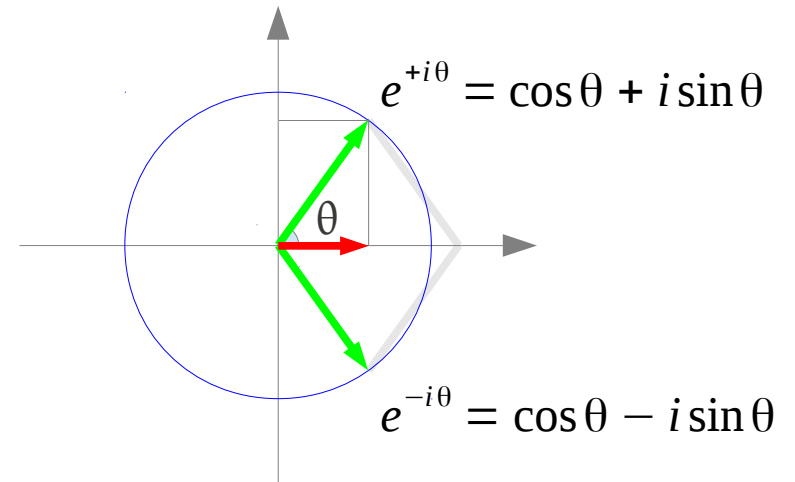
# sin and cos

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

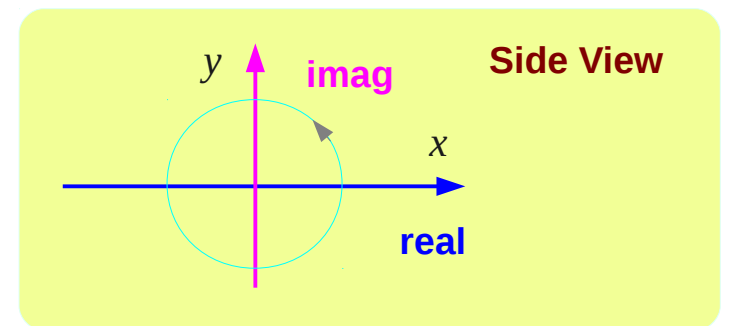
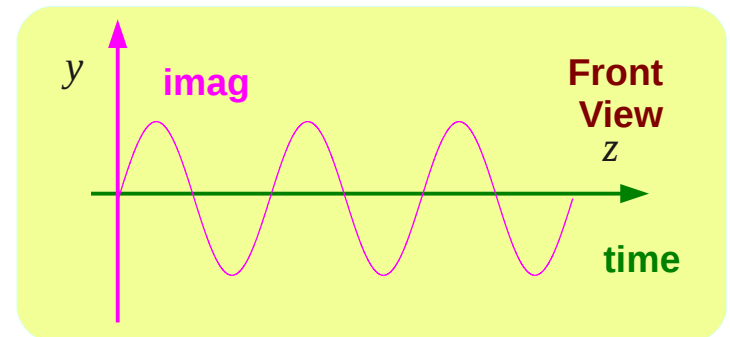
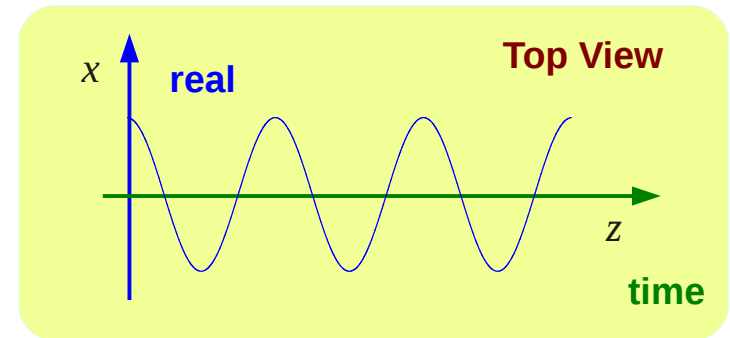
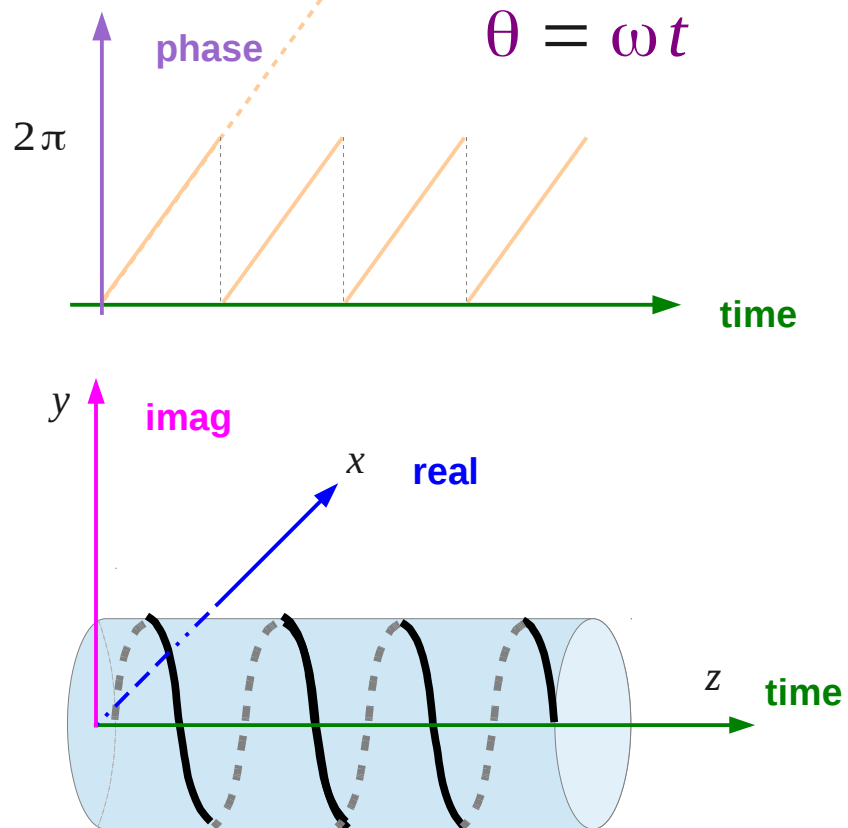
$$\Re\{e^{i\theta}\} = \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\Im\{e^{i\theta}\} = \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$



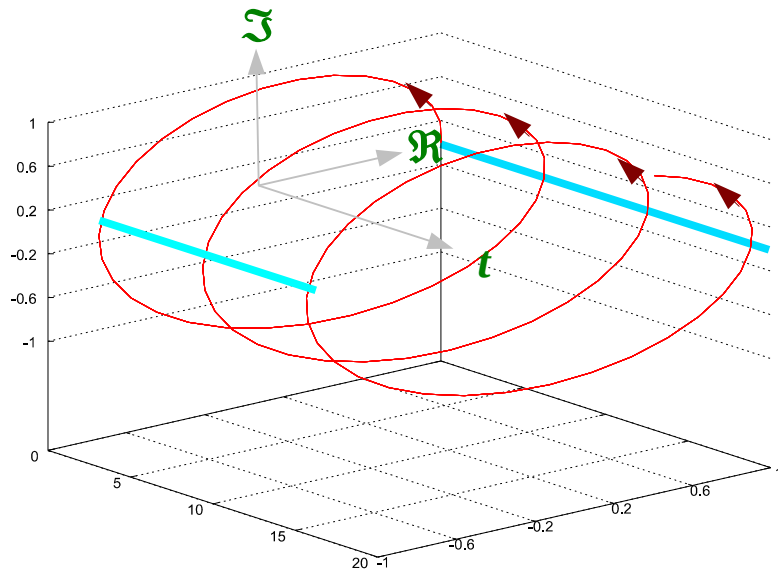
# Complex Exponential

$$e^{+j\omega t} = \cos \omega t + j \sin \omega t$$



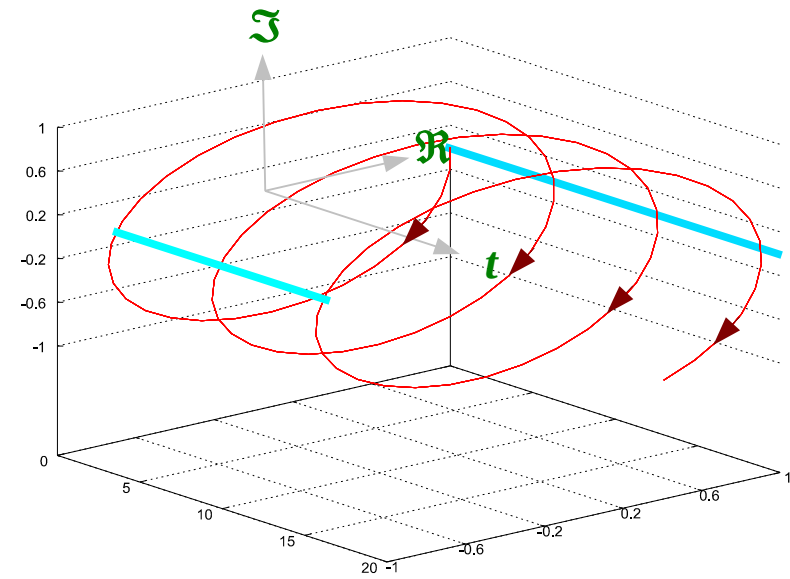


# Conjugate Complex Exponential



$$e^{+j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)$$

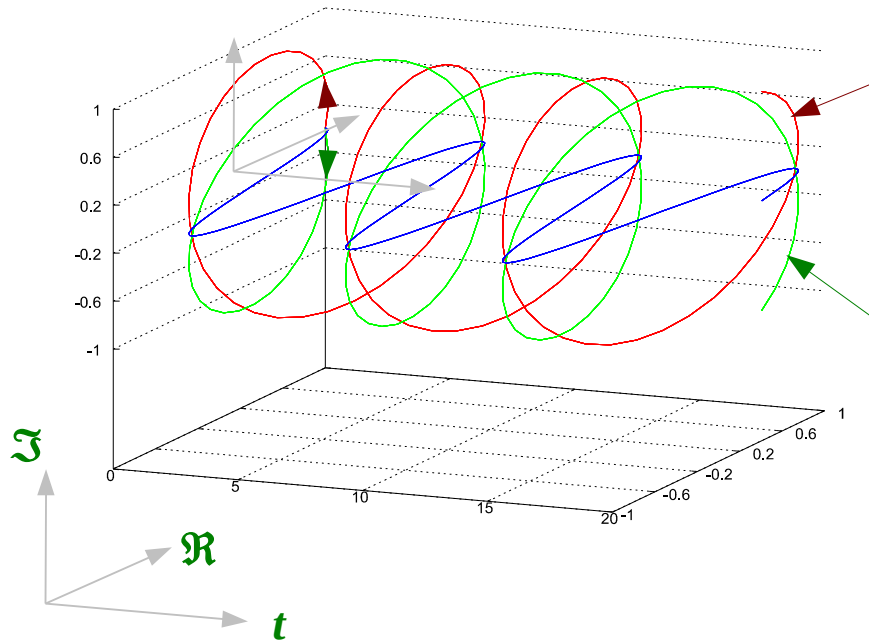
$$e^{+jt} = \cos(t) + j \sin(t) \quad (\omega_0 = 1)$$



$$e^{-j\omega_0 t} = \cos(\omega_0 t) - j \sin(\omega_0 t)$$

$$e^{-jt} = \cos(t) - j \sin(t) \quad (\omega_0 = 1)$$

# Cos( $\omega_0 t$ )



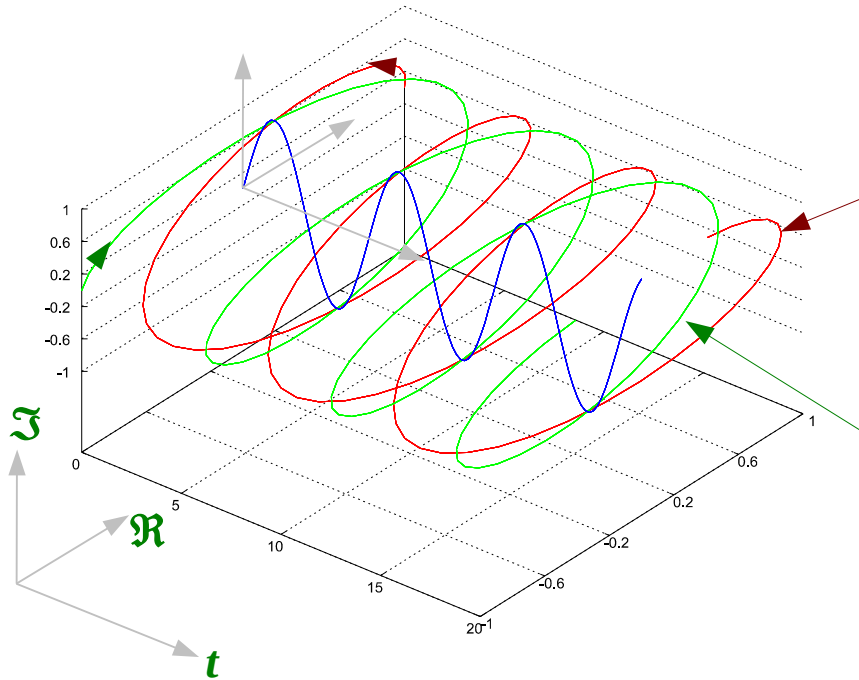
$$e^{+jt} = \cos(t) + j \sin(t)$$

$$(e^{+jt} + e^{-jt}) = 2 \cos(t)$$

$$e^{-jt} = \cos(t) - j \sin(t)$$

$$\begin{aligned} x(t) &= A \cos(\omega_0 t) \\ &= \frac{A}{2} e^{+j\omega_0 t} + \frac{A}{2} e^{-j\omega_0 t} \end{aligned}$$

# Sin( $\omega_0 t$ )



$$e^{+jt} = \cos(t) + j \sin(t)$$

$$(e^{+jt} - e^{-jt}) = 2j \sin(t)$$

$$-e^{-jt} = -\cos(t) + j \sin(t)$$

$$\begin{aligned} x(t) &= A \sin(\omega_0 t) \\ &= \frac{A}{2j} e^{+j\omega_0 t} - \frac{A}{2j} e^{-j\omega_0 t} \end{aligned}$$

# Complex Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$

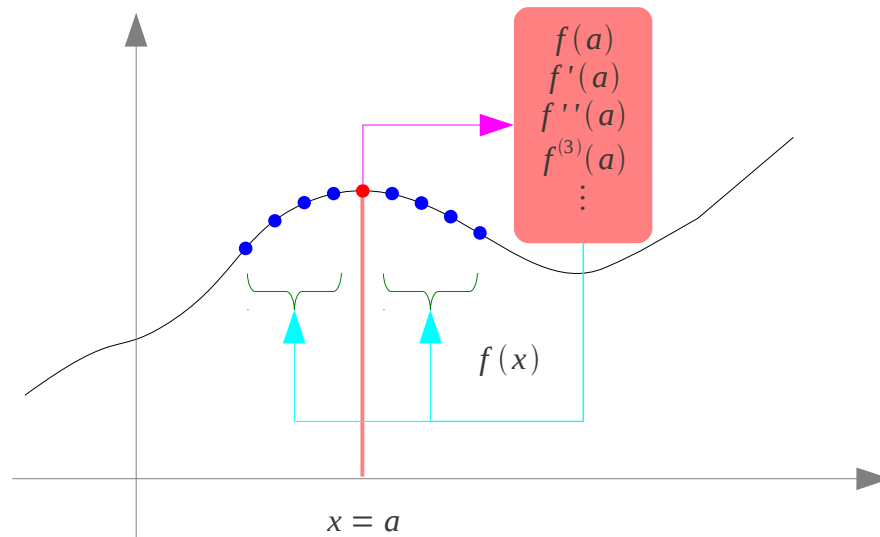
$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \dots$$

$$= \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots \right) + i \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots \right)$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

# Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$





# Power Series Expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

# Complex Arithmetic

$$z = a + ib$$

$$w = c + id$$

$$z+w = (a+c) + i(b+d)$$

$$z = a + ib$$

$$w = c + id$$

$$zw = (ac-bd) + i(ad+bc)$$

$$z = a + ib$$

$$w = c + id$$

$$z-w = (a-c) + i(b-d)$$

$$z = a + ib$$

$$w = c + id$$

$$\frac{z}{w} = \left( \frac{a+ib}{c+id} \right)$$

$$= \left( \frac{a+ib}{c+id} \right) \left( \frac{c-id}{c-id} \right)$$

$$= \left( \frac{ac+bd}{c^2+d^2} \right) + i \left( \frac{-ad+bc}{c^2+d^2} \right)$$



# Complex Conjugate

$$z = x + iy = \Re\{z\} + i\Im\{z\}$$

$$\bar{z} = x - iy = \Re\{z\} - i\Im\{z\}$$

$$\Re\{z\} = \frac{1}{2}(z + \bar{z})$$

$$\Im\{z\} = \frac{1}{2i}(z - \bar{z})$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{z - w} = \bar{z} - \bar{w}$$

$$\overline{\bar{z}w} = z\bar{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{x^2 + y^2} = \frac{\bar{z}}{|z|^2}$$

# Complex Power (1)

$$a = e^{\log_e a} = e^{\ln a}$$

$$a^b = (e^{\log_e a})^b = (e^{\ln a})^b = e^{b \ln a}$$

$$\begin{aligned} a^{ib} &= (e^{\log_e a})^{ib} = (e^{\ln a})^{ib} = e^{ib \ln a} \\ &= \cos(b \ln a) + i \sin(b \ln a) \end{aligned}$$

$$a^{c+ib} = a^c (e^{\log_e a})^{ib} = a^c (e^{\ln a})^{ib} = a^c e^{ib \ln a}$$

## Complex Power (2)

$$a = e^{\ln a}$$

$$a^b = e^{b \ln a}$$

$$a^{ib} = e^{ib \ln a} = [\cos(b \ln a) + i \sin(b \ln a)]$$

$$a^{c+ib} = a^c e^{ib \ln a} = a^c [\cos(b \ln a) + i \sin(b \ln a)]$$

## References

- [1] <http://en.wikipedia.org/>
- [2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [3] E. Kreyszig, “Advanced Engineering Mathematics”
- [4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
- [5] [www.chem.arizona.edu/~salzmanr/480a](http://www.chem.arizona.edu/~salzmanr/480a)